MULTIGRADED RINGS, DIAGONAL SUBALGEBRAS, AND RATIONAL SINGULARITIES

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1. Introduction

We study the properties of F-rationality and F-regularity in multigraded rings and their diagonal subalgebras. The main focus is on diagonal subalgebras of bigraded rings; these constitute an interesting class of rings since they arise naturally as homogeneous coordinate rings of blow-ups of projective varieties.

Let $X$ be a projective variety over a field $K$, with homogeneous coordinate ring $A$. Let $a \subseteq A$ be a homogeneous ideal, and $V \subseteq X$ the closed subvariety defined by $a$. For $g$ an integer, we use $a_g$ to denote the $K$-vector space consisting of homogeneous elements of $a$ of degree $g$. If $g \gg 0$, then $a_g$ defines a very ample complete linear system on the blow-up of $X$ along $V$, and hence $K[a_g]$ is a homogeneous coordinate ring for this blow-up. Since the ideals $a^h$ define the same subvariety $V$, the rings $K[(a^h)_g]$ are homogeneous coordinate rings for the blow-up provided $g \gg h > 0$.

Suppose that $A$ is a standard $\mathbb{N}$-graded $K$-algebra, and consider the $\mathbb{N}^2$-grading on the Rees algebra $A[\mathfrak{a}t]$, where $\deg rt^j = (i, j)$ for $r \in A_i$. The connection with diagonal subalgebras stems from the fact that if $a^h$ is generated by elements of degree less than or equal to $g$, then

$$K[(a^h)_g] \cong \bigoplus_{k \geq 0} A[\mathfrak{a}t]_{(gk, hk)}.$$ 

Using $\Delta = (g, h)\mathbb{Z}$ to denote the $(g, h)$-diagonal in $\mathbb{Z}^2$, the diagonal subalgebra $A[\mathfrak{a}t]_{\Delta} = \bigoplus_{k} A[\mathfrak{a}t]_{(gk, hk)}$ is a homogeneous coordinate ring for the blow-up of $\text{Proj} A$ along the subvariety defined by $a$, whenever $g \gg h > 0$.

The papers [GG, GGH, GGP, Tr] use diagonal subalgebras in studying blow-ups of projective space at finite sets of points. For $A$ a polynomial ring and $a$ a homogeneous ideal, the ring theoretic properties of $K[a_g]$ are studied by Simis, Trung, and Valla in [STV] by realizing $K[a_g]$ as a diagonal subalgebra of the Rees algebra $A[\mathfrak{a}t]$. In particular, they determine when $K[a_g]$ is Cohen-Macaulay for $a$ a complete intersection ideal generated by forms of equal degree, and also for $a$ the

Date: March 24, 2008.
2000 Mathematics Subject Classification. Primary 13A02; Secondary 13A35, 13H10, 14B15.
A.S. was supported in part by grants from the National Science Foundation.
ideal of maximal minors of a generic matrix. Some of their results are extended by Conca, Herzog, Trung, and Valla as in the following theorem:

**Theorem 1.1.** [CHTV, Theorem 4.6] Let $K[x_1, \ldots, x_m]$ be a polynomial ring over a field, and let $\mathfrak{a}$ be a complete intersection ideal minimally generated by forms of degrees $d_1, \ldots, d_r$. Fix positive integers $g$ and $h$ with $g/h > d = \max\{d_1, \ldots, d_r\}$.

Then $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay if and only if $g > (h-1)d - m + \sum_{j=1}^{r} d_j$.

When $A$ is a polynomial ring and $\mathfrak{a}$ an ideal for which $A[\mathfrak{a}^t]$ is Cohen-Macaulay, Lavila-Vidal [Lv1, Theorem 4.5] proved that the diagonal subalgebras $K[(\mathfrak{a}^h)_g]$ are Cohen-Macaulay for $g \gg h \gg 0$, thereby settling a conjecture from [CHTV]. In [CH] Cutkosky and Herzog obtain affirmative answers regarding the existence of a constant $c$ such that $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay whenever $g \geq ch$. For more work on the Cohen-Macaulay and Gorenstein properties of diagonal subalgebras, see [HHR, Hy2, Lv2], and [LvZ].

As a motivating example for some of the results of this paper, consider a polynomial ring $A = K[x_1, \ldots, x_m]$ and an ideal $\mathfrak{a} = (z_1, z_2)$ generated by relatively prime forms $z_1$ and $z_2$ of degree $d$. Setting $\Delta = (d+1, 1)\mathbb{Z}$, the diagonal subalgebra $A[\mathfrak{a}^t]_\Delta$ is a homogeneous coordinate ring for the blow-up of $\text{Proj } A = \mathbb{P}^{m-1}$ along the subvariety defined by $\mathfrak{a}$. The Rees algebra $A[\mathfrak{a}^t]$ has a presentation

$$R = K[x_1, \ldots, x_m, y_1, y_2]/(y_2 z_1 - y_1 z_2),$$

where $\deg x_i = (1, 0)$ and $\deg y_j = (d, 1)$, and consequently $R_\Delta$ is the subalgebra of $R$ generated by the elements $x_iy_j$. When $K$ has characteristic zero and $z_1$ and $z_2$ are general forms of degree $d$, the results of Section 3 imply that $R_\Delta$ has rational singularities if and only if $d \leq m$, and that it is of F-regular type if and only if $d < m$. As a consequence, we obtain large families of rings of the form $R_\Delta$, standard graded over a field, which have rational singularities, but which are not of F-regular type.

It is worth pointing out that if $R$ is an $\mathbb{N}^2$-graded ring over an infinite field $R_{(0,0)} = K$, and $\Delta = (g, h)\mathbb{Z}$ for coprime positive integers $g$ and $h$, then $R_\Delta$ is the ring of invariants of the torus $K^*$ acting on $R$ via

$$\lambda: r \mapsto \lambda^{hi-gj} r \quad \text{where } \lambda \in K^* \text{ and } r \in R_{(i,j)}.$$

Consequently there exist torus actions on hypersurfaces for which the rings of invariants have rational singularities but are not of F-regular type.

In Section 4 we use diagonal subalgebras to construct standard graded normal rings $R$, with isolated singularities, for which $H^2_{m}(R)_0 = 0$ and $H^2_{m}(R)_1 \neq 0$. If $S$ is the localization of such a ring $R$ at its homogeneous maximal ideal, then, by Danilov’s results, the divisor class group of $S$ is a finitely generated abelian group, though $S$ does not have a discrete divisor class group. Such rings $R$ are also of interest in view of the results of [RSS], where it is proved that the image of
$H^2_m(R)_0$ in $H^2_m(R^+_0)$ is annihilated by elements of $R^+_0$ of arbitrarily small positive degree; a corresponding result for $H^2_m(R)_1$ is not known at this point, and the rings constructed in Section 4 constitute interesting test cases.

Section 2 summarizes some notation and conventions for multigraded rings and modules. In Section 3 we carry out an analysis of diagonal subalgebras of bigraded hypersurfaces; this uses results on rational singularities and F-regular rings proved in Sections 5 and 6 respectively.

The authors would like to thank Shiro Goto and Ken-ichi Yoshida for their valuable comments.

2. Preliminaries

In this section, we provide a brief treatment of multigraded rings and modules; see [GW1, GW2, HHR], and [HIO] for further details.

By an $\mathbb{N}_r$-graded ring we mean a ring $R = \bigoplus_{n \in \mathbb{N}_r} R_n$, which is finitely generated over the subring $R_0$. If $(R_0, m)$ is a local ring, then $R$ has a unique homogeneous maximal ideal $\mathfrak{M} = mR + R_+$, where $R_+ = \oplus_{n \neq 0} R_n$.

For $m = (m_1, \ldots, m_r)$ and $n = (n_1, \ldots, n_r) \in \mathbb{Z}_r$, we say $n > m$ (resp. $n \geq m$) if $n_i > m_i$ (resp. $n_i \geq m_i$) for each $i$.

Let $M$ be a $\mathbb{Z}_r$-graded $R$-module. For $m \in \mathbb{Z}_r$, we set $M_{\geq m} = \bigoplus_{n \geq m} M_n$, which is a $\mathbb{Z}_r$-graded submodule of $M$. One writes $M(m)$ for the $\mathbb{Z}_r$-graded $R$-module with shifted grading $[M(m)]_n = M_{m+n}$ for each $n \in \mathbb{Z}_r$.

Let $M$ and $N$ be $\mathbb{Z}_r$-graded $R$-modules. Then $\text{Hom}_R(M, N)$ is the $\mathbb{Z}_r$-graded module with $[\text{Hom}_R(M, N)]_n$ being the abelian group consisting of degree preserving $R$-linear homomorphisms from $M$ to $N(n)$.

The functor $\text{Ext}^i_R(M, -)$ is the $i$-th derived functor of $\text{Hom}_R(M, -)$ in the category of $\mathbb{Z}_r$-graded $R$-modules. When $M$ is finitely generated, $\text{Ext}^i_R(M, N)$ and $\text{Ext}^i_R(M, N)$ agree as underlying $R$-modules. For a homogeneous ideal $\mathfrak{a}$ of $R$, the local cohomology modules of $M$ with support in $\mathfrak{a}$ are the $\mathbb{Z}_r$-graded modules

$$H^i_\mathfrak{a}(M) = \lim_{\longrightarrow} \text{Ext}^i_R(R/\mathfrak{a}^n, M).$$

Let $\varphi: \mathbb{Z}_r \rightarrow \mathbb{Z}_s$ be a homomorphism of abelian groups satisfying $\varphi(\mathbb{N}_r) \subseteq \mathbb{N}_s$. We write $R^{\varphi}$ for the ring $R$ with the $\mathbb{N}_s$-grading where

$$[R^{\varphi}]_n = \bigoplus_{\varphi(m)=n} R_m.$$
If $M$ is a $\mathbb{Z}^r$-graded $R$-module, then $M^\varphi$ is the $\mathbb{Z}^s$-graded $R^\varphi$-module with

$$[M^\varphi]_n = \bigoplus_{\varphi(m) = n} M_m. $$

The change of grading functor $(-)^\varphi$ is exact; by [HHR, Lemma 1.1] one has

$$H^i_{\mathbb{N}^r}(M)^\varphi = H^i_{\mathbb{N}^s}(M^\varphi).$$

Consider the projections $\varphi_i: \mathbb{Z}^r \rightarrow \mathbb{Z}$ with $\varphi_i(m_1, \ldots, m_r) = m_i$, and set

$$a(R^\varphi) = \max \{ a \in \mathbb{Z} | [H^i_{\mathbb{N}^r}(R)^{\varphi_i}]_a \neq 0 \};$$

this is the $\alpha$-invariant of the $\mathbb{N}$-graded ring $R^\varphi$ in the sense of Goto and Watanabe [GW1]. As in [HHR], the multigraded $\alpha$-invariant of $R$ is

$$a(R) = (a(R^{\varphi_1}), \ldots, a(R^{\varphi_s})).$$

Let $R$ be a $\mathbb{Z}^2$-graded ring and let $g, h$ be positive integers. The subgroup $\Delta = (g, h)\mathbb{Z}$ is a diagonal in $\mathbb{Z}^2$, and the corresponding diagonal subalgebra of $R$ is

$$R_\Delta = \bigoplus_{k \in \mathbb{Z}} R_{(gk, hk)}.$$

Similarly, if $M$ is a $\mathbb{Z}^2$-graded $R$-module, we set

$$M_\Delta = \bigoplus_{k \in \mathbb{Z}} M_{(gk, hk)},$$

which is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $R_\Delta$.

**Lemma 2.1.** Let $A$ and $B$ be $\mathbb{N}$-graded normal rings, finitely generated over a field $A_0 = K = B_0$. Set $T = A \otimes_K B$. Let $g$ and $h$ be positive integers and set $\Delta = (g, h)\mathbb{Z}$. Let $a$, $b$, and $m$ denote the homogeneous maximal ideals of $A$, $B$, and $T_\Delta$ respectively. Then, for each $q \geq 0$ and $i, j, k \in \mathbb{Z}$, one has

$$H^q_m(T(i, j)_\Delta)_k = (A_{i+gk} \otimes H^q_b(B)_{j+hk}) \oplus (H^q_a(A)_{i+gk} \otimes B_{j+hk}) \oplus \bigoplus_{q_1 + q_2 = q+1} (H^q_a(A)_{i+gk} \otimes H^q_b(B)_{j+hk}).$$

**Proof.** Let $A^{(g)}$ and $B^{(h)}$ denote the respective Veronese subrings of $A$ and $B$. Set

$$A^{(g, i)} = \bigoplus_{k \in \mathbb{Z}} A_{i+gk} \quad \text{and} \quad B^{(h, j)} = \bigoplus_{k \in \mathbb{Z}} B_{j+hk}.$$ 

These are graded modules over $A^{(g)}$ and $B^{(h)}$ respectively, and

$$T(i, j)_\Delta = \bigoplus_{k \in \mathbb{Z}} A_{i+gk} \otimes_K B_{j+hk} = A^{(g, i)} \# B^{(h, j)}.$$ 

The ideal $A_+^{(g)} A$ is $a$-primary; likewise, $B_+^{(h)} B$ is $b$-primary. The Künneth formula for local cohomology, [GW1, Theorem 4.1.5], now gives the desired result. \ 

**Notation 2.2.** We use bold letters to denote lists of elements, e.g., $z = z_1, \ldots, z_s$ and $\gamma = \gamma_1, \ldots, \gamma_s$. 
3. Diagonal subalgebras of bigraded hypersurfaces

We prove the following theorem about diagonal subalgebras of \( \mathbb{N}^2 \)-graded hypersurfaces. The proof uses results proved later in Sections 5 and 6.

**Theorem 3.1.** Let \( K \) be a field, let \( m, n \) be integers with \( m, n \geq 2 \), and let
\[
\mathcal{R} = K[x_1, \ldots, x_m, y_1, \ldots, y_n]/(f)
\]
be a normal \( \mathbb{N}^2 \)-graded hypersurface where \( \deg x_i = (1, 0) \), \( \deg y_j = (0, 1) \), and \( \deg f = (d, e) > (0, 0) \). For positive integers \( g \) and \( h \), set \( \Delta = (g, h)\mathbb{Z} \). Then:

1. The ring \( \mathcal{R}_\Delta \) is Cohen-Macaulay if and only if \( \lfloor (d - m)/g \rfloor < e/h \) and \( \lfloor (e - n)/h \rfloor < d/g \). In particular, if \( d < m \) and \( e < n \), then \( \mathcal{R}_\Delta \) is Cohen-Macaulay for each diagonal \( \Delta \).

2. The graded canonical module of \( \mathcal{R}_\Delta \) is \( \mathcal{R}(d - m, e - n)_\Delta \). Hence \( \mathcal{R}_\Delta \) is Gorenstein if and only if \( (d - m)/g = (e - n)/h \), and this is an integer.

If \( K \) has characteristic zero, and \( f \) is a generic polynomial of degree \( (d, e) \), then:

3. The ring \( \mathcal{R}_\Delta \) has rational singularities if and only if it is Cohen-Macaulay and \( d < m \) or \( e < n \).

4. The ring \( \mathcal{R}_\Delta \) is of \( F \)-regular type if and only if \( d < m \) and \( e < n \).

For \( m, n \geq 3 \) and \( \Delta = (1, 1)\mathbb{Z} \), the properties of \( \mathcal{R}_\Delta \), as determined by \( m, n, d, e \), are summarized in Figure 1.

![Figure 1. Properties of \( \mathcal{R}_\Delta \) for \( \Delta = (1, 1)\mathbb{Z} \).](image-url)
Remark 3.2. Let \( m, n \geq 2 \). A generic hypersurface of degree \((d, e) > (0, 0)\) in \( m, n \) variables is normal precisely when
\[
m > \min(2, d) \quad \text{and} \quad n > \min(2, e).
\]
Suppose that \( m = 2 = n \), and that \( f \) is nonzero. Then \( \dim \mathcal{R}_\Delta = 2 \); since \( \mathcal{R}_\Delta \) is generated over a field by elements of equal degree, \( \mathcal{R}_\Delta \) is of F-regular type if and only if it has rational singularities; see [Wa2]. This is the case precisely if
\[
d = 1, \quad e \leq h + 1, \quad \text{or} \quad e = 1, \quad d \leq g + 1.
\]

Following a suggestion of Hara, the case \( n = 2 \) and \( e = 1 \) was used in [Si, Example 7.3] to construct examples of standard graded rings with rational singularities which are not of F-regular type.

Proof of Theorem 3.1. Set \( A = K[x], B = K[y], \) and \( T = A \otimes_K B \). By Lemma 2.1, \( H^q_m(T(\Delta)) = 0 \) for \( q \neq m + n - 1 \). The local cohomology exact sequence induced by
\[
0 \longrightarrow T(-d, -e)_{\Delta} \longrightarrow T_{\Delta} \longrightarrow \mathcal{R}_{\Delta} \longrightarrow 0
\]
therefore gives \( H^{q-1}_m(\mathcal{R}_{\Delta}) = H^q_m(T(-d, -e)_{\Delta}) \) for \( q \leq m + n - 2 \), and also shows that \( H^{m+n-2}_m(\mathcal{R}_{\Delta}) \) and \( H^{m+n-1}_m(\mathcal{R}_{\Delta}) \) are, respectively, the kernel and cokernel of
\[
H^{m+n-1}_m(T(-d, -e)_{\Delta}) \longrightarrow H^{m+n-1}_m(\mathcal{R}_{\Delta})
\]
\[
[H^m_a(A(-d)) \otimes H^n_b(B(-e))]_{\Delta} \longrightarrow [H^m_a(A) \otimes H^n_b(B)]_{\Delta}.
\]
The horizontal map above is surjective since its graded dual
\[
[A(d - m) \otimes B(e - n)]_{\Delta} \longleftarrow [A(-m) \otimes B(-n)]_{\Delta}
\]
\[
T(d - m, e - n)_{\Delta} \longleftarrow T(-m, -n)_{\Delta}
\]
is injective. In particular, \( \dim \mathcal{R}_{\Delta} = m + n - 2 \).

It follows from the above discussion that \( \mathcal{R}_{\Delta} \) is Cohen-Macaulay if and only if \( H^q_m(T(-d, -e)_{\Delta}) = 0 \) for each \( q \leq m + n - 2 \). By Lemma 2.1, this is the case if and only if, for each integer \( k \), one has
\[
A_{-d+gk} \otimes H^n_b(B)_{-e+hk} = 0 = H^m_a(A)_{-d+gk} \otimes B_{-e+hk}.
\]
Hence \( \mathcal{R}_{\Delta} \) is Cohen-Macaulay if and only if there is no integer \( k \) satisfying
\[
d/g \leq k \leq (e - n)/h \quad \text{or} \quad e/h \leq k \leq (d - m)/g,
\]
which completes the proof of (1).

For (2), note that the graded canonical module of \( \mathcal{R}_{\Delta} \) is the graded dual of \( H^{m+n-2}_m(\mathcal{R}_{\Delta}) \), and hence that it equals
\[
coker(T(-m, -n)_{\Delta} \longrightarrow T(d - m, e - n)_{\Delta}) = \mathcal{R}(d - m, e - n)_{\Delta}.
\]
This module is principal if and only if \( R(d - m, e - n)_\Delta = R_\Delta(a) \) for some integer \( a \), i.e., \( d - m = ga \) and \( e - n = ha \).

When \( f \) is a general polynomial of degree \((d, e)\), the ring \( R_\Delta \) has an isolated singularity. Also, \( R_\Delta \) is normal since it is a direct summand of the normal ring \( R \). By Theorem 5.1, \( R_\Delta \) has rational singularities precisely if it is Cohen-Macaulay and \( a(R_\Delta) < 0 \); this proves (3).

It remains to prove (4). If \( d < m \) and \( e < n \), then Theorem 5.2 implies that \( R \) has rational singularities. By Theorem 6.2, it follows that for almost all primes \( p \), the characteristic \( p \) models \( R_p \) of \( R \) are \( F \)-rational hypersurfaces which, therefore, are \( F \)-regular. Alternatively, \( R_\Delta \) is a generic hypersurface of degree \((d, e) < (m, n)\), so Theorem 6.5 implies that \( R_p \) is \( F \)-regular. Since \( (R_p)_\Delta \) is a direct summand of \( R_p \), it follows that \( (R_p)_\Delta \) is \( F \)-regular. The rings \( (R_p)_\Delta \) are characteristic \( p \) models of \( R_\Delta \), so we conclude that \( R_\Delta \) is of \( F \)-regular type.

Suppose \( R_\Delta \) has \( F \)-regular type, and let \( (R_p)_\Delta \) be a characteristic \( p \) model which is \( F \)-regular. Fix an integer \( k > d/g \). Then Proposition 6.3 implies that there exists an integer \( q = p^e \) such that

\[
\text{rank}_K ((R_p)_\Delta)_k \leq \text{rank}_K \left[ H_{m+n-2}^{m+n-2}((\omega(q))^\Delta) \right]_k,
\]

where \( \omega \) is the graded canonical module of \( (R_p)_\Delta \). Using (2), we see that

\[
H_{m+n-2}^{m+n-2}((\omega(q))^\Delta) = H_{m+n-2}^{m+n-2}(R_p(qd - qm, qe - qn)_\Delta).
\]

Let \( T_p \) be a characteristic \( p \) model for \( T \) such that \( T_p/fT_p = R_p \). Multiplication by \( f \) on \( T_p \) induces a local cohomology exact sequence

\[
\cdots \longrightarrow H_{m_p}^{m+n-2}(T_p(qd - qm, qe - qn)_\Delta) \longrightarrow H_{m_p}^{m+n-2}(R_p(qd - qm, qe - qn)_\Delta) \longrightarrow \cdots.
\]

Since \( H_{m_p}^{m+n-2}(T_p(qd - qm, qe - qn)_\Delta) \) vanishes by Lemma 2.1, we conclude that

\[
\text{rank}_K ((R_p)_\Delta)_k \leq \text{rank}_K \left[ H_{m_p}^{m+n-2}(T_p(qd - qm - d, qe - qn - e)_\Delta) \right]_k = \text{rank}_K H_{m_p}^{n-1}(A_p)_{qd-qm-d+gk} \otimes H_{bq}^{n-1}(B_p)_{qe-qn-e+hk}.
\]

Hence \( qd - qm - d + gk < 0 \); as \( d - gk < 0 \), we conclude \( d < m \). Similarly, \( e < n \). □

We conclude this section with an example where a local cohomology module of a standard graded ring is not rigid in the sense that \( H^2_m(R)_0 = 0 \) while \( H^2_m(R)_1 \neq 0 \). Further such examples are constructed in Section 4.

**Proposition 3.3.** Let \( K \) be a field and let

\[
R = K[x_1, x_2, x_3, y_1, y_2]/(f)
\]

where \( \deg x_i = (1, 0) \), \( \deg y_j = (0, 1) \), and \( \deg f = (d, e) \) for \( d \geq 4 \) and \( e \geq 1 \). Let \( g \) and \( h \) be positive integers such that \( g \leq d - 3 \) and \( h \geq e \), and set \( \Delta = (g, h) \mathbb{Z} \). Then \( H^2_m(R_\Delta)_0 = 0 \) and \( H^2_m(R_\Delta)_1 \neq 0 \).
Proof. Using the resolution of $\mathcal{R}$ over the polynomial ring $T$ as in the proof of Theorem 3.1, we have an exact sequence
\[ H^2_m(T_\Delta) \longrightarrow H^2_m(\mathcal{R}_\Delta) \longrightarrow H^3_m(T(-d,-e)_\Delta) \longrightarrow H^3_m(T_\Delta). \]
Lemma 2.1 implies that $H^2_m(\mathcal{R}_\Delta) = 0 = H^3_m(T_\Delta)$. Hence, again by Lemma 2.1,
\[ H^2_m(R_\Delta)_0 = H^3(A)_{-d} \otimes B_{-e} = 0 \quad \text{and} \quad H^2_m(R_\Delta)_1 = H^3(A)_{g-d} \otimes B_{h-e} \neq 0. \]

4. Non-rigid local cohomology modules

We construct examples of standard graded normal rings $R$ over $\mathbb{C}$, with only isolated singularities, for which $H^2_m(R)_0 = 0$ and $H^2_m(R)_1 \neq 0$. Let $S$ be the localization of such a ring $R$ at its homogeneous maximal ideal. By results of Danilov [Da1, Da2], Theorem 4.1 below, it follows that the divisor class group of $S$ is finitely generated, though $S$ does not have a discrete divisor class group, i.e., the natural map $\text{Cl}(S) \longrightarrow \text{Cl}(S[[t]])$ is not bijective. Here, remember that if $A$ is a Noetherian normal domain, then so is $A[[t]]$.

**Theorem 4.1.** Let $R$ be a standard graded normal ring, which is finitely generated as an algebra over $R_0 = \mathbb{C}$. Assume, moreover, that $X = \text{Proj} R$ is smooth. Set $(S, m)$ to be the local ring of $R$ at its homogeneous maximal ideal, and $\hat{S}$ to be the $m$-adic completion of $S$. Then

1. the group $\text{Cl}(S)$ is finitely generated if and only if $H^1(X, \mathcal{O}_X) = 0$;
2. the map $\text{Cl}(S) \longrightarrow \text{Cl}(\hat{S})$ is bijective if and only if $H^1(X, \mathcal{O}_X(i)) = 0$ for each integer $i \geq 1$; and
3. the map $\text{Cl}(S) \longrightarrow \text{Cl}(S[[t]])$ is bijective if and only if $H^1(X, \mathcal{O}_X(i)) = 0$ for each integer $i \geq 0$.

The essential point in our construction is in the following proposition:

**Theorem 4.2.** Let $A$ be a Cohen-Macaulay ring of dimension $d \geq 2$, which is a standard graded algebra over a field $K$. For $s \geq 2$, let $z_1, \ldots, z_s$ be a regular sequence in $A$, consisting of homogeneous elements of equal degree, say $k$. Consider the Rees ring $\mathcal{R} = A[z_1 t, \ldots, z_s t]$ with the $\mathbb{Z}^2$-grading where $\deg x = (n,0)$ for $x \in A_n$, and $\deg z_i t = (0,1)$.

Let $\Delta = (g,h)\mathbb{Z}$ where $g, h$ are positive integers, and let $m$ denote the homogeneous maximal ideal of $\mathcal{R}_\Delta$. Then:

1. $H^2_m(\mathcal{R}_\Delta) = 0$ if $q \neq d - s + 1, d$; and
2. $H^{d-s+1}_m(\mathcal{R}_\Delta) i \neq 0$ if and only if $1 \leq i \leq (a + ks - k)/g$, where $a$ is the $a$-invariant of $A$.

In particular, $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $g > a + ks - k$. 

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Example 4.3. For $d \geq 3$, let $A = \mathbb{C}[x_0, \ldots, x_d]/(f)$ be a standard graded hypersurface such that $\text{Proj} A$ is smooth over $\mathbb{C}$. Take general $k$-forms $z_1, \ldots, z_{d-1} \in A$, and consider the Rees ring $R = A[z_1t, \ldots, z_{d-1}t]$. Since $(z) \subseteq A$ is a radical ideal, $\text{gr}(z), A \cong A/(z)[y_1, \ldots, y_{d-1}]$ is a reduced ring, and therefore $R = A[z_1t, \ldots, z_{d-1}t]$ is integrally closed in $A[t]$. Since $A$ is normal, so is $R$. Note that $\text{Proj} R_\Delta$ is the blow-up of $\text{Proj} A$ at the subvariety defined by $(z)$, i.e., at $k^d/(\text{deg} f)$ points. It follows that $\text{Proj} R_\Delta$ is smooth over $\mathbb{C}$. Hence $R_\Delta$ is a standard graded $\mathbb{C}$-algebra, which is normal and has an isolated singularity.

If $\Delta = (g, h)\mathbb{Z}$ is a diagonal with $1 \leq g \leq \text{deg} f + k(d-2) - (d+1)$ and $h \geq 1$, then Theorem 4.2 implies that

$$H^2_m(R_\Delta) |_0 = 0 \quad \text{and} \quad H^2_m(R_\Delta)_1 \neq 0.$$

The rest of this section is devoted to proving Theorem 4.2. We may assume that the base field $K$ is infinite. Then one can find linear forms $x_1, \ldots, x_{d-s}$ in $A$ such that $x_1, \ldots, x_{d-s}, z_1, \ldots, z_s$ is a maximal $A$-regular sequence.

We will use the following lemma; the notation is as in Theorem 4.2.

Lemma 4.4. Let $a$ be the homogeneous maximal ideal of $A$. Set $I = (z_1, \ldots, z_s)A$. Let $r$ be a positive integer.

1. $H^q_a(I^r) = 0$ if $q \neq d - s + 1, d$.
2. Assume $d > s$. Then, $H^{d-s+1}_a(I^r)_i \neq 0$ if and only if $i \leq a + ks + rk - k$.
3. Assume $d = s$. Then, $H^{d-s+1}_a(I^r)_i \neq 0$ if and only if $0 \leq i \leq a + ks + rk - k$.

Proof. Recall that $A$ and $A/I^r$ are Cohen-Macaulay rings of dimension $d$ and $d - s$, respectively. By the exact sequence

$$0 \rightarrow I^r \rightarrow A \rightarrow A/I^r \rightarrow 0$$

we obtain

$$H^q_a(I^r) = \begin{cases} H^q_a(A) & \text{if } q = d \\ H^{d-s}_a(A/I^r) & \text{if } q = d - s + 1 \\ 0 & \text{if } q \neq d - s + 1, d, \end{cases}$$

which proves (1).

Next we prove (2) and (3). Since $A/I^r$ is a standard graded Cohen-Macaulay ring of dimension $d - s$, it is enough to show that the $a$-invariant of this ring equals $a + ks + rk - k$. This is straightforward if $r = 1$, and we proceed by induction. Consider the exact sequence

$$0 \rightarrow I^r/I^{r+1} \rightarrow A/I^{r+1} \rightarrow A/I^r \rightarrow 0.$$

Since $z_1, \ldots, z_s$ is a regular sequence of $k$-forms, $I^r/I^{r+1}$ is isomorphic to $((A/I)(-rk))^{(d-s+1)}$. 


Thus, we have the following exact sequence:

\[ 0 \rightarrow H^d_{\alpha}(A/I)(-rk)^{(\ell+1)^{\text{even}}}) \rightarrow H^d_{\alpha}(A/I^{\ell+1}) \rightarrow H^d_{\alpha}(A/I^{r+1}) \rightarrow 0. \]

The \( \alpha \)-invariant of \((A/I)(-rk)\) equals \(a + ks + rk\), and that of \(A/I^{r+1}\) is \(a + ks + rk - k\) by the inductive hypothesis. Thus, \(A/I^{r+1}\) has \(\alpha\)-invariant \(a + ks + rk\). \(\square\)

**Proof of Theorem 4.2.** Let \(B = K[y_1, \ldots, y_s]\) be a polynomial ring, and set

\[ T = A \otimes_K B = A[y_1, \ldots, y_s]. \]

Consider the \(\mathbb{Z}^2\)-grading on \(T\) where \(\text{deg } x = (n, 0)\) for \(x \in A_n\), and \(\text{deg } y_i = (0, 1)\) for each \(i\). One has a surjective homomorphism of graded rings

\[ T \rightarrow R = A[z_1 t, \ldots, z_s t] \quad \text{where } y_i \mapsto z_i t, \]

and this induces an isomorphism

\[ R \cong T/I_2(\frac{z_1}{y_1} \cdots \frac{z_s}{y_s}). \]

The minimal free resolution of \(R\) over \(T\) is given by the Eagon-Northcott complex

\[ 0 \rightarrow F^{-(s-1)} \rightarrow F^{-(s-2)} \rightarrow \cdots \rightarrow F^0 \rightarrow 0, \]

where \(F^0 = T(0, 0)\), and \(F^{-i}\) for \(1 \leq i \leq s - 1\) is the direct sum of \(\binom{s}{i+1}\) copies of

\[ T(-k, -i) \oplus T(-2k, -(i-1)) \oplus \cdots \oplus T(-ik, -1). \]

Let \(n\) be the homogeneous maximal ideal of \(T_\Delta\). One has the spectral sequence:

\[ E_2^{p,q} = H^p(H^q_\Delta(F_\Delta^*)) \Rightarrow H^{p+q}_m(R_\Delta). \]

Let \(G\) be the set of \((n, m)\) such that \(T(n, m)\) appears in the Eagon-Northcott complex above, i.e., the elements of \(G\) are

\[ (0, 0), \]
\[ (k, -1), \]
\[ (k, -2), (2k, -1), \]
\[ (k, -3), (2k, -2), (3k, -1), \]
\[ \vdots \]
\[ (k, -(s-1)), \ldots (s-1)k, -1). \]

Let \(a\) and \(b\) be the homogeneous maximal ideal of \(A\) and \(B\) respectively. For integers \(n\) and \(m\), the Künneth formula gives

\[ H^n_\alpha(T(n, m)) \]
\[ = H^n_\alpha(A(n) \otimes_K B(m)) \]
\[ = (H^n_\alpha(A(n)) \otimes B(m)) \oplus (A(n) \otimes H^n_\alpha(B(m))) \oplus \bigoplus_{i+j=q+1} H^n_\alpha(A(n)) \otimes H^n_\alpha(B(m)) \]
\[ = H^n_\alpha(T(n, m)) \oplus H^n_\alpha(T(n, m)) \oplus \bigoplus_{i+j=q+1} H^n_\alpha(A(n)) \otimes_K H^n_\alpha(B(m)). \]
As \( A \) and \( B \) are Cohen-Macaulay of dimension \( d \) and \( s \) respectively, it follows that
\[
H^n_d(F^\bullet) = 0 \quad \text{if } q \neq s, d, d + s - 1.
\]
In the case where \( d > s \), one has
\[
H^n_d(F^\bullet) = H^n_b(F^\bullet) \quad \text{and} \quad H^n_a(F^\bullet) = H^n_d(F^\bullet),
\]
and if \( d = s \), then
\[
H^n_d(F^\bullet) = H^n_a(F^\bullet) \oplus H^n_b(F^\bullet).
\]
We claim \( H^n_b(F^\bullet)_\Delta = 0 \). If not, there exists \((n,m) \in G \) and \( \ell \in \mathbb{Z} \) such that
\[
H^n_b(T(n,m))_{(g\ell,h\ell)} \neq 0.
\]
This implies that
\[
H^n_b(T(n,m))_{(g\ell,h\ell)} = A(n)_{g\ell} \otimes_K H^n_b(B(m))_{h\ell} = A_{n+g\ell} \otimes_K H^n_b(B)_{m+h\ell}
\]
is nonzero, so
\[
n + g\ell \geq 0 \quad \text{and} \quad m + h\ell \leq -s,
\]
and hence
\[
-n \leq \ell \leq -\frac{s + m}{h}.
\]
But \((n,m) \in G\), so \( n \leq 0 \) and \( m \geq -(s - 1) \), implying that
\[
0 \leq \ell \leq -\frac{1}{h},
\]
which is not possible. This proves that \( H^n_b(F^\bullet)_\Delta = 0 \). Thus, we have
\[
H^n_a(F^\bullet)_\Delta = \begin{cases} 
0 & \text{if } q \neq d, d + s - 1, \\
H^n_d(F^\bullet)_\Delta & \text{if } q = d.
\end{cases}
\]
It follows that
\[
E^{p,q}_2 = H^p(H^n_a(F^\bullet)) = E^{p,q}_\infty
\]
for each \( p \) and \( q \). Therefore,
\[
H^i_m(R_\Delta) = E^{i-d,d}_2 = H^{i-d}(H^n_a(F^\bullet)_\Delta) = H^{i-d}(H^n_d(F^\bullet)_\Delta) = H^i_a(R)_\Delta
\]
for \( d - s + 1 \leq i \leq d - 1 \), and
\[
H^i_m(R_\Delta) = 0 \quad \text{for } i < d - s + 1.
\]

We next study \( H^i_a(R) \). Since
\[
R = A \oplus I(k) \oplus I^2(2k) \oplus \cdots \oplus I^r(rk) \oplus \cdots,
\]
we have
\[
H^i_a(R) = H^i_a(A) \oplus H^i_a(I)(k) \oplus H^i_a(I^2)(2k) \oplus \cdots \oplus H^i_a(I^r)(rk) \oplus \cdots.
\]
Theorem 4.2 (1) now follow using Lemma 4.4 (1).

Assume that \( d > s \). Then, by Lemma 4.4 (2), \( H^{d-s+1}_a(I^r(rk))_i \neq 0 \) if and only if \( i \leq a + ks - k \).
Assume that \( d = s \). Then, by Lemma 4.4 (3), \( H^d_{a}(I^r(rk)) \neq 0 \) if and only if \( -rk \leq i \leq a + ks - k \).

In each case, \( H^d_{a}(R) \neq 0 \) if and only if \( 1 \leq i \leq a + ks - k \).

\[ \square \]

5. Rational singularities

Let \( R \) be a normal domain, essentially of finite type over a field of characteristic zero, and consider a desingularization \( f: Z \rightarrow \text{Spec} R \), i.e., a proper birational morphism with \( Z \) a nonsingular variety. One says \( R \) has rational singularities if \( R^i \cdot O_Z = 0 \) for each \( i \geq 1 \); this does not depend on the choice of the desingularization \( f \). For \( \mathbb{N} \)-graded rings, one has the following criterion due to Flenner [Fl] and Watanabe [Wa1].

**Theorem 5.1.** Let \( R \) be a normal \( \mathbb{N} \)-graded ring which is finitely generated over a field \( R_0 \) of characteristic zero. Then \( R \) has rational singularities if and only if it is Cohen-Macaulay, \( a(R) < 0 \), and the localization \( R_p \) has rational singularities for each \( p \in \text{Spec} R \setminus \{ R_+ \} \).

When \( R \) has an isolated singularity, the above theorem gives an effective criterion for determining if \( R \) has rational singularities. However, a multigraded hypersurface typically does not have an isolated singularity, and the following variation turns out to be useful:

**Theorem 5.2.** Let \( R \) be a normal \( \mathbb{N}^r \)-graded ring such that \( R_0 \) is a local ring essentially of finite type over a field of characteristic zero, and \( R \) is generated over \( R_0 \) by elements

\[ x_{11}, x_{12}, \ldots, x_{1t}, \quad x_{21}, x_{22}, \ldots, x_{2t_2}, \quad \ldots, \quad x_{r1}, x_{r2}, \ldots, x_{rt_r}, \]

where \( \deg x_{ij} \) is a positive integer multiple of the \( i \)-th unit vector \( e_i \in \mathbb{N}^r \). Then \( R \) has rational singularities if and only if

1. \( R \) is Cohen-Macaulay,
2. \( R_p \) has rational singularities for each \( p \) belonging to

\[ \text{Spec} R \setminus (V(x_{11}, x_{12}, \ldots, x_{1t}) \cup \cdots \cup V(x_{r1}, x_{r2}, \ldots, x_{rt})), \]

and

3. \( a(R) < 0 \), i.e., \( a(R^p) < 0 \) for each coordinate projection \( \varphi_i: \mathbb{N}^r \rightarrow \mathbb{N} \).

Before proceeding with the proof, we record some preliminary results.

**Remark 5.3.** Let \( R \) be an \( \mathbb{N} \)-graded ring. We use \( R^T \) to denote the Rees algebra with respect to the filtration \( F_n = R_{\geq n} \), i.e.,

\[ R^T = F_0 \oplus F_1 T \oplus F_2 T^2 \oplus \cdots. \]
When considering \( \text{Proj} \, R^\oplus \), we use the \( \mathbb{N} \)-grading on \( R^\oplus \) where \([R^\oplus]_n = F_n T^n\). The inclusion \( R = [R^\oplus]_0 \hookrightarrow R^\oplus \) gives a map

\[
\text{Proj} \, R^\oplus \xrightarrow{f} \text{Spec} \, R.
\]

Also, the inclusions \( R_n \hookrightarrow F_n \) give rise to an injective homomorphism of graded rings \( R \hookrightarrow R^\oplus \), which induces a surjection

\[
\text{Proj} \, R^\oplus \twoheadrightarrow \text{Proj} \, R.
\]

**Lemma 5.4.** Let \( R \) be an \( \mathbb{N} \)-graded ring which is finitely generated over \( R_0 \), and assume that \( R_0 \) is essentially of finite type over a field of characteristic zero.

If \( R_p \) has rational singularities for all primes \( p \in \text{Spec} \, R \setminus V(R_+) \), then \( \text{Proj} \, R^\oplus \) has rational singularities.

**Proof.** Note that \( \text{Proj} \, R^\oplus \) is covered by affine open sets \( D_+(r T^n) \) for integers \( n \geq 1 \) and homogeneous elements \( r \in R_{\geq n} \). Consequently, it suffices to check that \([R^\oplus]_0 \) has rational singularities. Next, note that

\[
[R^\oplus]_0 = R + \frac{1}{r} [R]_{\geq n} + \frac{1}{r^2} [R]_{\geq 2n} + \cdots.
\]

In the case \( \text{deg} \, r > n \), the ring above is simply \( R_r \), which has rational singularities by the hypothesis of the lemma. If \( \text{deg} \, r = n \), then

\[
[R^\oplus]_0 = [R_r]_{\geq 0}.
\]

The \( \mathbb{Z} \)-graded ring \( R_r \) has rational singularities and so, by [Wa1, Lemma 2.5], the ring \([R_r]_{\geq 0}\) has rational singularities as well. \( \square \)

**Lemma 5.5.** [Hy2, Lemma 2.3] Let \( R \) be an \( \mathbb{N} \)-graded ring which is finitely generated over a local ring \((R_0, m)\). Suppose \([H^i_{m+R_+}(R)]_{\geq 0} = 0 \) for all \( i \geq 0 \). Then, for all ideals \( a \) of \( R_0 \), one has

\[
[H^i_a + R_+ (R)]_{\geq 0} = 0 \quad \text{for all } i \geq 0.
\]

We are now in a position to prove the following theorem, which is a variation of [Fl, Satz 3.1], [Wa1, Theorem 2.2], and [Hy1, Theorem 1.5].

**Theorem 5.6.** Let \( R \) be an \( \mathbb{N} \)-graded normal ring which is finitely generated over \( R_0 \), and assume that \( R_0 \) is a local ring essentially of finite type over a field of characteristic zero. Then \( R \) has rational singularities if and only if

(1) \( R \) is Cohen-Macaulay,

(2) \( R_p \) has rational singularities for all \( p \in \text{Spec} \, R \setminus V(R_+) \), and

(3) \( a(R) < 0 \).

**Proof.** It is straightforward to see that conditions (1)–(3) hold when \( R \) has rational singularities, and we focus on the converse. Consider the morphism

\[
Y = \text{Proj} \, R^\oplus \xrightarrow{f} \text{Spec} \, R
\]
as in Remark 5.3. Let \( g : Z \longrightarrow Y \) be a desingularization of \( Y \); the composition

\[
Z \xrightarrow{g} Y \xrightarrow{f} \text{Spec } R
\]

is then a desingularization of \( \text{Spec } R \). Note that \( Y = \text{Proj } R^\natural \) has rational singularities by Lemma 5.4, so

\[
g_* \mathcal{O}_Z = \mathcal{O}_Y \quad \text{and} \quad R^q g_* \mathcal{O}_Z = 0 \quad \text{for all } q \geq 1.
\]

Consequently the Leray spectral sequence

\[
E_2^{p,q} = H^p(Y, R^q g_* \mathcal{O}_Z) = H^{p+q}(Z, \mathcal{O}_Z)
\]

degenerates, and we get \( H^p(Z, \mathcal{O}_Z) = H^p(Y, \mathcal{O}_Y) \) for all \( p \geq 1 \). Since \( \text{Spec } R \) is affine, we also have \( R^p(g \circ f)_* \mathcal{O}_Z = H^p(Z, \mathcal{O}_Z) \). To prove that \( R \) has rational singularities, it now suffices to show that \( H^p(Y, \mathcal{O}_Y) = 0 \) for all \( p \geq 1 \). Consider the map \( \pi : Y \longrightarrow X = \text{Proj } R \). We have

\[
H^p(Y, \mathcal{O}_Y) = H^p(X, \pi_* \mathcal{O}_X) = \bigoplus_{n \geq 0} H^p(X, \mathcal{O}_X(n)) = [H^1_{R^+}(R)]_{p \geq 0}.
\]

By condition (1), we have \( [H^1_{m+R^+}(R)]_{p \geq 0} = 0 \) for all \( p \geq 0 \), and so Lemma 5.5 implies that \( [H^1_{R^+}(R)]_{p \geq 0} = 0 \) for all \( p \geq 0 \) as desired. \( \square \)

**Proof of theorem 5.2.** If \( R \) has rational singularities, it is easily seen that conditions (1)–(3) must hold. For the converse, we proceed by induction on \( r \). The case \( r = 1 \) is Theorem 5.6 established above, so assume \( r \geq 2 \). It suffices to show that \( R_\mathfrak{M} \) has rational singularities where \( \mathfrak{M} \) is the homogeneous maximal ideal of \( R \). Set

\[
\mathfrak{m} = \mathfrak{M} \cap [R^\natural]_0,
\]

and consider the \( \mathbb{N} \)-graded ring \( S \) obtained by inverting the multiplicative set \( [R^\natural]_0 \setminus \mathfrak{m} \) in \( R^\natural \). Since \( R_{\mathfrak{M}} \) is a localization of \( S \), it suffices to show that \( S \) has rational singularities. Note that \( a(S) = a(R^\natural) \), which is a negative integer by (1). Using Theorem 5.6, it is therefore enough to show that \( R_\mathfrak{P} \) has rational singularities for all \( \mathfrak{P} \in \text{Spec } R \setminus V(x_{r1}, x_{r2}, \ldots, x_{rt}) \). Fix such a prime \( \mathfrak{P} \), and let

\[
\psi : \mathbb{Z}^r \longrightarrow \mathbb{Z}^{r-1}
\]

be the projection to the first \( r-1 \) coordinates. Note that \( R_\mathfrak{P} \) is the ring \( R \) regraded such that \( \deg x_{rj} = 0 \), and the degrees of \( x_{ij} \) for \( i < r \) are unchanged. Set

\[
\mathfrak{p} = \mathfrak{P} \cap [R_\psi^\natural]_0,
\]

and let \( T \) be the ring obtained by inverting the multiplicative set \( [R_\psi^\natural]_0 \setminus \mathfrak{p} \) in \( R_\psi \). It suffices to show that \( T \) has rational singularities. Note that \( T \) is an \( \mathbb{N}^{r-1} \)-graded ring defined over a local ring \( (T_0, \mathfrak{p}) \), and that it has homogeneous maximal ideal \( \mathfrak{p} + bT \) where

\[
b = (R_\psi^\natural)_+ = (x_{ij} \mid i < r)R.
\]
Using the inductive hypothesis, it remains to verify that $a(T) < 0$. By condition (1), for all integers $1 \leq j \leq r - 1$, we have

$$[H^i_{\mathfrak{m}}(R^{\varphi_j})]_{\geq 0} = 0 \quad \text{for all } i \geq 0,$$

and using Lemma 5.5 it follows that

$$[H^i_{p^b}(R^{\varphi_j})]_{\geq 0} = 0 \quad \text{for all } i \geq 0.$$

Consequently $a(T^{\varphi_j}) < 0$ for $1 \leq j \leq r - 1$, which completes the proof. □

6. F-regularity

For the theory of tight closure, we refer to the papers [HH1, HH2] and [HH3]. We summarize results about F-rational and F-regular rings:

**Theorem 6.1.** The following hold for rings of prime characteristic.

1. Regular rings are F-regular.
2. Direct summands of F-regular rings are F-regular.
3. F-rational rings are normal; an F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.
4. F-rational Gorenstein rings are F-regular.
5. Let $R$ be an $\mathbb{N}$-graded ring which is finitely generated over a field $R_0$. If $R$ is weakly F-regular, then it is F-regular.

**Proof.** For (1) and (2) see [HH1, Theorem 4.6] and [HH1, Proposition 4.12] respectively; (3) is part of [HH2, Theorem 4.2], and for (4) see [HH2, Corollary 4.7]. Lastly, (5) is [LS, Corollary 4.4]. □

The characteristic zero aspects of tight closure are developed in [HH4]. Let $K$ be a field of characteristic zero. A finitely generated $K$-algebra $R = K[x_1, \ldots, x_m]/a$ is of F-regular type if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$, and a finitely generated free $A$-algebra

$$R_A = A[x_1, \ldots, x_m]/a_A,$$

such that $R \cong R_A \otimes_A K$ and, for all maximal ideals $\mu$ in a Zariski dense subset of Spec $A$, the fiber rings $R_A \otimes_A A/\mu$ are F-regular rings of characteristic $p > 0$. Similarly, $R$ is of F-rational type if for a dense subset of $\mu$, the fiber rings $R_A \otimes_A A/\mu$ are F-rational. Combining results from [Ha, HW, MS, Sm] one has:

**Theorem 6.2.** Let $R$ be a ring which is finitely generated over a field of characteristic zero. Then $R$ has rational singularities if and only if it is of F-rational type. If $R$ is $\mathbb{Q}$-Gorenstein, then it has log terminal singularities if and only if it is of F-regular type.
Proposition 6.3. Let $K$ be a field of characteristic $p > 0$, and $R$ an $\mathbb{N}$-graded normal ring which is finitely generated over $R_0 = K$. Let $\omega$ denote the graded canonical module of $R$, and set $d = \dim R$.

Suppose $R$ is F-regular. Then, for each integer $k$, there exists $q = p^r$ such that
\[
\text{rank}_K R_k \leq \text{rank}_K [H^d_m(\omega^{(q)})]_k.
\]

Proof. If $d \leq 1$, then $R$ is regular and the assertion is elementary. Assume $d \geq 2$.

Let $\xi \in [H^d_m(\omega)]_0$ be an element which generates the socle of $H^d_m(\omega)$. Since the map $\omega^{[q]} \to \omega^{(q)}$ is an isomorphism in codimension one, $F^e(\xi)$ may be viewed as an element of $H^d_m(\omega^{(q)})$ as in [Wa2].

Fix an integer $k$. For each $e \in \mathbb{N}$, set $V_e$ to be the kernel of the vector space homomorphism
\[
(6.3.1) \quad R_k \longrightarrow [H^d_m(\omega^{(p^e)})]_k, \quad \text{where } c \longrightarrow cF^e(\xi).
\]

If $cF^{e+1}(\xi) = 0$, then $F(cF^e(\xi)) = c^p F^{e+1}(\xi) = 0$; since $R$ is F-pure, it follows that $cF^e(\xi) = 0$. Consequently the vector spaces $V_e$ form a descending sequence
\[
V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots.
\]

The hypothesis that $R$ is F-regular implies $\bigcap_e V_e = 0$. Since each $V_e$ has finite rank, $V_e = 0$ for $e \gg 0$. Hence the homomorphism (6.3.1) is injective for $e \gg 0$. \hfill \Box

We next record tight closure properties of general $\mathbb{N}$-graded hypersurfaces. The results for F-purity are essentially worked out in [HR].

Theorem 6.4. Let $A = K[x_1, \ldots, x_m]$ be a polynomial ring over a field $K$ of positive characteristic. Let $d$ be a nonnegative integer, and set $M = \binom{d+m-1}{d} - 1$. Consider the affine space $\mathbb{A}_K^M$ parameterizing the degree $d$ forms in $A$ in which $x_1^d$ occurs with coefficient 1.

Let $U$ be the subset of $\mathbb{A}_K^M$ corresponding to the forms $f$ for which $A/fA$ F-pure. Then $U$ is a Zariski open set, and it is nonempty if and only if $d \leq m$.

Let $V$ be the set corresponding to forms $f$ for which $A/fA$ is F-regular. Then $V$ contains a nonempty Zariski open set if $d < m$, and is empty otherwise.

Proof. The set $U$ is Zariski open by [HR, page 156] and it is empty if $d > m$ by [HR, Proposition 5.18]. If $d \leq m$, the square-free monomial $x_1 \cdots x_d$ defines an F-pure hypersurface $A/(x_1 \cdots x_d)$. A linear change of variables yields the polynomial
\[
f = x_1(x_1 + x_2) \cdots (x_1 + x_d)
\]
in which $x_1^d$ occurs with coefficient 1. Hence $U$ is nonempty for $d \leq m$.

If $d \geq m$, then $A/fA$ has $a$-invariant $d - m \geq 0$ so $A/fA$ is not F-regular. Suppose $d < m$. Consider the set $W \subseteq \mathbb{A}_K^M$ parameterizing the forms $f$ for which $A/fA$ is F-pure and $(A/fA)_{x_1}$ is regular; $W$ is a nonempty open subset of $\mathbb{A}_K^M$. Let $f$ correspond to a point of $W$. The element $\varpi_1 \in A/fA$ has a power which
is a test element; since $A/fA$ is F-pure, it follows that $\mathfrak{p}_1$ is a test element. Note that $\mathfrak{p}_2, \ldots, \mathfrak{p}_m$ is a homogeneous system of parameters for $A/fA$ and that $\mathfrak{p}_1^{d-1}$ generates the socle modulo $(\mathfrak{p}_2, \ldots, \mathfrak{p}_m)$. Hence the ring $A/fA$ is F-regular if and only if there exists a power $q$ of the prime characteristic $p$ such that

$$x_1^{(d-1)q+1} \notin (x_2^q, \ldots, x_m^q, f)A.$$ 

The set of such $f$ corresponds to an open subset of $W$; it remains to verify that this subset is nonempty. For this, consider

$$f = x_1^d + x_2 \cdots x_{d+1},$$

which corresponds to a point of $W$, and note that $A/fA$ is F-regular since

$$x_1^{(d-1)p+1} \notin (x_2^p, \ldots, x_m^p, f)A.$$ 

These ideas carry over to multi-graded hypersurfaces; we restrict below to the bigraded case. The set of forms in $K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ of degree $(d, e)$ in which $x_1^q y_1^e$ occurs with coefficient 1 is parametrized by the affine space $\mathbb{A}_K^n$ where $N = (d+e-1)(e+n-1) - 1$.

**Theorem 6.5.** Let $B = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over a field $K$ of positive characteristic. Consider the $\mathbb{N}^2$-grading on $B$ with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$. Let $d, e$ be nonnegative integers, and consider the affine space $\mathbb{A}_K^n$ parameterizing forms of degree $(d, e)$ in which $x_1^q y_1^e$ occurs with coefficient 1.

Let $U$ be the subset of $\mathbb{A}_K^n$ corresponding to forms $f$ for which $B/fB$ is F-pure. Then $U$ is a Zariski open set, and it is nonempty if and only if $d \leq m$ and $e \leq n$.

Let $V$ be the set corresponding to forms $f$ for which $B/fB$ is F-regular. Then $V$ contains a nonempty Zariski open set if $d < m$ and $e < n$, and is empty otherwise.

**Proof.** The argument for F-purity is similar to the proof of Theorem 6.4; if $d \leq m$ and $e \leq n$, then the polynomial $x_1 \cdots x_d y_1 \cdots y_e$ defines an F-pure hypersurface.

If $B/fB$ is F-regular, then $a(B/fB) < 0$ implies $d < m$ and $e < n$. Conversely, if $d < m$ and $e < n$, then there is a nonempty open set $W$ corresponding to forms $f$ for which the hypersurface $B/fB$ is F-pure and $(B/fB)_{x_1 y_1}$ is regular. In this case, $x_1 y_1^e \in B/fB$ is a test element. The socle modulo the parameter ideal $(x_1 - y_1, x_2, \ldots, x_m, y_2, \ldots, y_n)B/fB$ is generated by $x_1^{d+e-1}$, so $B/fB$ is F-regular if and only if there exists a power $q = p^r$ such that

$$x_1^{(d+e-1)q+1} \notin (x_1^q - y_1^q, x_2^q, \ldots, x_m^q, y_2^q, \ldots, y_n^q, f)B.$$ 

The subset of $W$ corresponding to such $f$ is open; it remains to verify that it is nonempty. For this, use $f = x_1^q y_1^e + x_2 \cdots x_{d+1} y_2 \cdots y_{e+1}$. 

$\square$
References


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