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GRAPH-THEORETIC ALBANESE MAPS REVISITED

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1. Introduction

In [6], [5], we introduced the notion of Albanese maps in the graph-theoretic context (see also [7], [8]). An Albanese map is a harmonic map of a finite graph as a 1-dimensional singular space into a flat torus which, together with the flat metric, is characterized by a minimizing property for certain energy functional, and is related to asymptotic behaviors of random walks on crystal lattices. On the other hand, the notion of Abel-Jacobi maps was brought in graph theory by R. Bacher, P. De La Harpe, and T. Nagnibeda [1] (see also [3]). A graph version of Abel-Jacobi maps is a certain class of harmonic functions defined on vertices with values in finite abelian groups. The aim of this note is to give a relationship between these notions.

2. Albanese maps

We first explain Albanese maps in a bit different way from the original one given in [8].

Let $X = (V, E)$ be a finite graph with a set of vertices V and a set of oriented edges E. By $o(e)$ (resp. $t(e)$) we denote the origin (resp. terminus) of $e \in E$. The symbol $\overline{e}$ stands for the inverse edge of e. Define the bilinear form on $C_1(X, \mathbb{Z})$, the group of 1-chains on X, by

$$\langle e, e' \rangle = \begin{cases} 1 & (e = e') \\ -1 & (e = \overline{e'}) \\ 0 & (otherwise), \end{cases} \tag{1}$$

where $e, e' \in E$, oriented edges in X. This extends to an inner product on $C_1(X, \mathbb{R})$, and is restricted to the homology group $H_1(X, \mathbb{R}) = \mathrm{Ker}\, \partial$, where $\partial : C_1(X) \longrightarrow C_0(X)$ is the boundary map. The *Albanese torus* $\mathbb{A}(X)$ is defined to be the flat torus $H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$ with the flat metric induced from this inner product.

The Albanese map $\Phi^{al} : X \longrightarrow \mathbb{A}(X)$ is defined as follows. Let $P : C_1(X, \mathbb{R}) \longrightarrow H_1(X_0, \mathbb{R})$ be the orthogonal projection. Fix a reference point $x_0 \in V$, and let $c = (e_1, \cdots, e_n)$ be a path with $o(c) = x_0, t(c) = x$. Then put $\Phi^{al}(x_0) = \mathbf{0}$ and

$$\Phi^{al}(x) = P(e_1 + \cdots + e_n) = P(e_1) + \cdots + P(e_n) \pmod{H_1(X, \mathbb{Z})}.$$

If $c' = (e'_1, \cdots, e'_m)$ be another path joining x_0 and x, then

$$e_1 + \cdots + e_n - (e'_1 + \cdots + e'_m) \in H_1(X, \mathbb{Z}),$$

so that

$$P(e_1 + \cdots + e_n) = P(e'_1) + \cdots + P(e'_m) \pmod{H_1(X, \mathbb{Z})}.$$

Hence Φ^{al} as a map from V into $\mathbb{A}(X)$ is well-defined. We extend Φ^{al} to edges as a piecewise linear maps. The map $\Phi^{al} : X \longrightarrow \mathbb{A}(X)$ obtained in this way is a harmonic map in the sense that

$$\Delta\Phi^{al}(x) = \sum_{e \in E_x} \big[\Phi^{al}(te) - \Phi^{al}(oe)\big] = \mathbf{0},$$

where $E_x = \{e \in E;\ o(e) = x\}$. In fact, for any closed path $c = (e_1, \cdots, e_n)$ in X,

$$\sum_{e \in E_x} \langle e, c \rangle = 0$$

since, if $t(e_i) = o(e_{i+1}) = x$, then $\langle e_i, c\rangle + \langle e_{i+1}, c\rangle = 0$. Hence

$$\sum_{e \in E_x} e \in H_1(X, \mathbb{R})^{\perp},$$

and $\Delta\Phi^{al}(x) = P\big(\sum_{e \in E_x} e\big) = \mathbf{0}$.

3. Abel-Jacobi maps into finite abelian groups

There are several definitions of Abel-Jacobi maps. We take up a definiton which resembles the classical one in algebraic geometry.

Define the group of divisors of degree zero by

$$\mathrm{Div}^0(X) = \{\sum_{x \in V} a_x x \in C_0(X, \mathbb{Z}) |\ \sum_x a_x = 0\}$$

and the group of principal divisors by

$$\mathrm{Prin}(X) = \partial\partial^*\big(C_0(X, \mathbb{Z})\big)$$

where ∂^* is the adjoint of ∂ with respect to the inner products on $C_0(X, \mathbb{R})$

$$x \cdot y = \begin{cases} 1 & (x = y) \\ 0 & (x \neq y) \end{cases}$$

and the one on $C_1(X,\mathbb{R})$ defined in the previous section. The *Picard group* is defined as

$$\mathrm{Pic}(X) = \mathrm{Div}^0(X)/\mathrm{Prin}(X).$$

The order $|\mathrm{Pic}(X)|$ coincides with $\kappa(X)$, the number of spanning trees in X. The *discrete Abel-Jacobi map* $\Phi^{aj} : V \longrightarrow \mathrm{Pic}(X)$ is defined by

$$\Phi^{aj}(x) = [x - x_0].$$

4. Discrete Albanese tori and Abel's theorem

Let us now establish a relationship between Albanese maps and discrete Abel-Jacobi maps.

The homology group $H_1(X,\mathbb{Z})$ with coefficients in $\mathbb{Z}$ is an integral lattice in $H_1(X,\mathbb{R})$ with respect to the inner product (1). Denote by $H_1(X,\mathbb{Z})^{\#}$ the dual lattice of $H_1(X,\mathbb{Z})$ in $H_1(X,\mathbb{R})$. Since the lattice $H_1(X,\mathbb{Z})$ is integral, we have $H_1(X,\mathbb{Z}) \subset H_1(X,\mathbb{Z})^{\#}$. The discrete Albanese torus $A(X)$ is defined to be $H_1(X,\mathbb{Z})^{\#}/H_1(X,\mathbb{Z})$ which is identified with a finite subgroup of $\mathbb{A}(X)$.

For any $e \in E$ and $\alpha \in H_1(X,\mathbb{Z})$, we find $\langle P(e),\alpha\rangle = \langle e, P(\alpha)\rangle = \langle e,\alpha\rangle \in \mathbb{Z}$, and hence $P(e) \in H_1(X,\mathbb{Z})^{\#}$. Thus we have

Lemma 4.1. *Let Φ^{al} be the Albanese map of X into $\mathbb{A}(X)$. Then $\Phi(V) \subset A(X)$.*

We shall call $\Phi^{al}|V : V \longrightarrow A(X)$ the *discrete Albanese map.*

In order to prove that $\Phi^{ab}(V)$ generates $A(X)$, take a spanning tree T of X, and let $e_1,\dots,e_b$ ($b = \text{rank } H_1(X,\mathbb{Z})$) be all edges not in T. Then $P(e_1),\dots,P(e_b)$ constitute a $\mathbb{Z}$-basis of $H_1(X,\mathbb{Z})^{\#}$ since, if we take circuits $c_1,\dots,c_b$ in X such that c_i contains e_i, then $\{c_1,\dots,c_b\}$ is a $\mathbb{Z}$-basis of $H_1(X,\mathbb{Z})$, and $\langle c_i, P(e_j)\rangle = \langle P(c_i), e_j\rangle = \langle c_i, e_j\rangle = \delta_{ij}$.

Theorem 4.1. (*A discrete version of Abel's theorem*) *The correspondence $x \in V \mapsto \Phi^{al}(x) \in A(X)$ induces an isomorphism φ of* Pic(X) *onto $A(X)$ such that $\varphi \circ \Phi^{aj} = \Phi^{al}$.*

Proof. This is actually a consequence of the universality of Abel-Jacobi maps (cf. [2]). For the completeness, we will give a proof.

Define the homomorphism $\varphi : \mathrm{Div}^0(X) \longrightarrow A(X)$ by setting $\varphi(x - x_0) = \Phi^{al}(x)$ (note that $\{x - x_0;\ x \neq x_0 \in V\}$ is a $\mathbb{Z}$-basis of $\mathrm{Div}^0(X)$). On the other hand, an easy computation leads us to

$$\partial\partial^*\Big(\sum_{x\in V} a_x x\Big) = -\sum_{x\in V} a_x \sum_{e\in E_x} \big(t(e) - o(e)\big),$$

and hence

$$\varphi\Big(\partial\partial^*\Big(\sum_{x\in V} a_x x\Big)\Big) = -\sum_{x\in V} a_x \sum_{e\in E_x} \big(\Phi^{al}(t(e)) - \Phi^{al}(o(e))\big) = 0.$$

which implies that φ induces a homomorphism $\varphi : \mathrm{Pic}(X) \longrightarrow A(X)$. From what we have seen above, φ is surjective.

To check that φ is an isomorphism, it is enough to see that $|A(X)| = \kappa(X)$. For this, we take a look at the following exact sequence

$$0 \to A(X) \to \mathbb{A}(X) \to H_1(X.\mathbb{R})/H_1(X,\mathbb{Z})^\# \to 0.$$

We therefore have the following identity for the order of $A(X)$.

$$|A(X)| = \mathrm{vol}(\mathbb{A}(X))/\mathrm{vol}(H_1(X.\mathbb{R})/H_1(X,\mathbb{Z})^\#).$$

We also have

$$\mathrm{vol}(H_1(X.\mathbb{R})/H_1(X,\mathbb{Z})^\#) = \mathrm{vol}(\mathbb{A}(X))^{-1},$$

and hence we obtain

$$|A(X)| = \mathrm{vol}(\mathbb{A}(X))^2.$$

It is known ([5]) that $\mathrm{vol}(\mathbb{A}(X))^2$ coincides with $\kappa(X)$, and hence $|A(X)| = \kappa(X)$.

A non-degenerate symmetric bilinear form on $A(X)$ with values in $\mathbb{Q}/\mathbb{Z}$ is induced from the inner product on $H_1(X,\mathbb{R})$. Thinking of this form as an analogue of "polarization", one may ask whether the Torelli type theorem holds in the discrete realm. More specifically, one asks whether two regular graphs X_1 and X_2 with the same degree are isomorphic when there exists a group isomorphism between $A(X_1)$ and $A(X_2)$ preserving polarizations[1].

References

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[1]This problem was suggested by Kenichi Yoshikawa. If we would remove the conditions on degree and polarizations, there are many examples of X_1, X_2 with $A(X_1) \cong A(X_2)$ ([1]).

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