MULTIGRADED RINGS, DIAGONAL SUBALGEBRAS, AND RATIONAL SINGULARITIES

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1. Introduction

We study the properties of F-rationality and F-regularity in multigraded rings and their diagonal subalgebras. The main focus is on diagonal subalgebras of bigraded rings: these constitute an interesting class of rings since they arise naturally as homogeneous coordinate rings of blow-ups of projective varieties.

Let X be a projective variety over a field K, with homogeneous coordinate ring A. Let $\mathfrak{a} \subset A$ be a homogeneous ideal, and $V \subset X$ the closed subvariety defined by $\mathfrak{a}$. For g an integer, we use $\mathfrak{a}_g$ to denote the K-vector space consisting of homogeneous elements of $\mathfrak{a}$ of degree g. If $g \gg 0$, then $\mathfrak{a}_g$ defines a very ample complete linear system on the blow-up of X along V, and hence $K[\mathfrak{a}_g]$ is a homogeneous coordinate ring for this blow-up. Since the ideals $\mathfrak{a}^h$ define the same subvariety V, the rings $K[(\mathfrak{a}^h)_g]$ are homogeneous coordinate ring for the blow-up provided $g \gg h > 0$.

Suppose that A is a standard $\mathbb{N}$-graded K-algebra, and consider the $\mathbb{N}^2$-grading on the Rees algebra $A[\mathfrak{a}t]$, where $\deg rt^j = (i, j)$ for $r \in A_i$. The connection with diagonal subalgebras stems from the fact that if $\mathfrak{a}^h$ is generated by elements of degree less than or equal to g, then

$$K[(\mathfrak{a}^h)_g] \cong \bigoplus_{k \geqslant 0} A[\mathfrak{a}t]_{(gk,hk)} \,.$$

Using $\Delta = (g, h)\mathbb{Z}$ to denote the (g, h)-*diagonal* in $\mathbb{Z}^2$, the *diagonal subalgebra* $A[\mathfrak{a}t]_\Delta = \oplus_k A[\mathfrak{a}t]_{(gk,hk)}$ is a homogeneous coordinate ring for the blow-up of $\operatorname{Proj} A$ along the subvariety defined by $\mathfrak{a}$, whenever $g \gg h > 0$.

The papers [GG, GGH, GGP, Tr] use diagonal subalgebras in studying blow-ups of projective space at finite sets of points. For A a polynomial ring and $\mathfrak{a}$ a homogeneous ideal, the ring theoretic properties of $K[\mathfrak{a}_g]$ are studied by Simis, Trung, and Valla in [STV] by realizing $K[\mathfrak{a}_g]$ as a diagonal subalgebra of the Rees algebra $A[\mathfrak{a}t]$. In particular, they determine when $K[\mathfrak{a}_g]$ is Cohen-Macaulay for $\mathfrak{a}$ a complete intersection ideal generated by forms of equal degree, and also for $\mathfrak{a}$ the

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ideal of maximal minors of a generic matrix. Some of their results are extended by Conca, Herzog, Trung, and Valla as in the following theorem:

Theorem 1.1. [CHTV, Theorem 4.6] *Let $K[x_1, \dots, x_m]$ be a polynomial ring over a field, and let $\mathfrak{a}$ be a complete intersection ideal minimally generated by forms of degrees $d_1, \dots, d_r$. Fix positive integers g and h with $g/h > d = \max\{d_1, \dots, d_r\}$.*

Then $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay if and only if $g > (h-1)d - m + \sum_{j=1}^{r} d_j$.

When A is a polynomial ring and $\mathfrak{a}$ an ideal for which $A[\mathfrak{a}t]$ is Cohen-Macaulay, Lavila-Vidal [Lv1, Theorem 4.5] proved that the diagonal subalgebras $K[(\mathfrak{a}^h)_g]$ are Cohen-Macaulay for $g \gg h \gg 0$, thereby settling a conjecture from [CHTV]. In [CH] Cutkosky and Herzog obtain affirmative answers regarding the existence of a constant c such that $K[(\mathfrak{a}^h)_g]$ is Cohen-Macaulay whenever $g \geqslant ch$. For more work on the Cohen-Macaulay and Gorenstein properties of diagonal subalgebras, see [HHR, Hy2, Lv2], and [LvZ].

As a motivating example for some of the results of this paper, consider a polynomial ring $A = K[x_1, \dots, x_m]$ and an ideal $\mathfrak{a} = (z_1, z_2)$ generated by relatively prime forms z_1 and z_2 of degree d. Setting $\Delta = (d+1, 1)\mathbb{Z}$, the diagonal subalgebra $A[\mathfrak{a}t]_\Delta$ is a homogeneous coordinate ring for the blow-up of $\operatorname{Proj} A = \mathbb{P}^{m-1}$ along the subvariety defined by $\mathfrak{a}$. The Rees algebra $A[\mathfrak{a}t]$ has a presentation

$$\mathcal{R} = K[x_1, \dots, x_m, y_1, y_2]/(y_2 z_1 - y_1 z_2)\,,$$

where $\deg x_i = (1, 0)$ and $\deg y_j = (d, 1)$, and consequently $\mathcal{R}_\Delta$ is the subalgebra of $\mathcal{R}$ generated by the elements $x_i y_j$. When K has characteristic zero and z_1 and z_2 are general forms of degree d, the results of Section 3 imply that $\mathcal{R}_\Delta$ has rational singularities if and only if $d \leqslant m$, and that it is of F-regular type if and only if $d < m$. As a consequence, we obtain large families of rings of the form $\mathcal{R}_\Delta$, standard graded over a field, which have rational singularities, but which are not of F-regular type.

It is worth pointing out that if $\mathcal{R}$ is an $\mathbb{N}^2$-graded ring over an infinite field $\mathcal{R}_{(0,0)} = K$, and $\Delta = (g, h)\mathbb{Z}$ for coprime positive integers g and h, then $\mathcal{R}_\Delta$ is the ring of invariants of the torus K^* acting on $\mathcal{R}$ via

$$\lambda\colon r \longmapsto \lambda^{hi-gj} r \qquad \text{where } \lambda \in K^* \text{ and } r \in \mathcal{R}_{(i,j)}\,.$$

Consequently there exist torus actions on hypersurfaces for which the rings of invariants have rational singularities but are not of F-regular type.

In Section 4 we use diagonal subalgebras to construct standard graded normal rings R, with isolated singularities, for which $H^2_{\mathfrak{m}}(R)_0 = 0$ and $H^2_{\mathfrak{m}}(R)_1 \neq 0$. If S is the localization of such a ring R at its homogeneous maximal ideal, then, by Danilov's results, the divisor class group of S is a finitely generated abelian group, though S does not have a discrete divisor class group. Such rings R are also of interest in view of the results of [RSS], where it is proved that the image of

$H^2_{\mathfrak{m}}(R)_0$ in $H^2_{\mathfrak{m}}(R^+)$ is annihilated by elements of R^+ of arbitrarily small positive degree; a corresponding result for $H^2_{\mathfrak{m}}(R)_1$ is not known at this point, and the rings constructed in Section 4 constitute interesting test cases.

Section 2 summarizes some notation and conventions for multigraded rings and modules. In Section 3 we carry out an analysis of diagonal subalgebras of bigraded hypersurfaces; this uses results on rational singularities and F-regular rings proved in Sections 5 and 6 respectively.

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2. Preliminaries

In this section, we provide a brief treatment of multigraded rings and modules; see [GW1, GW2, HHR], and [HIO] for further details.

By an $\mathbb{N}^r$-*graded ring* we mean a ring

$$\mathcal{R} = \bigoplus_{\boldsymbol{n}\in\mathbb{N}^r} \mathcal{R}_{\boldsymbol{n}}\,,$$

which is finitely generated over the subring $\mathcal{R}_{\mathbf{0}}$. If $(\mathcal{R}_{\mathbf{0}}, \mathfrak{m})$ is a local ring, then $\mathcal{R}$ has a unique homogeneous maximal ideal $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$, where $\mathcal{R}_+ = \oplus_{\boldsymbol{n}\neq\mathbf{0}}\mathcal{R}_{\boldsymbol{n}}$.

For $\boldsymbol{m} = (m_1, \dots, m_r)$ and $\boldsymbol{n} = (n_1, \dots, n_r)$ in $\mathbb{Z}^r$, we say $\boldsymbol{n} > \boldsymbol{m}$ (resp. $\boldsymbol{n} \geqslant \boldsymbol{m}$) if $n_i > m_i$ (resp. $n_i \geqslant m_i$) for each i.

Let M be a $\mathbb{Z}^r$-graded $\mathcal{R}$-module. For $\boldsymbol{m} \in \mathbb{Z}^r$, we set

$$M_{\geqslant \boldsymbol{m}} = \bigoplus_{\boldsymbol{n}\geqslant\boldsymbol{m}} M_{\boldsymbol{n}}\,,$$

which is a $\mathbb{Z}^r$-graded submodule of M. One writes $M(\boldsymbol{m})$ for the $\mathbb{Z}^r$-graded $\mathcal{R}$-module with shifted grading $[M(\boldsymbol{m})]_{\boldsymbol{n}} = M_{\boldsymbol{m}+\boldsymbol{n}}$ for each $\boldsymbol{n} \in \mathbb{Z}^r$.

Let M and N be $\mathbb{Z}^r$-graded $\mathcal{R}$-modules. Then $\underline{\mathrm{Hom}}_{\mathcal{R}}(M, N)$ is the $\mathbb{Z}^r$-graded module with $[\underline{\mathrm{Hom}}_{\mathcal{R}}(M, N)]_{\boldsymbol{n}}$ being the abelian group consisting of degree preserving $\mathcal{R}$-linear homomorphisms from M to $N(\boldsymbol{n})$.

The functor $\underline{\mathrm{Ext}}^i_{\mathcal{R}}(M, -)$ is the i-th derived functor of $\underline{\mathrm{Hom}}_{\mathcal{R}}(M, -)$ in the category of $\mathbb{Z}^r$-graded $\mathcal{R}$-modules. When M is finitely generated, $\underline{\mathrm{Ext}}^i_{\mathcal{R}}(M, N)$ and $\mathrm{Ext}^i_{\mathcal{R}}(M, N)$ agree as underlying $\mathcal{R}$-modules. For a homogeneous ideal $\mathfrak{a}$ of $\mathcal{R}$, the local cohomology modules of M with support in $\mathfrak{a}$ are the $\mathbb{Z}^r$-graded modules

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n} \underline{\mathrm{Ext}}^i_{\mathcal{R}}(\mathcal{R}/\mathfrak{a}^n, M)\,.$$

Let $\varphi\colon \mathbb{Z}^r \longrightarrow \mathbb{Z}^s$ be a homomorphism of abelian groups satisfying $\varphi(\mathbb{N}^r) \subseteq \mathbb{N}^s$. We write $\mathcal{R}^{\varphi}$ for the ring $\mathcal{R}$ with the $\mathbb{N}^s$-grading where

$$[\mathcal{R}^{\varphi}]_{\boldsymbol{n}} = \bigoplus_{\varphi(\boldsymbol{m})=\boldsymbol{n}} \mathcal{R}_{\boldsymbol{m}}\,.$$

If M is a $\mathbb{Z}^r$-graded $\mathcal{R}$-module, then M^φ is the $\mathbb{Z}^s$-graded $\mathcal{R}^\varphi$-module with

$$[M^\varphi]_{\boldsymbol{n}} = \bigoplus_{\varphi(\boldsymbol{m})=\boldsymbol{n}} M_{\boldsymbol{m}}\,.$$

The change of grading functor $(-)^\varphi$ is exact; by [HHR, Lemma 1.1] one has

$$H^i_{\mathfrak{M}}(M)^\varphi = H^i_{\mathfrak{M}^\varphi}(M^\varphi)\,.$$

Consider the projections $\varphi_i\colon \mathbb{Z}^r \longrightarrow \mathbb{Z}$ with $\varphi_i(m_1,\dots,m_r) = m_i$, and set

$$a(\mathcal{R}^{\varphi_i}) = \max\left\{a \in \mathbb{Z} \mid \left[H^{\dim \mathcal{R}}_{\mathfrak{M}}(\mathcal{R})^{\varphi_i}\right]_a \neq 0\right\};$$

this is the a-invariant of the $\mathbb{N}$-graded ring $\mathcal{R}^{\varphi_i}$ in the sense of Goto and Watanabe [GW1]. As in [HHR], the *multigraded $\boldsymbol{a}$-invariant* of $\mathcal{R}$ is

$$\boldsymbol{a}(\mathcal{R}) = \big(a(\mathcal{R}^{\varphi_1}),\dots,a(\mathcal{R}^{\varphi_r})\big)\,.$$

Let $\mathcal{R}$ be a $\mathbb{Z}^2$-graded ring and let g,h be positive integers. The subgroup $\Delta = (g,h)\mathbb{Z}$ is a *diagonal* in $\mathbb{Z}^2$, and the corresponding *diagonal subalgebra* of $\mathcal{R}$ is

$$\mathcal{R}_\Delta = \bigoplus_{k\in\mathbb{Z}} \mathcal{R}_{(gk,hk)}\,.$$

Similarly, if M is a $\mathbb{Z}^2$-graded $\mathcal{R}$-module, we set

$$M_\Delta = \bigoplus_{k\in\mathbb{Z}} M_{(gk,hk)}\,,$$

which is a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\mathcal{R}_\Delta$.

Lemma 2.1. *Let A and B be $\mathbb{N}$-graded normal rings, finitely generated over a field $A_0 = K = B_0$. Set $T = A \otimes_K B$. Let g and h be positive integers and set $\Delta = (g,h)\mathbb{Z}$. Let $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{m}$ denote the homogeneous maximal ideals of A, B, and T_Δ respectively. Then, for each $q \geqslant 0$ and $i,j,k \in \mathbb{Z}$, one has*

$$H^q_{\mathfrak{m}}\big(T(i,j)_\Delta\big)_k = \big(A_{i+gk} \otimes H^q_{\mathfrak{b}}(B)_{j+hk}\big) \oplus \big(H^q_{\mathfrak{a}}(A)_{i+gk} \otimes B_{j+hk}\big) \oplus \bigoplus_{q_1+q_2=q+1} \big(H^{q_1}_{\mathfrak{a}}(A)_{i+gk} \otimes H^{q_2}_{\mathfrak{b}}(B)_{j+hk}\big)\,.$$

Proof. Let $A^{(g)}$ and $B^{(h)}$ denote the respective Veronese subrings of A and B. Set

$$A^{(g,i)} = \bigoplus_{k\in\mathbb{Z}} A_{i+gk} \qquad \text{and} \qquad B^{(h,j)} = \bigoplus_{k\in\mathbb{Z}} B_{j+hk}\,.$$

These are graded modules over $A^{(g)}$ and $B^{(h)}$ respectively, and

$$T(i,j)_\Delta = \bigoplus_{k\in\mathbb{Z}} A_{i+gk} \otimes_K B_{j+hk} = A^{(g,i)} \,\#\, B^{(h,j)}\,.$$

The ideal $A^{(g)}_+ A$ is $\mathfrak{a}$-primary; likewise, $B^{(h)}_+ B$ is $\mathfrak{b}$-primary. The Künneth formula for local cohomology, [GW1, Theorem 4.1.5], now gives the desired result. □

Notation 2.2. We use bold letters to denote lists of elements, e.g., $\boldsymbol{z} = z_1,\dots,z_s$ and $\boldsymbol{\gamma} = \gamma_1,\dots,\gamma_s$.

3. Diagonal subalgebras of bigraded hypersurfaces

We prove the following theorem about diagonal subalgebras of $\mathbb{N}^2$-graded hypersurfaces. The proof uses results proved later in Sections 5 and 6.

Theorem 3.1. *Let K be a field, let m, n be integers with $m, n \geqslant 2$, and let*

$$\mathcal{R} = K[x_1, \dots, x_m, y_1, \dots, y_n]/(f)$$

be a normal $\mathbb{N}^2$-graded hypersurface where $\deg x_i = (1,0)$, $\deg y_j = (0,1)$, and $\deg f = (d,e) > (0,0)$. For positive integers g and h, set $\Delta = (g,h)\mathbb{Z}$. Then:

(1) *The ring $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $\lfloor (d-m)/g \rfloor < e/h$ and $\lfloor (e-n)/h \rfloor < d/g$. In particular, if $d < m$ and $e < n$, then $\mathcal{R}_\Delta$ is Cohen-Macaulay for each diagonal Δ.*

(2) *The graded canonical module of $\mathcal{R}_\Delta$ is $\mathcal{R}(d-m, e-n)_\Delta$. Hence $\mathcal{R}_\Delta$ is Gorenstein if and only if $(d-m)/g = (e-n)/h$, and this is an integer.*

If K has characteristic zero, and f is a generic polynomial of degree (d,e), then:

(3) *The ring $\mathcal{R}_\Delta$ has rational singularities if and only if it is Cohen-Macaulay and $d < m$ or $e < n$.*

(4) *The ring $\mathcal{R}_\Delta$ is of F-regular type if and only if $d < m$ and $e < n$.*

For $m, n \geqslant 3$ and $\Delta = (1,1)\mathbb{Z}$, the properties of $\mathcal{R}_\Delta$, as determined by m, n, d, e, are summarized in Figure 1.

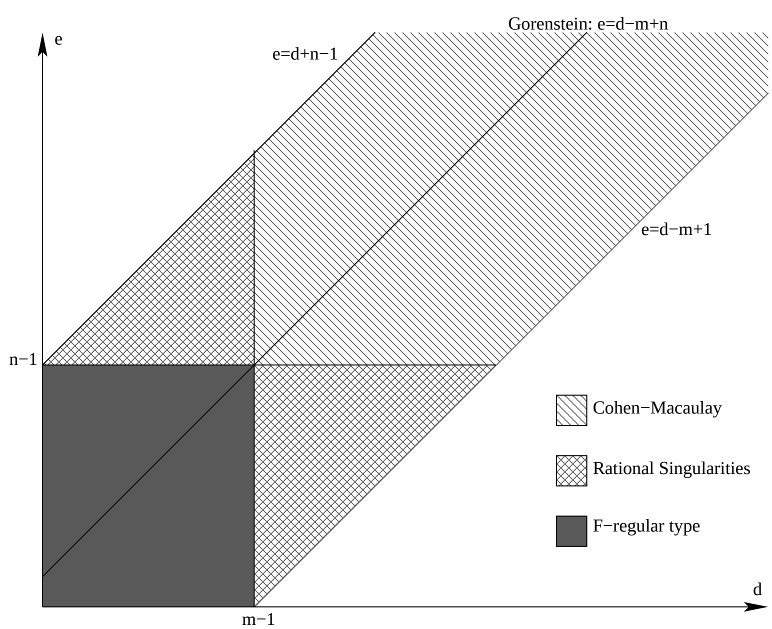

Figure 1. Properties of $\mathcal{R}_\Delta$ for $\Delta = (1,1)\mathbb{Z}$.

Remark 3.2. Let $m, n \geqslant 2$. A generic hypersurface of degree $(d,e) > (0,0)$ in m, n variables is normal precisely when

$$m > \min(2,d) \qquad \text{and} \qquad n > \min(2,e)\,.$$

Suppose that $m = 2 = n$, and that f is nonzero. Then $\dim \mathcal{R}_\Delta = 2$; since $\mathcal{R}_\Delta$ is generated over a field by elements of equal degree, $\mathcal{R}_\Delta$ is of F-regular type if and only if it has rational singularities; see [Wa2]. This is the case precisely if

$$\begin{aligned} &d = 1,\ e \leqslant h+1\,, \quad \text{or} \\ &e = 1,\ d \leqslant g+1\,. \end{aligned}$$

Following a suggestion of Hara, the case $n = 2$ and $e = 1$ was used in [Si, Example 7.3] to construct examples of standard graded rings with rational singularities which are not of F-regular type.

Proof of Theorem 3.1. Set $A = K[\boldsymbol{x}]$, $B = K[\boldsymbol{y}]$, and $T = A \otimes_K B$. By Lemma 2.1, $H^q_{\mathfrak{m}}(T_\Delta) = 0$ for $q \neq m+n-1$. The local cohomology exact sequence induced by

$$0 \longrightarrow T(-d,-e)_\Delta \xrightarrow{\ f\ } T_\Delta \longrightarrow \mathcal{R}_\Delta \longrightarrow 0$$

therefore gives $H^{q-1}_{\mathfrak{m}}(\mathcal{R}_\Delta) = H^q_{\mathfrak{m}}(T(-d,-e)_\Delta)$ for $q \leqslant m+n-2$, and also shows that $H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta)$ and $H^{m+n-1}_{\mathfrak{m}}(\mathcal{R}_\Delta)$ are, respectively, the kernel and cokernel of

$$\begin{array}{ccc} H^{m+n-1}_{\mathfrak{m}}(T(-d,-e)_\Delta) & \xrightarrow{\ f\ } & H^{m+n-1}_{\mathfrak{m}}(T_\Delta) \\ \| & & \| \\ [H^m_{\mathfrak{a}}(A(-d)) \otimes H^n_{\mathfrak{b}}(B(-e))]_\Delta & \xrightarrow{\ f\ } & [H^m_{\mathfrak{a}}(A) \otimes H^n_{\mathfrak{b}}(B)]_\Delta\,. \end{array}$$

The horizontal map above is surjective since its graded dual

$$\begin{array}{ccc} [A(d-m) \otimes B(e-n)]_\Delta & \xleftarrow{\ f\ } & [A(-m) \otimes B(-n)]_\Delta \\ \| & & \| \\ T(d-m,e-n)_\Delta & \xleftarrow{\ f\ } & T(-m,-n)_\Delta \end{array}$$

is injective. In particular, $\dim \mathcal{R}_\Delta = m+n-2$.

It follows from the above discussion that $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $H^q_{\mathfrak{m}}(T(-d,-e)_\Delta) = 0$ for each $q \leqslant m+n-2$. By Lemma 2.1, this is the case if and only if, for each integer k, one has

$$A_{-d+gk} \otimes H^n_{\mathfrak{b}}(B)_{-e+hk} = 0 = H^m_{\mathfrak{a}}(A)_{-d+gk} \otimes B_{-e+hk}\,.$$

Hence $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if there is no integer k satisfying

$$d/g \leqslant k \leqslant (e-n)/h \qquad \text{or} \qquad e/h \leqslant k \leqslant (d-m)/g\,,$$

which completes the proof of (1).

For (2), note that the graded canonical module of $\mathcal{R}_\Delta$ is the graded dual of $H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_\Delta)$, and hence that it equals

$$\operatorname{coker}\left(T(-m,-n)_\Delta \xrightarrow{\ f\ } T(d-m,e-n)_\Delta\right) = \mathcal{R}(d-m,e-n)_\Delta\,.$$

This module is principal if and only if $\mathcal{R}(d-m, e-n)_{\Delta} = \mathcal{R}_{\Delta}(a)$ for some integer a, i.e., $d - m = ga$ and $e - n = ha$.

When f is a general polynomial of degree (d, e), the ring $\mathcal{R}_{\Delta}$ has an isolated singularity. Also, $\mathcal{R}_{\Delta}$ is normal since it is a direct summand of the normal ring $\mathcal{R}$. By Theorem 5.1, $\mathcal{R}_{\Delta}$ has rational singularities precisely if it is Cohen-Macaulay and $a(\mathcal{R}_{\Delta}) < 0$; this proves (3).

It remains to prove (4). If $d < m$ and $e < n$, then Theorem 5.2 implies that $\mathcal{R}$ has rational singularities. By Theorem 6.2, it follows that for almost all primes p, the characteristic p models $\mathcal{R}_p$ of $\mathcal{R}$ are F-rational hypersurfaces which, therefore, are F-regular. Alternatively, $\mathcal{R}_p$ is a generic hypersurface of degree $(d, e) < (m, n)$, so Theorem 6.5 implies that $\mathcal{R}_p$ is F-regular. Since $(\mathcal{R}_p)_{\Delta}$ is a direct summand of $\mathcal{R}_p$, it follows that $(\mathcal{R}_p)_{\Delta}$ is F-regular. The rings $(\mathcal{R}_p)_{\Delta}$ are characteristic p models of $\mathcal{R}_{\Delta}$, so we conclude that $\mathcal{R}_{\Delta}$ is of F-regular type.

Suppose $\mathcal{R}_{\Delta}$ has F-regular type, and let $(\mathcal{R}_p)_{\Delta}$ be a characteristic p model which is F-regular. Fix an integer $k > d/g$. Then Proposition 6.3 implies that there exists an integer $q = p^e$ such that

$$\operatorname{rank}_K ((\mathcal{R}_p)_{\Delta})_k \leqslant \operatorname{rank}_K [H^{m+n-2}_{\mathfrak{m}}(\omega^{(q)})]_k ,$$

where ω is the graded canonical module of $(\mathcal{R}_p)_{\Delta}$. Using (2), we see that

$$H^{m+n-2}_{\mathfrak{m}}(\omega^{(q)}) = H^{m+n-2}_{\mathfrak{m}}(\mathcal{R}_p(qd - qm, qe - qn)_{\Delta}) .$$

Let T_p be a characteristic p model for T such that $T_p/fT_p = \mathcal{R}_p$. Multiplication by f on T_p induces a local cohomology exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{m+n-2}_{\mathfrak{m}_p}(T_p(qd-qm, qe-qn)_{\Delta}) &\longrightarrow H^{m+n-2}_{\mathfrak{m}_p}(\mathcal{R}_p(qd-qm, qe-qn)_{\Delta}) \\ &\longrightarrow H^{m+n-1}_{\mathfrak{m}_p}(T_p(qd-qm-d, qe-qn-e)_{\Delta}) \longrightarrow \cdots . \end{aligned}$$

Since $H^{m+n-2}_{\mathfrak{m}_p}(T_p(qd-qm, qe-qn)_{\Delta})$ vanishes by Lemma 2.1, we conclude that

$$\begin{aligned} \operatorname{rank}_K ((\mathcal{R}_p)_{\Delta})_k &\leqslant \operatorname{rank}_K [H^{m+n-1}_{\mathfrak{m}_p}(T_p(qd-qm-d, qe-qn-e)_{\Delta})]_k \\ &= \operatorname{rank}_K H^m_{\mathfrak{a}_p}(A_p)_{qd-qm-d+gk} \otimes H^n_{\mathfrak{b}_q}(B_p)_{qe-qn-e+hk} . \end{aligned}$$

Hence $qd-qm-d+gk < 0$; as $d - gk < 0$, we conclude $d < m$. Similarly, $e < n$. $\square$

We conclude this section with an example where a local cohomology module of a standard graded ring is not rigid in the sense that $H^2_{\mathfrak{m}}(R)_0 = 0$ while $H^2_{\mathfrak{m}}(R)_1 \neq 0$. Further such examples are constructed in Section 4.

Proposition 3.3. *Let K be a field and let*

$$\mathcal{R} = K[x_1, x_2, x_3, y_1, y_2]/(f)$$

where $\deg x_i = (1, 0)$, $\deg y_j = (0, 1)$, and $\deg f = (d, e)$ for $d \geqslant 4$ and $e \geqslant 1$. Let g and h be positive integers such that $g \leqslant d - 3$ and $h \geqslant e$, and set $\Delta = (g, h)\mathbb{Z}$. Then $H^2_{\mathfrak{m}}(\mathcal{R}_{\Delta})_0 = 0$ and $H^2_{\mathfrak{m}}(\mathcal{R}_{\Delta})_1 \neq 0$.

Proof. Using the resolution of $\mathcal{R}$ over the polynomial ring T as in the proof of Theorem 3.1, we have an exact sequence

$$H^2_{\mathfrak{m}}(T_\Delta) \longrightarrow H^2_{\mathfrak{m}}(\mathcal{R}_\Delta) \longrightarrow H^3_{\mathfrak{m}}(T(-d,-e)_\Delta) \longrightarrow H^3_{\mathfrak{m}}(T_\Delta)\,.$$

Lemma 2.1 implies that $H^2_{\mathfrak{m}}(T_\Delta) = 0 = H^3_{\mathfrak{m}}(T_\Delta)$. Hence, again by Lemma 2.1,

$$H^2_{\mathfrak{m}}(\mathcal{R}_\Delta)_0 = H^3(A)_{-d} \otimes B_{-e} = 0 \quad \text{and} \quad H^2_{\mathfrak{m}}(\mathcal{R}_\Delta)_1 = H^3(A)_{g-d} \otimes B_{h-e} \neq 0\,. \ \square$$

4. Non-rigid local cohomology modules

We construct examples of standard graded normal rings R over $\mathbb{C}$, with only isolated singularities, for which $H^2_{\mathfrak{m}}(R)_0 = 0$ and $H^2_{\mathfrak{m}}(R)_1 \neq 0$. Let S be the localization of such a ring R at its homogeneous maximal ideal. By results of Danilov [Da1, Da2], Theorem 4.1 below, it follows that the divisor class group of S is finitely generated, though S does not have a discrete divisor class group, i.e., the natural map $\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S[[t]])$ is not bijective. Here, remember that if A is a Noetherian normal domain, then so is $A[[t]]$.

Theorem 4.1. *Let R be a standard graded normal ring, which is finitely generated as an algebra over $R_0 = \mathbb{C}$. Assume, moreover, that $X = \operatorname{Proj} R$ is smooth. Set $(S, \mathfrak{m})$ to be the local ring of R at its homogeneous maximal ideal, and $\widehat{S}$ to be the $\mathfrak{m}$-adic completion of S. Then*

1. *the group $\mathrm{Cl}(S)$ is finitely generated if and only if $H^1(X, \mathcal{O}_X) = 0$;*
2. *the map $\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(\widehat{S})$ is bijective if and only if $H^1(X, \mathcal{O}_X(i)) = 0$ for each integer $i \geqslant 1$; and*
3. *the map $\mathrm{Cl}(S) \longrightarrow \mathrm{Cl}(S[[t]])$ is bijective if and only if $H^1(X, \mathcal{O}_X(i)) = 0$ for each integer $i \geqslant 0$.*

The essential point in our construction is in the following proposition:

Theorem 4.2. *Let A be a Cohen-Macaulay ring of dimension $d \geqslant 2$, which is a standard graded algebra over a field K. For $s \geqslant 2$, let $z_1, \dots, z_s$ be a regular sequence in A, consisting of homogeneous elements of equal degree, say k. Consider the Rees ring $\mathcal{R} = A[z_1 t, \dots, z_s t]$ with the $\mathbb{Z}^2$-grading where $\deg x = (n, 0)$ for $x \in A_n$, and $\deg z_i t = (0,1)$.*

Let $\Delta = (g, h)\mathbb{Z}$ where g, h are positive integers, and let $\mathfrak{m}$ denote the homogeneous maximal ideal of $\mathcal{R}_\Delta$. Then:

1. *$H^q_{\mathfrak{m}}(\mathcal{R}_\Delta) = 0$ if $q \neq d - s + 1, d$; and*
2. *$H^{d-s+1}_{\mathfrak{m}}(\mathcal{R}_\Delta)_i \neq 0$ if and only if $1 \leqslant i \leqslant (a + ks - k)/g$, where a is the a-invariant of A.*

In particular, $\mathcal{R}_\Delta$ is Cohen-Macaulay if and only if $g > a + ks - k$.

Example 4.3. For $d \geqslant 3$, let $A = \mathbb{C}[x_0, \dots, x_d]/(f)$ be a standard graded hypersurface such that $\operatorname{Proj} A$ is smooth over $\mathbb{C}$. Take general k-forms $z_1, \dots, z_{d-1} \in A$, and consider the Rees ring $\mathcal{R} = A[z_1 t, \dots, z_{d-1} t]$. Since $(\boldsymbol{z}) \subset A$ is a radical ideal,

$$\operatorname{gr}((\boldsymbol{z}), A) \cong A/(\boldsymbol{z})[y_1, \dots, y_{d-1}]$$

is a reduced ring, and therefore $\mathcal{R} = A[z_1 t, \dots, z_{d-1} t]$ is integrally closed in $A[t]$. Since A is normal, so is $\mathcal{R}$. Note that $\operatorname{Proj} \mathcal{R}_\Delta$ is the blow-up of $\operatorname{Proj} A$ at the subvariety defined by $(\boldsymbol{z})$, i.e., at $k^{d-1}(\deg f)$ points. It follows that $\operatorname{Proj} \mathcal{R}_\Delta$ is smooth over $\mathbb{C}$. Hence $\mathcal{R}_\Delta$ is a standard graded $\mathbb{C}$-algebra, which is normal and has an isolated singularity.

If $\Delta = (g, h)\mathbb{Z}$ is a diagonal with $1 \leqslant g \leqslant \deg f + k(d-2) - (d+1)$ and $h \geqslant 1$, then Theorem 4.2 implies that

$$H^2_{\mathfrak{m}}(\mathcal{R}_\Delta)_0 = 0 \quad \text{ and } \quad H^2_{\mathfrak{m}}(\mathcal{R}_\Delta)_1 \neq 0 \,.$$

The rest of this section is devoted to proving Theorem 4.2. We may assume that the base field K is infinite. Then one can find linear forms $x_1, \dots, x_{d-s}$ in A such that $x_1, \dots, x_{d-s}, z_1, \dots, z_s$ is a maximal A-regular sequence.

We will use the following lemma; the notation is as in Theorem 4.2.

Lemma 4.4. *Let $\mathfrak{a}$ be the homogeneous maximal ideal of A. Set $I = (z_1, \dots, z_s)A$. Let r be a positive integer.*

(1) $H^q_{\mathfrak{a}}(I^r) = 0$ *if* $q \neq d - s + 1, d$.
(2) *Assume $d > s$. Then,* $H^{d-s+1}_{\mathfrak{a}}(I^r)_i \neq 0$ *if and only if* $i \leqslant a + ks + rk - k$.
(3) *Assume $d = s$. Then,* $H^{d-s+1}_{\mathfrak{a}}(I^r)_i \neq 0$ *if and only if* $0 \leqslant i \leqslant a + ks + rk - k$.

Proof. Recall that A and A/I^r are Cohen-Macaulay rings of dimension d and $d - s$, respectively. By the exact sequence

$$0 \longrightarrow I^r \longrightarrow A \longrightarrow A/I^r \longrightarrow 0$$

we obtain

$$H^q_{\mathfrak{a}}(I^r) = \begin{cases} H^d_{\mathfrak{a}}(A) & \text{if } q = d \\ H^{d-s}_{\mathfrak{a}}(A/I^r) & \text{if } q = d - s + 1 \\ 0 & \text{if } q \neq d - s + 1, d \,, \end{cases}$$

which proves (1).

Next we prove (2) and (3). Since A/I^r is a standard graded Cohen-Macaulay ring of dimension $d - s$, it is enough to show that the a-invariant of this ring equals $a + ks + rk - k$. This is straightforward if $r = 1$, and we proceed by induction. Consider the exact sequence

$$0 \longrightarrow I^r/I^{r+1} \longrightarrow A/I^{r+1} \longrightarrow A/I^r \longrightarrow 0 \,.$$

Since $z_1, \dots, z_s$ is a regular sequence of k-forms, I^r/I^{r+1} is isomorphic to

$$\big((A/I)(-rk)\big)^{\binom{s-1+r}{r}} \,.$$

Thus, we have the following exact sequence:

$$0 \longrightarrow H^{d-s}_{\mathfrak{a}}\big((A/I)(-rk)\big)^{\binom{s-1+r}{r}} \longrightarrow H^{d-s}_{\mathfrak{a}}(A/I^{r+1}) \longrightarrow H^{d-s}_{\mathfrak{a}}(A/I^{r}) \longrightarrow 0\,.$$

The a-invariant of $(A/I)(-rk)$ equals $a+ks+rk$, and that of A/I^r is $a+ks+rk-k$ by the inductive hypothesis. Thus, A/I^{r+1} has a-invariant $a+ks+rk$. ☐

Proof of Theorem 4.2. Let $B = K[y_1, \dots, y_s]$ be a polynomial ring, and set

$$T = A \otimes_K B = A[y_1, \dots, y_s]\,.$$

Consider the $\mathbb{Z}^2$-grading on T where $\deg x = (n,0)$ for $x \in A_n$, and $\deg y_i = (0,1)$ for each i. One has a surjective homomorphism of graded rings

$$T \longrightarrow \mathcal{R} = A[z_1t, \dots, z_st] \qquad \text{where } y_i \longmapsto z_it\,,$$

and this induces an isomorphism

$$\mathcal{R} \cong T/I_2\left(\begin{smallmatrix} z_1 & \cdots & z_s \\ y_1 & \cdots & y_s \end{smallmatrix}\right)\,.$$

The minimal free resolution of $\mathcal{R}$ over T is given by the Eagon-Northcott complex

$$0 \longrightarrow F^{-(s-1)} \longrightarrow F^{-(s-2)} \longrightarrow \cdots \longrightarrow F^0 \longrightarrow 0\,,$$

where $F^0 = T(0,0)$, and F^{-i} for $1 \leqslant i \leqslant s-1$ is the direct sum of $\binom{s}{i+1}$ copies of

$$T(-k,-i) \oplus T(-2k,-(i-1)) \oplus \cdots \oplus T(-ik,-1)\,.$$

Let $\mathfrak{n}$ be the homogeneous maximal ideal of T_Δ. One has the spectral sequence:

$$E_2^{p,q} = H^p(H^q_{\mathfrak{n}}(F^\bullet_\Delta)) \Longrightarrow H^{p+q}_{\mathfrak{m}}(\mathcal{R}_\Delta)\,.$$

Let G be the set of (n,m) such that $T(n,m)$ appears in the Eagon-Northcott complex above, i.e., the elements of G are

$$\begin{gathered} (0,0)\,, \\ (-k,-1), \\ (-k,-2),\ (-2k,-1)\,, \\ (-k,-3),\ (-2k,-2),\ (-3k,-1)\,, \\ \vdots \\ (-k,-(s-1)), \qquad \dots \qquad (-(s-1)k,-1)\,. \end{gathered}$$

Let $\mathfrak{a}$ and $\mathfrak{b}$ be the homogeneous maximal ideal of A and B respectively. For integers n and m, the Künneth formula gives

$$\begin{aligned} &H^q_{\mathfrak{n}}(T(n,m)) \\ &\quad= H^q_{\mathfrak{n}}(A(n) \otimes_K B(m)) \\ &\quad= \big(H^q_{\mathfrak{a}}(A(n)) \otimes B(m)\big) \oplus \big(A(n) \otimes H^q_{\mathfrak{b}}(B(m))\big) \oplus \bigoplus_{i+j=q+1} H^i_{\mathfrak{a}}(A(n)) \otimes H^j_{\mathfrak{b}}(B(m)) \\ &\quad= H^q_{\mathfrak{a}}(T(n,m)) \oplus H^q_{\mathfrak{b}}(T(n,m)) \oplus \bigoplus_{i+j=q+1} H^i_{\mathfrak{a}}(A(n)) \otimes_K H^j_{\mathfrak{b}}(B(m))\,. \end{aligned}$$

As A and B are Cohen-Macaulay of dimension d and s respectively, it follows that

$$H^q_{\mathfrak{n}}(F^\bullet) = 0 \qquad \text{if } q \neq s, d, d+s-1\,.$$

In the case where $d > s$, one has

$$H^s_{\mathfrak{n}}(F^\bullet) = H^s_{\mathfrak{b}}(F^\bullet) \quad \text{ and } \quad H^d_{\mathfrak{n}}(F^\bullet) = H^d_{\mathfrak{a}}(F^\bullet)\,,$$

and if $d = s$, then

$$H^d_{\mathfrak{n}}(F^\bullet) = H^d_{\mathfrak{a}}(F^\bullet) \oplus H^s_{\mathfrak{b}}(F^\bullet)\,.$$

We claim $H^s_{\mathfrak{b}}(F^\bullet)_\Delta = 0$. If not, there exists $(n,m) \in G$ and $\ell \in \mathbb{Z}$ such that

$$H^s_{\mathfrak{b}}(T(n,m))_{(g\ell,h\ell)} \neq 0\,.$$

This implies that

$$H^s_{\mathfrak{b}}(T(n,m))_{(g\ell,h\ell)} = A(n)_{g\ell} \otimes_K H^s_{\mathfrak{b}}(B(m))_{h\ell} = A_{n+g\ell} \otimes_K H^s_{\mathfrak{b}}(B)_{m+h\ell}$$

is nonzero, so

$$n + g\ell \geqslant 0 \quad \text{ and } \quad m + h\ell \leqslant -s\,,$$

and hence

$$-\frac{n}{g} \leqslant \ell \leqslant -\frac{s+m}{h}\,.$$

But $(n,m) \in G$, so $n \leqslant 0$ and $m \geqslant -(s-1)$, implying that

$$0 \leqslant \ell \leqslant -\frac{1}{h}\,,$$

which is not possible. This proves that $H^s_{\mathfrak{b}}(F^\bullet)_\Delta = 0$. Thus, we have

$$H^q_{\mathfrak{n}}(F^\bullet)_\Delta = \begin{cases} 0 & \text{if } q \neq d, d+s-1, \\ H^d_{\mathfrak{a}}(F^\bullet)_\Delta & \text{if } q = d. \end{cases}$$

It follows that

$$E_2^{p,q} = H^p(H^q_{\mathfrak{n}}(F^\bullet_\Delta)) = E_\infty^{p,q}$$

for each p and q. Therefore,

$$H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) = E_2^{i-d,d} = H^{i-d}(H^d_{\mathfrak{n}}(F^\bullet_\Delta)) = H^{i-d}(H^d_{\mathfrak{a}}(F^\bullet)_\Delta) = H^i_{\mathfrak{a}}(\mathcal{R})_\Delta$$

for $d-s+1 \leqslant i \leqslant d-1$, and

$$H^i_{\mathfrak{m}}(\mathcal{R}_\Delta) = 0 \qquad \text{for } i < d-s+1\,.$$

We next study $H^i_{\mathfrak{a}}(\mathcal{R})$. Since

$$\mathcal{R} = A \oplus I(k) \oplus I^2(2k) \oplus \cdots \oplus I^r(rk) \oplus \cdots\,,$$

we have

$$H^i_{\mathfrak{a}}(\mathcal{R}) = H^i_{\mathfrak{a}}(A) \oplus H^i_{\mathfrak{a}}(I)(k) \oplus H^i_{\mathfrak{a}}(I^2)(2k) \oplus \cdots \oplus H^i_{\mathfrak{a}}(I^r)(rk) \oplus \cdots\,.$$

Theorem 4.2 (1) now follow using Lemma 4.4 (1).

Assume that $d > s$. Then, by Lemma 4.4 (2), $H^{d-s+1}_{\mathfrak{a}}(I^r(rk))_i \neq 0$ if and only if $i \leqslant a + ks - k$.

Assume that $d = s$. Then, by Lemma 4.4 (3), $H_{\mathfrak{a}}^{d-s+1}(I^r(rk))_i \neq 0$ if and only if $-rk \leqslant i \leqslant a + ks - k$.

In each case, $H_{\mathfrak{a}}^{d-s+1}(\mathcal{R})_{(gi,hi)} \neq 0$ if and only if

$$1 \leqslant i \leqslant \frac{a+ks-k}{g}. \qquad \square$$

5. Rational singularities

Let R be a normal domain, essentially of finite type over a field of characteristic zero, and consider a *desingularization* $f\colon Z \longrightarrow \operatorname{Spec} R$, i.e., a proper birational morphism with Z a nonsingular variety. One says R has *rational singularities* if $R^i f_* \mathcal{O}_Z = 0$ for each $i \geqslant 1$; this does not depend on the choice of the desingularization f. For $\mathbb{N}$-graded rings, one has the following criterion due to Flenner [Fl] and Watanabe [Wa1].

Theorem 5.1. *Let R be a normal $\mathbb{N}$-graded ring which is finitely generated over a field R_0 of characteristic zero. Then R has rational singularities if and only if it is Cohen-Macaulay, $a(R) < 0$, and the localization $R_{\mathfrak{p}}$ has rational singularities for each $\mathfrak{p} \in \operatorname{Spec} R \smallsetminus \{R_+\}$.*

When R has an isolated singularity, the above theorem gives an effective criterion for determining if R has rational singularities. However, a multigraded hypersurface typically does not have an isolated singularity, and the following variation turns out to be useful:

Theorem 5.2. *Let R be a normal $\mathbb{N}^r$-graded ring such that $R_{\mathbf{0}}$ is a local ring essentially of finite type over a field of characteristic zero, and R is generated over $R_{\mathbf{0}}$ by elements*

$$x_{11}, x_{12}, \dots, x_{1t_1}, \quad x_{21}, x_{22}, \dots, x_{2t_2}, \quad \dots, \quad x_{r1}, x_{r2}, \dots, x_{rt_r},$$

where $\deg x_{ij}$ is a positive integer multiple of the i-th unit vector $e_i \in \mathbb{N}^r$. Then R has rational singularities if and only if

1. *R is Cohen-Macaulay,*
2. *$R_{\mathfrak{p}}$ has rational singularities for each $\mathfrak{p}$ belonging to*

   $$\operatorname{Spec} R \setminus \big(V(x_{11}, x_{12}, \dots, x_{1t_1}) \cup \dots \cup V(x_{r1}, x_{r2}, \dots, x_{rt_r})\big), \quad \text{and}$$
3. *$\boldsymbol{a}(R) < \mathbf{0}$, i.e., $a(R^{\varphi_i}) < 0$ for each coordinate projection $\varphi_i\colon \mathbb{N}^r \longrightarrow \mathbb{N}$.*

Before proceeding with the proof, we record some preliminary results.

Remark 5.3. Let R be an $\mathbb{N}$-graded ring. We use $R^\natural$ to denote the Rees algebra with respect to the filtration $F_n = R_{\geqslant n}$, i.e.,

$$R^\natural = F_0 \oplus F_1 T \oplus F_2 T^2 \oplus \cdots.$$

When considering $\operatorname{Proj} R^\natural$, we use the $\mathbb{N}$-grading on $R^\natural$ where $[R^\natural]_n = F_nT^n$. The inclusion $R = [R^\natural]_0 \hookrightarrow R^\natural$ gives a map

$$\operatorname{Proj} R^\natural \xrightarrow{f} \operatorname{Spec} R\,.$$

Also, the inclusions $R_n \hookrightarrow F_n$ give rise to an injective homomorphism of graded rings $R \hookrightarrow R^\natural$, which induces a surjection

$$\operatorname{Proj} R^\natural \xrightarrow{\pi} \operatorname{Proj} R\,.$$

Lemma 5.4. *Let R be an $\mathbb{N}$-graded ring which is finitely generated over R_0, and assume that R_0 is essentially of finite type over a field of characteristic zero.*

If $R_\mathfrak{p}$ has rational singularities for all primes $\mathfrak{p} \in \operatorname{Spec} R \smallsetminus V(R_+)$, then $\operatorname{Proj} R^\natural$ has rational singularities.

Proof. Note that $\operatorname{Proj} R^\natural$ is covered by affine open sets $D_+(rT^n)$ for integers $n \geqslant 1$ and homogeneous elements $r \in R_{\geqslant n}$. Consequently, it suffices to check that $[R^\natural_{rT^n}]_0$ has rational singularities. Next, note that

$$\big[R^\natural_{rT^n}\big]_0 = R + \frac{1}{r}[R]_{\geqslant n} + \frac{1}{r^2}[R]_{\geqslant 2n} + \cdots\,.$$

In the case $\deg r > n$, the ring above is simply R_r, which has rational singularities by the hypothesis of the lemma. If $\deg r = n$, then

$$\big[R^\natural_{rT^n}\big]_0 = \big[R_r\big]_{\geqslant 0}\,.$$

The $\mathbb{Z}$-graded ring R_r has rational singularities and so, by [Wa1, Lemma 2.5], the ring $[R_r]_{\geqslant 0}$ has rational singularities as well. □

Lemma 5.5. [Hy2, Lemma 2.3] *Let R be an $\mathbb{N}$-graded ring which is finitely generated over a local ring $(R_0, \mathfrak{m})$. Suppose $[H^i_{\mathfrak{m}+R_+}(R)]_{\geqslant 0} = 0$ for all $i \geqslant 0$. Then, for all ideals $\mathfrak{a}$ of R_0, one has*

$$\big[H^i_{\mathfrak{a}+R_+}(R)\big]_{\geqslant 0} = 0 \qquad \text{for all } i \geqslant 0\,.$$

We are now in a position to prove the following theorem, which is a variation of [Fl, Satz 3.1], [Wa1, Theorem 2.2], and [Hy1, Theorem 1.5].

Theorem 5.6. *Let R be an $\mathbb{N}$-graded normal ring which is finitely generated over R_0, and assume that R_0 is a local ring essentially of finite type over a field of characteristic zero. Then R has rational singularities if and only if*

(1) *R is Cohen-Macaulay,*
(2) *$R_\mathfrak{p}$ has rational singularities for all $\mathfrak{p} \in \operatorname{Spec} R \smallsetminus V(R_+)$, and*
(3) *$a(R) < 0$.*

Proof. It is straightforward to see that conditions (1)–(3) hold when R has rational singularities, and we focus on the converse. Consider the morphism

$$Y = \operatorname{Proj} R^\natural \xrightarrow{f} \operatorname{Spec} R$$

as in Remark 5.3. Let $g\colon Z \longrightarrow Y$ be a desingularization of Y; the composition

$$Z \xrightarrow{g} Y \xrightarrow{f} \operatorname{Spec} R$$

is then a desingularization of $\operatorname{Spec} R$. Note that $Y = \operatorname{Proj} R^\natural$ has rational singularities by Lemma 5.4, so

$$g_*\mathcal{O}_Z = \mathcal{O}_Y \quad \text{ and } \quad R^q g_*\mathcal{O}_Z = 0 \qquad \text{for all } q \geqslant 1\,.$$

Consequently the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q g_*\mathcal{O}_Z) \Longrightarrow H^{p+q}(Z, \mathcal{O}_Z)$$

degenerates, and we get $H^p(Z, \mathcal{O}_Z) = H^p(Y, \mathcal{O}_Y)$ for all $p \geqslant 1$. Since $\operatorname{Spec} R$ is affine, we also have $R^p(g \circ f)_*\mathcal{O}_Z = H^p(Z, \mathcal{O}_Z)$. To prove that R has rational singularities, it now suffices to show that $H^p(Y, \mathcal{O}_Y) = 0$ for all $p \geqslant 1$. Consider the map $\pi\colon Y \longrightarrow X = \operatorname{Proj} R$. We have

$$H^p(Y, \mathcal{O}_Y) = H^p(X, \pi_*\mathcal{O}_X) = \bigoplus_{n \geqslant 0} H^p(X, \mathcal{O}_X(n)) = \left[H^{p+1}_{R_+}(R)\right]_{\geqslant 0}.$$

By condition (1), we have $[H^p_{\mathfrak{m}+R_+}(R)]_{\geqslant 0} = 0$ for all $p \geqslant 0$, and so Lemma 5.5 implies that $[H^p_{R_+}(R)]_{\geqslant 0} = 0$ for all $p \geqslant 0$ as desired. □

Proof of theorem 5.2. If R has rational singularities, it is easily seen that conditions (1)–(3) must hold. For the converse, we proceed by induction on r. The case $r = 1$ is Theorem 5.6 established above, so assume $r \geqslant 2$. It suffices to show that $R_{\mathfrak{M}}$ has rational singularities where $\mathfrak{M}$ is the homogeneous maximal ideal of R. Set

$$\mathfrak{m} = \mathfrak{M} \cap \left[R^{\varphi_r}\right]_0,$$

and consider the $\mathbb{N}$-graded ring S obtained by inverting the multiplicative set $[R^{\varphi_r}]_0 \smallsetminus \mathfrak{m}$ in R^{φ_r}. Since $R_{\mathfrak{M}}$ is a localization of S, it suffices to show that S has rational singularities. Note that $a(S) = a(R^{\varphi_r})$, which is a negative integer by (1). Using Theorem 5.6, it is therefore enough to show that $R_{\mathfrak{P}}$ has rational singularities for all $\mathfrak{P} \in \operatorname{Spec} R \smallsetminus V(x_{r1}, x_{r2}, \dots, x_{rt_r})$. Fix such a prime $\mathfrak{P}$, and let

$$\psi\colon \mathbb{Z}^r \longrightarrow \mathbb{Z}^{r-1}$$

be the projection to the first $r-1$ coordinates. Note that R^ψ is the ring R regraded such that $\deg x_{rj} = 0$, and the degrees of x_{ij} for $i < r$ are unchanged. Set

$$\mathfrak{p} = \mathfrak{P} \cap \left[R^\psi\right]_{\mathbf{0}},$$

and let T be the ring obtained by inverting the multiplicative set $[R^\psi]_{\mathbf{0}} \smallsetminus \mathfrak{p}$ in R^ψ. It suffices to show that T has rational singularities. Note that T is an $\mathbb{N}^{r-1}$-graded ring defined over a local ring $(T_{\mathbf{0}}, \mathfrak{p})$, and that it has homogeneous maximal ideal $\mathfrak{p} + \mathfrak{b}T$ where

$$\mathfrak{b} = (R^\psi)_+ = (x_{ij} \mid i < r)R\,.$$

Using the inductive hypothesis, it remains to verify that $\boldsymbol{a}(T) < \mathbf{0}$. By condition (1), for all integers $1 \leqslant j \leqslant r-1$, we have

$$\left[H^i_{\mathfrak{M}}(R)^{\varphi_j}\right]_{\geqslant 0} = 0 \qquad \text{for all } i \geqslant 0\,,$$

and using Lemma 5.5 it follows that

$$\left[H^i_{\mathfrak{p}+\mathfrak{b}}(R)^{\varphi_j}\right]_{\geqslant 0} = 0 \qquad \text{for all } i \geqslant 0\,.$$

Consequently $a(T^{\varphi_j}) < 0$ for $1 \leqslant j \leqslant r-1$, which completes the proof. □

6. F-regularity

For the theory of tight closure, we refer to the papers [HH1, HH2] and [HH3]. We summarize results about F-rational and F-regular rings:

Theorem 6.1. *The following hold for rings of prime characteristic.*

1. *Regular rings are F-regular.*
2. *Direct summands of F-regular rings are F-regular.*
3. *F-rational rings are normal; an F-rational ring which is a homomorphic image of a Cohen-Macaulay ring is Cohen-Macaulay.*
4. *F-rational Gorenstein rings are F-regular.*
5. *Let R be an $\mathbb{N}$-graded ring which is finitely generated over a field R_0. If R is weakly F-regular, then it is F-regular.*

Proof. For (1) and (2) see [HH1, Theorem 4.6] and [HH1, Proposition 4.12] respectively; (3) is part of [HH2, Theorem 4.2], and for (4) see [HH2, Corollary 4.7], Lastly, (5) is [LS, Corollary 4.4]. □

The characteristic zero aspects of tight closure are developed in [HH4]. Let K be a field of characteristic zero. A finitely generated K-algebra $R = K[x_1, \ldots, x_m]/\mathfrak{a}$ is of *F-regular type* if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$, and a finitely generated free A-algebra

$$R_A = A[x_1, \ldots, x_m]/\mathfrak{a}_A\,,$$

such that $R \cong R_A \otimes_A K$ and, for all maximal ideals μ in a Zariski dense subset of $\operatorname{Spec} A$, the fiber rings $R_A \otimes_A A/\mu$ are F-regular rings of characteristic $p > 0$. Similarly, R is of *F-rational type* if for a dense subset of μ, the fiber rings $R_A \otimes_A A/\mu$ are F-rational. Combining results from [Ha, HW, MS, Sm] one has:

Theorem 6.2. *Let R be a ring which is finitely generated over a field of characteristic zero. Then R has rational singularities if and only if it is of F-rational type. If R is $\mathbb{Q}$-Gorenstein, then it has log terminal singularities if and only if it is of F-regular type.*

Proposition 6.3. *Let K be a field of characteristic $p > 0$, and R an $\mathbb{N}$-graded normal ring which is finitely generated over $R_0 = K$. Let ω denote the graded canonical module of R, and set $d = \dim R$.*

Suppose R is F-regular. Then, for each integer k, there exists $q = p^e$ such that

$$\operatorname{rank}_K R_k \leqslant \operatorname{rank}_K [H^d_{\mathfrak{m}}(\omega^{(q)})]_k .$$

Proof. If $d \leqslant 1$, then R is regular and the assertion is elementary. Assume $d \geqslant 2$. Let $\xi \in [H^d_{\mathfrak{m}}(\omega)]_0$ be an element which generates the socle of $H^d_{\mathfrak{m}}(\omega)$. Since the map $\omega^{[q]} \longrightarrow \omega^{(q)}$ is an isomorphism in codimension one, $F^e(\xi)$ may be viewed as an element of $H^d_{\mathfrak{m}}(\omega^{(q)})$ as in [Wa2].

Fix an integer k. For each $e \in \mathbb{N}$, set V_e to be the kernel of the vector space homomorphism

$$R_k \longrightarrow \left[H^d_{\mathfrak{m}}\left(\omega^{(p^e)}\right)\right]_k , \qquad \text{where } c \longmapsto cF^e(\xi) . \tag{6.3.1}$$

If $cF^{e+1}(\xi) = 0$, then $F(cF^e(\xi)) = c^p F^{e+1}(\xi) = 0$; since R is F-pure, it follows that $cF^e(\xi) = 0$. Consequently the vector spaces V_e form a descending sequence

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots .$$

The hypothesis that R is F-regular implies $\bigcap_e V_e = 0$. Since each V_e has finite rank, $V_e = 0$ for $e \gg 0$. Hence the homomorphism (6.3.1) is injective for $e \gg 0$. $\square$

We next record tight closure properties of general $\mathbb{N}$-graded hypersurfaces. The results for F-purity are essentially worked out in [HR].

Theorem 6.4. *Let $A = K[x_1, \dots, x_m]$ be a polynomial ring over a field K of positive characteristic. Let d be a nonnegative integer, and set $M = \binom{d+m-1}{d} - 1$. Consider the affine space $\mathbb{A}^M_K$ parameterizing the degree d forms in A in which x_1^d occurs with coefficient 1.*

Let U be the subset of $\mathbb{A}^M_K$ corresponding to the forms f for which A/fA F-pure. Then U is a Zariski open set, and it is nonempty if and only if $d \leqslant m$.

Let V be the set corresponding to forms f for which A/fA is F-regular. Then V contains a nonempty Zariski open set if $d < m$, and is empty otherwise.

Proof. The set U is Zariski open by [HR, page 156] and it is empty if $d > m$ by [HR, Proposition 5.18]. If $d \leqslant m$, the square-free monomial $x_1 \cdots x_d$ defines an F-pure hypersurface $A/(x_1 \cdots x_d)$. A linear change of variables yields the polynomial

$$f = x_1(x_1 + x_2) \cdots (x_1 + x_d)$$

in which x_1^d occurs with coefficient 1. Hence U is nonempty for $d \leqslant m$.

If $d \geqslant m$, then A/fA has a-invariant $d - m \geqslant 0$ so A/fA is not F-regular. Suppose $d < m$. Consider the set $W \subseteq \mathbb{A}^M_K$ parameterizing the forms f for which A/fA is F-pure and $(A/fA)_{\overline{x}_1}$ is regular; W is a nonempty open subset of $\mathbb{A}^M_K$. Let f correspond to a point of W. The element $\overline{x}_1 \in A/fA$ has a power which is a test element; since A/fA is F-pure, it follows that $\overline{x}_1$ is a test element. Note that $\overline{x}_2, \ldots, \overline{x}_m$ is a homogeneous system of parameters for A/fA and that $\overline{x}_1^{d-1}$ generates the socle modulo $(\overline{x}_2, \ldots, \overline{x}_m)$. Hence the ring A/fA is F-regular if and only if there exists a power q of the prime characteristic p such that

$$x_1^{(d-1)q+1} \notin (x_2^q, \ldots, x_m^q, f)A\,.$$

The set of such f corresponds to an open subset of W; it remains to verify that this subset is nonempty. For this, consider

$$f = x_1^d + x_2 \cdots x_{d+1}\,,$$

which corresponds to a point of W, and note that A/fA is F-regular since

$$x_1^{(d-1)p+1} \notin (x_2^p, \ldots, x_m^p, f)A\,. \qquad \square$$

These ideas carry over to multi-graded hypersurfaces; we restrict below to the bigraded case. The set of forms in $K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ of degree (d, e) in which $x_1^d y_1^e$ occurs with coefficient 1 is parametrized by the affine space $\mathbb{A}_K^N$ where $N = \binom{d+m-1}{d}\binom{e+n-1}{e} - 1$.

Theorem 6.5. *Let $B = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over a field K of positive characteristic. Consider the $\mathbb{N}^2$-grading on B with $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$. Let d, e be nonnegative integers, and consider the affine space $\mathbb{A}_K^N$ parameterizing forms of degree (d, e) in which $x_1^d y_1^e$ occurs with coefficient 1.*

Let U be the subset of $\mathbb{A}_K^N$ corresponding to forms f for which B/fB is F-pure. Then U is a Zariski open set, and it is nonempty if and only if $d \leqslant m$ and $e \leqslant n$.

Let V be the set corresponding to forms f for which B/fB is F-regular. Then V contains a nonempty Zariski open set if $d < m$ and $e < n$, and is empty otherwise.

Proof. The argument for F-purity is similar to the proof of Theorem 6.4; if $d \leqslant m$ and $e \leqslant n$, then the polynomial $x_1 \cdots x_d y_1 \cdots y_e$ defines an F-pure hypersurface.

If B/fB is F-regular, then $\boldsymbol{a}(B/fB) < \mathbf{0}$ implies $d < m$ and $e < n$. Conversely, if $d < m$ and $e < n$, then there is a nonempty open set W corresponding to forms f for which the hypersurface B/fB is F-pure and $(B/fB)_{\overline{x}_1\overline{y}_1}$ is regular. In this case, $\overline{x}_1\overline{y}_1 \in B/fB$ is a test element. The socle modulo the parameter ideal $(x_1 - y_1, x_2, \ldots, x_m, y_2, \ldots, y_n)B/fB$ is generated by $\overline{x}_1^{d+e-1}$, so B/fB is F-regular if and only if there exists a power $q = p^e$ such that

$$x_1^{(d+e-1)q+1} \notin (x_1^q - y_1^q, x_2^q, \ldots, x_m^q, y_2^q, \ldots, y_n^q, f)B\,.$$

The subset of W corresponding to such f is open; it remains to verify that it is nonempty. For this, use $f = x_1^d y_1^e + x_2 \cdots x_{d+1} y_2 \cdots y_{e+1}$. $\square$

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