RAY-SINGER ZETA FUNCTIONS FOR COMPACT FLAT MANIFOLDS

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1. Introduction

Let M be a compact oriented Riemannian manifold, and let $\rho : \pi_1(M) \longrightarrow U(N)$ be a representation of the fundamental group of M by unitary matrices. We denote by E_ρ the flat vector bundle associated with ρ, and by Δ_p^ρ the Laplacian acting on E_ρ-valued p-forms on M. The *Ray-Singer zeta function* is defined by

$$Z_\rho(s) = \sum_{p=0}^{n} (-1)^p p \zeta_p(s),$$

where

$$\zeta_p(s) = \Gamma(s)^{-1} \int_0^\infty t^{s-1} \big[\mathrm{tr}(e^{-t\Delta_p^\rho}) - \dim \operatorname{Ker} \Delta_p^\rho\big] dt.$$

We shall show that $Z_\rho(s)$ for a compact flat manifold is expressed in terms of the *Hurwits zeta function*

$$\zeta(s,\theta) = \sum_{n=0}^{\infty} (n+\theta)^{-s} \qquad (0 < \theta \leq 1).$$

The number θ appearing in the expression turns out to be closely connected with a certain class of closed geodesics. We may in particular evaluate the value $\frac{1}{2} Z'_\rho(0)$, which equals the logarithm of the *Reidemeister-Franz torsion* (W. Müller [3] and J. Cheeger [1]). A *trace formula* applied to flat manifolds plays a crucial role in our discussion (see [7], [10],[11],[12]).

2. Trace formulae

A compact orientable flat manifold M is expressed as $M = \mathbb{R}^n/\Gamma$ with a torsion free discrete subgroup of the group of orientation preserving motions of $\mathbb{R}^d$. There is a natural one-to-one correspondence between the set of conjugacy classes $[\gamma]$, $\gamma \in \Gamma$, and the set of free homotopy classes of maps of S^1 into M. We denote by $M_{[\gamma]}$ the set of closed geodesics $c : S^1 \longrightarrow M$ belonging to the homotopy class $[\gamma]$. The space $M_{[\gamma]}$ equipped with compact open topology is a compact connected manifold, and the map $M_{[\gamma]} \longrightarrow M$ defined by $c \mapsto c(0)$ is an immersion which induces a flat metric on $M_{[\gamma]}$ (see [8], [9]). The fundamental group of $M_{[\gamma]}$ is isomorphic to the centralizer Γ_γ of γ. We set ℓ_γ = length of $c \in M_{[\gamma]}$, which depends only on the class $[\gamma]$.

The following proposition is a straightforward generalization of the trace formula established in [7].

Proposition 2.1. *Let Δ_E be the Laplacian acting on sections of the flat vector bundle E associated with a representation $\rho : \Gamma \longrightarrow U(N)$. Then*

$$\text{tr}(e^{-t\Delta_E}) = \sum_{[\gamma]\in[\Gamma]} \text{tr}\ \rho(\gamma)\text{vol}(M_{[\gamma]})\alpha(\gamma)(4\pi t)^{-\dim\ M_{[\gamma]}/2}\exp(-\ell^2_{[\gamma]}/4t), \tag{1}$$

where $\alpha(\gamma) = \big|\det\big(A(\gamma) - I|\text{Im}(A(\gamma)-I)\big)\big|^{-1}$, $A(\gamma)$ *being the rotation part of the motion* γ.

As an illustration, consider the case $M = S^1 = \mathbb{R}/\mathbb{Z}$ and $\rho(n) = \exp 2\pi\sqrt{-1}\alpha n$. The eigenvalues of Δ_E in this case are $4\pi^2(n+\alpha)^2, n \in \mathbb{Z}$. Therefore the trace formula reduces to the classical summation formula

$$\sum_{n\in\mathbb{Z}}\exp\big(-4\pi^2(n+\alpha)^2 t\big) = (4\pi t)^{-1/2}\sum_{n\in\mathbb{Z}}\exp\big(2\pi\sqrt{-1}n\alpha - n^2/4t\big), \tag{2}$$

which is useful in later discussion.

Let $A : \Gamma \longrightarrow O(n)$ is the representation defined by taking the rotation part of each element in Γ. The p-th exterior product of the cotangent bundle, $\wedge^p T^*M$, is the flat bundle associated with the representation

$$\wedge^p A : \Gamma \longrightarrow O(\wedge^p\mathbb{R}^n),$$

so that the tensor product $E_\rho \otimes \wedge^p T^*M$ is the flat bundle associated with the representation $\rho\otimes\wedge^p A$. Applying the proposition above, we get

$$\begin{aligned}&\text{tr}(e^{-t\Delta^\rho_p})\\ = &\sum_{[\gamma]\in[\Gamma]} \text{tr}\ \rho(\gamma)\cdot\text{tr}\big(\wedge^p A(\gamma)\big)\text{vol}(M_{[\gamma]})\alpha(\gamma)(4\pi t)^{-\dim\ M_{[\gamma]}/2}\exp(-\ell^2_{[\gamma]}/4t).\end{aligned}$$

For brevity, we set

$$\begin{aligned}D_\rho(t) &= \sum_{p=0}^n(-1)^p p\ \text{tr}(e^{-t\Delta^\rho_p})\\ &= \sum_{[\gamma]\in[\Gamma]}\text{tr}\ \rho(\gamma)\cdot\Big(\sum_{p=0}^n(-1)^p p\ \text{tr}\big(\wedge^p A(\gamma)\big)\Big)\text{vol}(M_{[\gamma]})\alpha(\gamma)\\ &\qquad\times(4\pi t)^{-\dim\ M_{[\gamma]}/2}\exp(-\ell^2_{[\gamma]}/4t).\end{aligned}$$

To transform D_ρ still further, we divide the case into two parts.

(i) $n = 2k$. This being the case, the eigenvalues of $A(\gamma)$ are $e^{\pm\sqrt{-1}\theta_1}, \ldots, e^{\pm\sqrt{-1}\theta_k}$. Since $\det(I - xA) = \sum_{p=0}^n(-1)^p x^p\text{tr}(\wedge^p A)$, we have

$$\begin{aligned}\sum_{p=0}^n(-1)^p p\ \text{tr}\big(\wedge^p A(\gamma) &= \frac{d}{dx}\Big|_{x=1}\det\big(I - xA(\gamma)\big)\\ = \frac{d}{dx}\Big|_{x=1}\prod_{i=1}^k(x^2 - 2x\cos\theta_i + 1) &= k2^k\prod_{i=1}^k(1-\cos\theta_i).\end{aligned}$$

(ii) $n = 2k+1$. In this case, the eigenvalues of $A(\gamma)$ are $1, e^{\pm\sqrt{-1}\theta_1}, \ldots, e^{\pm\sqrt{-1}\theta_k}$. In the same way as (i), we find

$$\sum_{p=0}^n(-1)^p p\ \text{tr}\big(\wedge^p A(\gamma)\big) = -2^k\prod_{i=1}^k(1-\cos\theta_i).$$

We should note that, for any $\gamma \in \Gamma$, the kernel of $A(\gamma) - I$ is a non-zero vector subspace, and hence, in the case $n = 2k$, there exists some i with $\theta_i \in 2\pi\mathbb{Z}$. This implies $D_\rho \equiv 0$, and $Z_\rho \equiv 0$.

From now on, we shall confine ourselves to the case $n = 2k + 1$. Note that $\prod_{i=1}^{k}(1-\cos\theta_i) \neq 0$ if and only if $\dim \mathrm{Ker}\big(A(\gamma)-I\big) = 1$, or equivalently $\dim M_{[\gamma]} = 1$. On the other hand, we have

$$\alpha(\gamma) = \prod_{i=1}^{k}(2 - 2\cos\theta_i)^{-1}.$$

Consequently we have

Proposition 2.2.

$$D_\rho(t) = -\sum_{[\gamma]} \mathrm{tr}\ \rho(\gamma)\mathrm{vol}(M_{[\gamma]})(4\pi t)^{-1/2} \exp\big(-\ell_{[\gamma]}^2/4t\big),$$

where $[\gamma]$ *runs over all conjugacy classes with* $\dim M_{[\gamma]} = 1$.

Corollary 2.1. *If* $\dim M_{[\gamma]} \geq 2$ *for every* $[\gamma]$, *then*

$$\sum_{p=0}^{n}(-1)^p p \ \dim\ H^p(M, E_\rho) = 0.$$

3. Primitive geodesics

We will call a homotopy class $[\gamma]$ *isolated* if $\dim M_{[\gamma]} = 1$. Intuitively speaking, this is equivalent to that there is no way to deform $c \in M_{[\gamma]}$ as geodesics except for changing the parameter of c.

Lemma 3.1. *If* $[\gamma]$ *is isolated, then the ratio* $\ell_{[\gamma]}/\mathrm{vol}(M_{[\gamma]})$ *is a positive integer.*

Proof. Define the map $\widetilde{\omega} : S^1 \longrightarrow M_{[\gamma]}$ by $\widetilde{\omega}(s) = c_s$, where $c_s(t) = c(s + t)$. If we equip S^1 with the metric induced from $c : S^1 \longrightarrow M$, then $\widetilde{\omega}$ is a local isometry. Since $\mathrm{vol}(S^1) = \ell_{[\gamma]}$, and $\ell_{[\gamma]}/\mathrm{vol}(M_{[\gamma]})$ equals the degree of the covering map $\widetilde{\omega}$, we are done.

A class $[\gamma]$ is said to be *primitive* if $[\gamma]$ is isolated and $\ell_{[\gamma]}/\mathrm{vol}(M_{[\gamma]}) = 1$. The geometric meaning of this concept is the following: A closed geodesic is said to be *prime* if it is *not* an m-fold cover of another geodesic with $m > 1$. Here we define the m-fold cover c^m of c by $c^m(t) = c(mt)$. A class $[\gamma]$ is primitive if and only if $[\gamma]$ is isolated and a geodesic $c \in M_{[\gamma]}$ is prime. One may also give a group theoretic meaning. A class $[\gamma]$ is isolated if and only if Γ_γ is isomorphic to $\mathbb{Z}$. An isolated $[\gamma]$ is primitive if and only if γ generates Γ_γ.

Lemma 3.2. (1) *If* $[\gamma]$ *is primitive, then so is* $[\gamma^{-1}]$.

(2) *For any isolated class* $[\gamma]$, *there exist a unique primitive class* $[\mu]$ *and a positive integer* m *such that* $[\gamma] = [\mu^m]$.

Since (1) is obvious, we shall prove (2). Let $c \in M_{[\gamma]}$. There exist a unique prime closed geodesic c_1 and $m \geq 1$ with $c = c_1^m$. Suppose $c_1 \in M_{[\nu]}$ (and hence $[\gamma] = [\nu^m]$). The class $[\nu]$ is isolated because $1 = \dim M_{[\gamma]} \geq \dim M_{[\nu]} \geq 1$. Here we have used the fact that the map $M_{[\nu]} \longrightarrow M_{[\nu^m]} = M_{[\gamma]}$ given by $c \mapsto c^m$ is an immersion. Next suppose that there is another primitive class $[\nu']$ with $[(\nu')^{m'}] = [\gamma], m' \geq 1$.

Take $c' \in M_{[\nu']}$. Then one can find some $s \in \mathbb{R}$ with $c'(m't) = c(s+t), t \in \mathbb{R}$, which implies that $m = m'$ and $c'(t) = c_1(t+ms)$, so that $[\nu'] = [\nu]$.

In view of the lemma above, one can find a set of primitive classes $\{[\mu_\alpha]\}_{\alpha\in A}$ such that any isolated class $[\gamma]$ can be written uniquely as $[\gamma] = [\mu_\alpha^m]$ for some $\alpha \in A$ and some $m \in \mathbb{Z}$. Noting that $\mathrm{vol}(M_{[\mu_\alpha^m]}) = \ell_{[\mu_\alpha]}$, we have

$$D_\rho(t) = -(4\pi t)^{-1/2} \sum_{\alpha\in A} \sum_{h\in\mathbb{Z}}{}' \mathrm{tr}\ \rho(\mu_\alpha^h)\ell_{[\mu_\alpha]} \exp\big(-h^2\ell_{[\mu_\alpha]}^2/4t\big), \tag{3}$$

where, in the inner sum $\sum'$, h runs over all integers with isolated $[\mu_\alpha^h]$. From now on, we write ℓ_α for $\ell_{[\mu_\alpha]}$. In order to describe such integers h, we let $\{1, \exp\big(\pm 2\pi\sqrt{-1}b_{\alpha 1}/a_{\alpha 1}\big), \ldots, \exp\big(\pm 2\pi\sqrt{-1}b_{\alpha k}/a_{\alpha k}\big)\}$ be the eigenvalues of $A(\mu_\alpha)$, where $a_{\alpha j}, b_{\alpha j}$ $(j = 1, \ldots, k)$ are positive integers with $(a_{\alpha j}, b_{\alpha j}) = 1$ (co-prime). Since $\dim \mathrm{Ker}\,\big(A(\mu_\alpha) - I\big) = 1$, we have $a_{\alpha i} > 1$. Note that $[\mu_\alpha^h]$ is isolated if and only if $a_{\alpha j}$ is not a divisor of h for any $j = 1, \ldots, k$. Therefore the "Inclusion-Exclusion Principle" leads us to

$$\begin{aligned} &\sum_{h\in\mathbb{Z}}{}' \mathrm{tr}\ \rho(\mu_\alpha^h) \exp\big(-h^2\ell_\alpha^2/4t\big) \\ = &\sum_{h\in\mathbb{Z}} \mathrm{tr}\ \rho(\mu_\alpha^h) \exp\big(-h^2\ell_\alpha^2/4t\big) - \sum_{m=1}^{k}\sum_{h\in\mathbb{Z}} \mathrm{tr}\ \rho(\mu_\alpha^{h a_{\alpha m}}) \exp\big(-h^2 a_{\alpha m}^2 \ell_\alpha^2/4t\big) \\ &+ \sum_{1\le m_1<m_2\le k}\sum_{h\in\mathbb{Z}} \mathrm{tr}\ \rho(\mu_\alpha^{h[a_{\alpha m_1}, a_{\alpha m_2}]} \exp\big(-h^2[a_{\alpha m_1}, a_{\alpha m_2}]^2\ell_\alpha^2/4t\big) - \cdots, \end{aligned} \tag{4}$$

where the symbol $[p.q.r.\cdots]$ means the least common multiple of numbers $p.q.r.\ldots$.

We now let $\{\exp 2\pi\sqrt{-1}\theta_{\alpha 1}, \ldots, \exp 2\pi\sqrt{-1}\theta_{\alpha N}\}$ be the eigenvalues of $\rho(\mu_\alpha)$. Substituting these values for $\mathrm{tr}\ \rho(\mu_\alpha^h)$, we obtain

$$\begin{aligned} D_\rho(t) \ = \ &-\sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k} (4\pi t)^{-1/2}\ell_\alpha \\ &\cdot \sum_{h\in\mathbb{Z}} \exp\big(2\pi\sqrt{-1}\theta_{\alpha j} h[a_{\alpha m_1}, \ldots, a_{\alpha m_u}] - h^2[a_{\alpha m_1}, \ldots, a_{\alpha m_u}]^2\ell_\alpha^2/4t\big). \end{aligned}$$

Here, for $u = 0$, we understand $[a_{\alpha m_1}, \ldots, a_{\alpha m_u}]$ to be 1. This is the stage to use the summation formula (2) to get

$$\begin{aligned} D_\rho(t) \ = \ &-\sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k} [a_{\alpha m_1}, \ldots, a_{\alpha m_u}]^{-1} \\ &\cdot \sum_{h\in\mathbb{Z}} \exp\Big(-4\pi^2 \frac{\big(h - \theta_\alpha[a_{\alpha m_1}, \ldots, a_{\alpha m_u}]\big)^2 t}{[a_{\alpha m_1}, \ldots, a_{\alpha m_u}]^2\ell_\alpha^2}\Big). \end{aligned} \tag{5}$$

Proposition 3.1. *There exist only finite many primitive classes.*

Proof. In (5), we let ρ be the trivial representation $\mathbf{1}$. It should be noted that the series in the right hand side of (5) converges absolutely and each term is dominated by a positive $K_{\alpha j\ell m_1\cdots m_\ell}$, which does not depend on $t > 0$ and satisfies $\sum K_{\alpha j\ell m_1\cdots m_\ell} < \infty$. Therefore, we may first take the limit $(t \uparrow \infty)$ of each term in the series, and find

$$\lim_{t\to\infty} D_{\mathbf{1}}(t) = -\sum_{\alpha\in A}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]^{-1}.$$

On the other hand, we have

$$\lim_{t\to\infty} D_{\mathbf{1}}(t) = \sum_{p=0}^{n}(-1)^p p \dim \operatorname{Ker} \Delta_p = \sum_{p=0}^{n}(-1)^p p \dim H^p(M,\mathbb{R}).$$

We set, for a sequence of positive integers $a_1,\ldots,a_k$ with $a_i>1$,

$$[[a_1,\ldots,a_k]] = \sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k}[a_{m_1},\ldots,a_{m_u}]^{-1}.$$

Since, as shown below,

$$[[a_1,\ldots,a_k]] \ge \prod_{i=1}^{k}\left(1-{a_i}^{-1}\right)\ (\ge 2^{-k}), \tag{6}$$

and

$$\lim_{t\to\infty} D_{\mathbf{1}}(t) = -\sum_{\alpha\in A}[[a_{\alpha 1},\ldots,a_{\alpha k}]], \quad (a_{\alpha j}>1),$$

the set A is necessarily finite.

We shall prove (6) by induction on k. In the same time, we prove

$$[[\theta_1 a_1,\ldots,\theta_k a_k]] \ge [[a_1,\ldots,a_k]] \tag{7}$$

for positive integers $\theta_1,\ldots,\theta_k$. For $k=1$, we have $[[a_1]] = 1-{a_1}^{-1}$ and $[[\theta_1 a_1]] = 1-(\theta_1 a_1)^{-1} \ge 1-{a_1}^{-1} = [[a_1]]$. Suppose that our claim holds up to $k-1$. An easy computation leads us to

$$[[a_1,\ldots,a_k]] = [[a_2,\ldots,a_k]] - \frac{1}{a_1}\Big[\Big[\frac{[a_1,a_2]}{a_1},\ldots,\frac{[a_1,a_k]}{a_1}\Big]\Big]. \tag{8}$$

Hence, noting $a_i = \dfrac{[a_1,a_i]}{a_1}(a_1,a_i)$, and using the induction hypothesis, we get

$$[[a_1,\ldots,a_k]] \ge (1-{a_1}^{-1})[[a_2,\ldots,a_k]] \ge \prod_{i=1}^{k}(1-{a_i}^{-1}).$$

We shall show (7). For this, it is enough to check $[[\theta a_1,a_2,\ldots,a_k]] \ge [[a_1,\ldots,a_k]]$. Using again (8), we obtain

$$\begin{aligned}[[\theta a_1,a_2,\ldots,a_k]] - [[a_1,\ldots,a_k]] &= \frac{1}{a_1}\Big\{\Big[\Big[\frac{[a_1,a_2]}{a_1},\ldots,\frac{[a_1,a_k]}{a_1}\Big]\Big] \\ &\quad -\frac{1}{\theta}\Big[\Big[\frac{[\theta a_1,a_2]}{\theta a_1},\ldots,\frac{[\theta a_1,a_k]}{\theta a_1}\Big]\Big]\Big\} \\ &\ge \frac{1}{a_1}(1-\theta^{-1})\Big[\Big[\frac{[a_1,a_2]}{a_1},\ldots,\frac{[a_1,a_k]}{a_1}\Big]\Big] \ge 0,\end{aligned}$$

where we have used that

$$\frac{[a_1,a_i]}{a_1}\Big/\frac{[\theta a_1,a_i]}{\theta a_1} = \frac{(\theta a_1,a_i)}{(a_1,a_i)}$$

is a positive integer.

Corollary 3.1.

$$\begin{aligned}&\sum_{p=0}^{n}(-1)^p p \dim H^p(M,E_\rho)\\ =& \sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k}[a_{\alpha m_1},\dots,a_{\alpha m_u}]^{-1}\langle\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_u}]\rangle,\end{aligned}$$

where the symbol $\langle\cdot\rangle$ *denotes the characteristic function of* $\mathbb{Z}$ *in* $\mathbb{R}$*; that is,* $\langle a\rangle = 1$ *if* $a\in\mathbb{Z}$*, otherwise* $\langle a\rangle = 0$*. In particular*

$$\sum_{p=0}^{n}(-1)^p p \dim H^p(M,\mathbb{R}) = -\sum_{\alpha\in A}[[a_{\alpha 1},\dots,a_{\alpha k}]] \le 0.$$

4. Zeta functions and torsions

We are now in a position to calculate the Ray-Singer zeta function

$$Z_\rho(s) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\big(D_\rho(t) - \lim_{t\to\infty} D_\rho(t)\big)dt.$$

In virtue of the identity

$$\Gamma(s)^{-1}\int_0^\infty t^{s-1}\exp\Big(-4\pi^2\frac{(h-\alpha)^2 t}{a^2}\Big)dt = \Big(\frac{a^2}{4\pi^2}\Big)^s (h-\alpha)^{-2s} \quad (h\ne\alpha),$$

we obtain

$$\begin{aligned}Z_\rho(s) =& -\sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{\ell=0}^{k}(-1)^\ell \sum_{1\le m_1<\cdots<m_\ell\le k}[a_{\alpha m_1},\dots,a_{\alpha m_\ell}]^{2s-1}\big(\ell_\alpha/2\pi\big)^{2s}\\ &\times\sum_{h\in\mathbb{Z}}{}'\big(h-\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_\ell}]\big)^{-2s},\end{aligned}$$

where h runs over integers with $h\ne\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_\ell}]$. Therefore in terms of the Hurwitz zeta function $\zeta(s,\theta)$, we have

Theorem 4.1.

$$\begin{aligned}Z_\rho(s) =& -\sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k}[a_{\alpha m_1},\dots,a_{\alpha m_u}]^{2s-1}\big(\ell_\alpha/2\pi\big)^{2s}\\ &\times\Big\{\langle\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_u}]\rangle 2\zeta(2s)\\ &+(1-\langle\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_u}]\rangle)\times\Big(\zeta(2s,\langle\langle\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_u}]\rangle\rangle)\\ &+\zeta(2s,1-\langle\langle\theta_{\alpha j}[a_{\alpha m_1},\dots,a_{\alpha m_u}]\rangle\rangle)\Big)\Big\}\end{aligned}$$

where $\langle\langle\alpha\rangle\rangle = \alpha - [\alpha]$.

The expression of $Z_\rho(s)$ above allows us to compute the Reidemeister-Franz torsion. Since

$$\zeta(s,\theta) = \Big(\frac{1}{2}-\theta\Big) + \log\Big(\frac{\Gamma(\theta)}{\sqrt{2\pi}}\Big)s + \cdots$$

around $s=0$, we have

$$\begin{aligned} Z'_\rho(0) &= -2\sum_{\alpha\in A}\sum_{j=1}^{N}\sum_{u=0}^{k}(-1)^u \sum_{1\le m_1<\cdots<m_u\le k}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]^{-1} \\ & \times\Big\{-\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\log\Big([a_{\alpha m_1},\ldots,a_{\alpha m_u}]\frac{\ell_\alpha}{2\pi}\Big) \\ & +\big(1-\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\big)\log\Big(\frac{1}{2\pi}\Gamma\big(\langle\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\rangle\big) \\ & \times\Gamma\big(1-\langle\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\rangle\big)\Big)\Big\}, \end{aligned}$$

which, in view of the identity $\Gamma(x)\Gamma(1-x)=\dfrac{\pi}{\sin\pi x}$, leads us to

Theorem 4.2. *The torsion* $T_\rho(M)=\exp\Big(\dfrac{1}{2}Z'_\rho(0)\Big)$ *equals*

$$\begin{aligned} &\prod_{\alpha\in A}\prod_{j=1}^{N}\prod_{u=0}^{k}\prod_{1\le m_1<\cdots<m_u\le k} \\ &\Big([a_{\alpha m_1},\ldots,a_{\alpha m_u}]\frac{\ell_\alpha}{2\pi}\Big)^{(-1)^u\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle/[a_{\alpha m_1},\ldots,a_{\alpha m_u}]} \\ &\times\Big(2\sin\pi\langle\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\rangle\Big)^{(-1)^u\big(1-\langle\theta_{\alpha j}[a_{\alpha m_1},\ldots,a_{\alpha m_u}]\rangle\big)/[a_{\alpha m_1},\ldots,a_{\alpha m_u}]}. \end{aligned}$$

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