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AN OVERVIEW OF SUNADA'S WORK

ATSUSHI KATSUDA AND POLLY WEE SY

We are proud of publishing this special paper introducing excellent achievements by Prof. Sunada, a MIMS fellow.
(The authors are not MIMS members.)

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1. Brief Profile of Professor Toshikazu Sunada

Professor Toshikazu Sunada was born in Tokyo, Japan, on September 7, 1948, three years after the end of World War II. He dwelled and grew up in the suburb of Tokyo until the age of twenty-five. Sunada described himself in his childhood as an ordinary boy, somewhat introverted and showing no particular interest in any subjects taught in primary and junior high schools. According to his reminiscence, he sat absentmindedly all day long during class hours. He even confessed that arithmetic was then his instinctive dislike.

His zest for mathematics arose when he was a high school student and had a chance to read "History of Modern Mathematics" written by Takagi Teiji, a Japanese luminary who established the class field theory, a culmination of algebraic number theory. The book, including a vivid description of the lives of Gauss, Abel, and Galois together with the development of the theory of elliptic functions, was so fascinating that it led him to the ambition of becoming a mathematician. Since he had thought of himself as a literature-oriented person at that time, this was a major turning point in his life.

He thus decided to study mathematics and entered Tokyo Institute of Technology (TIT), which had a department of mathematics of moderate size. However, soon after his admission to the university (1968), his study was disrupted by student riot, a movement sweeping over universities around the world. During this period, classes were cancelled and the campus was locked out. Interested students of the mathematics department voluntarily requested their teachers to organize seminars outside the campus. The subjects they took up then were vector bundles and complex multiplications; topics which were not covered in the regular lectures for undergraduate courses. The enthusiasm for mathematics that the teachers demonstrated as well as their selfless effort to impart knowledge even in this extraordinary period has left a lasting impression on the mind of the young Sunada. This experience made him more ambitious to become a professional mathematician. When Sunada was in his senior year, his supervisor was Prof. Koji Shiga, who conducted one of the seminars mentioned above and is now Sunada's lifelong friend.

After his undergraduate studies, he was admitted to the graduate school of the University of Tokyo (UT) and soon began his research under the supervision of Prof. Mikio Ise. The most decisive moment for his future career came when he defended his master's thesis which consists of three different subjects in front of an examination committee. Since the time allocated for presentation to each student was short, he had to choose one out of the three subjects. Prof. Kunihiko Kodaira, a Fields Medal laureate, asked Sunada to explain in detail the other two subjects as well, even though his time was already up. Moreover, Kodaira made valuable comments to each subject. This happening was a big boost to Sunada's confidence in pursuing his dream.

Just after receiving his master's degree from UT in 1974, he was appointed as a research associate at Nagoya University (NU) where he was to stay for the next 15 years. This stay has made his dream to become a mathematician comes true. In 1977, he received his doctorate degree by submitting a dissertation to UT.

In 1979-80, he was invited as a guest researcher at Bonn University. He says that the two-years stay in Bonn was the most fruitful time in his life. During this period, he made the aqcuaintance of many active young mathematicians, and published a series of excellent papers ([S7], [S8], [S9], [S10]). And it was also during this period that his geometric model of number theory was conceived (see Subsection 2.4).

After his return to NU, he was promoted to associate professor in 1982. In 1985, he gave a beautiful construction of isospectral manifolds based on his geometric model of number theory. For this important contribution, he was subsequently awarded the Iyanaga Prize by the Mathematical Society of Japan in 1987.

Sunada became a full professor at NU in 1988. Three years after, he was appointed professor at UT (1991-1993) and thereafter, at Tohoku University (TU, 1993-2003) before he has finally settled down at Meiji University in 2003. Currently he is also professor emeritus of TU, a position held since 2003, and is affiliated with the newly-established Meiji Institute for Advanced Study of Mathematical Sciences in Tokyo. It is a rare case in Japan that a full professor transfers frequently from one university to another since there is almost no difference in the financial status. The motivation for his frequent movement was to seek better research environment. He frankly says "UT, one of the most prestigious universities in Japan, was worst in my experience as far as the human relation is concerned".

In the meanwhile, Sunada stayed for six months (1988) in Institut Hautes Études Scientifiques (IHES) as a guest professor, for a few months in Isaac Newton Institute at Cambridge as an organizer of a special project (2007), and for seven months in Max Planck Institute in Bonn (2008) as a visiting professor. In 2008, he held an Andrejewski Lecturership at Humboldt University in Berlin under the auspices of the Walter and Eva Andrejewski Foundation as a distinguished scholar. He also stayed in Mathematical Sciences Research Institute (MSRI) in Berkeley, Johns Hopkins University, Augsburg University, Institut Henri Poincaré (IHP), Tata Institute of Fundamental Research, Institut Mittag-Leffler, the Academy of Science in Beijing, National University of Singapore, and the University of the Philippines for short periods. His first stay in the Philippines (1986), which was the most exciting moment in all of his travels (where he has witnessed the peaceful People Power Revolution in Manila), was the beginning of his active involvement in the Southeast Asia regional mathematical activities.

Sunada gave an invited lecture at the International Congress of Mathematicians (ICM) in Kyoto in 1990, at the Third Asian Mathematical Conference (AMC) in Manila in 2000, and at the LMS South West and South Wales Regional Meeting in Cardiff, UK in 2007, to name a few. He was invited to numerous other international conferences and symposia as a keynote speaker.

His activities are not limited to teaching and research. He was chosen a member of the Kyoto Prize Selection Committee for three terms (1989, 1994, 2002) in the past 20 years. In 2008, he was appointed a panel member of the European Research Council, an organization set up to promote outstanding, frontier research in all areas of science and humanities throughout Europe. His other services to the mathematics community include his two-term board membership of the Mathematical Society of Japan and the membership of the IMU-CDE committee where he served for two consecutive terms. Moreover, he helped in the organization of several major conferences, including the celebrated Taniguchi Symposia, held in Asia as a member of steering, scientific or advisory committee.

Besides his many research publications, Sunada has written a number of mathematics books for the general public as well as textbooks for undergraduate and graduate students (most of which are in Japanese) ([SB1] - [SB9]), and enlightening essays which appeared in *Sugaku Seminar* (Mathematics Seminar) and other mathematical magazines. He has also been involved in the publication of several series of mathematical books, journals, and proceedings as an editor. Sunada is at present a member of the Editorial Board of a popular Japanese mathematical magazine, *Have Fun with Mathematics*, published by Kame-Shobo.

Although Sunada usually portrays himself as a geometer, we realize from his list of publications, that it is difficult to single out his specialization. In fact, Sunada's work covers complex analytic geometry, spectral geometry, dynamical systems, probability, and graph theory, some of which we shall explain in detail in the next section. Through his work, we would describe him as an extraordinary and talented man with enormous insight and technical power who is constantly generating new ideas and methods to form exciting and remarkable mathematical results.

2. **Sunada's work**

This section gives an overview of Sunada's achievement up to the age of 60. Because of the wide range of subjects Sunada has been involved in and his research's style of moving back and forth among subjects, this narration of Sunada's work shall be done in accordance with the subject matters and not in chronological order.

The organization of the subsections is as follows:

2.1. Complex analysis
2.2. Trace formulae
2.3. Density of states
2.4. Isospectral manifolds

2.1. **Complex analysis.** Under the guidance of his supervisor Professor Mikio Ise, Sunada succeeded in completing a master's thesis consisting of three different topics; namely, "Holomorphic equivalence problem of bounded Reinhaldt domains", "Implicit function theorem for non-linear elliptic operators", and "Random walks on a Riemannian manifold". The first one [S4] published in Mathematische Annalen in 1978 is considered his debut paper to the mathematical community. As he recalled in recent times, this work has given him self-confidence of his ability as a mathematician. In this paper, he generalized Thullen's classical result [50] asserting that a 2-dimensional bounded *Reinhaldt domain* [1] containing the origin is biholomorphic to one of the following domains provided that the orbit of the origin by the automophism group has positive dimension:

(1) $\{(z,w)\in \mathbf{C}^2;\ |z|<1,\ |w|<1\}$ (polydisc);
(2) $\{(z,w)\in \mathbf{C}^2;\ |z|^2+|w|^2<1\}$ (unit ball);
(3) $\{(z,w)\in \mathbf{C}^2;\ |z|^2+|w|^{2/p}<1\}$ $(p>0,\neq 1)$ (Thullen domain).

Here, the orbit for the Thullen domain is $\{(z,0);\ |z|<1\}$. An interesting aspect of the classification above is that even in the non-homogeneous case, the shape of the domain is explicitly described.

The main theorems in [S4] are stated below.

Theorem 2.1. *An n-dimensional bounded Reinhaldt domain D containing the origin is biholomorphic to the Reinhaldt domain $\tilde{D}$ in $\mathbf{C}^{n_1}\times\cdots\times\mathbf{C}^{n_s}\times\mathbf{C}^{m_1}\times\cdots\times\mathbf{C}^{m_t}$ which has the following characteristics:*

(1) *Let $\tilde{D}_0$ be the orbit of the origin for the action of the identity component of the automorphism group of $\tilde{D}$. Then $\tilde{D}_0=\{(\mathbf{z}_1,\dots,\mathbf{z}_s,\mathbf{w}_1,\dots,\mathbf{w}_t);\ \mathbf{z}_i\in\mathbf{C}^{n_i},\ \mathbf{w}_j\in\mathbf{C}^{m_j},\ |\mathbf{z}_1|<1,\dots,|\mathbf{z}_s|<1,\mathbf{w}_1=\cdots=\mathbf{w}_t=\mathbf{0}\}$.*

(2) *$\tilde{D}_1=\{(\mathbf{w}_1,\dots,\mathbf{w}_t);\ (\mathbf{0},\dots,\mathbf{0},\mathbf{w}_1,\dots,\mathbf{w}_t)\in\tilde{D}\}$ is a bounded Reinhaldt domain.*

(3) *$\tilde{D}$ is described in terms of $\tilde{D}_0$ and $\tilde{D}_1$ as*

$$\tilde{D} = \Big\{(\mathbf{z}_1,\dots,\mathbf{z}_s,\mathbf{w}_1,\dots,\mathbf{w}_t);\ (\mathbf{z}_1,\dots,\mathbf{z}_s)\in\tilde{D}_0,$$
$$\Big(\mathbf{w}_1\prod_{i=1}^{s}\big(1-|\mathbf{z}_i|^2\big)^{-p_{1i}/2},\dots,\mathbf{w}_t\prod_{i=1}^{s}\big(1-|\mathbf{z}_i|^2\big)^{-p_{ti}/2}\Big)\in\tilde{D}_1\Big\},$$

where p_{ij} are non-negative real numbers.

Theorem 2.2. *Two n-dimensional bounded Reinhardt domains D_1 and D_2 are mutually equivalent if and only if there exists a transformation $\varphi:\mathbf{C}^n\longrightarrow\mathbf{C}^n$ given by $z_i\mapsto r_i z_{\sigma(i)}$ ($r_i>0$ and σ being a permutation of the indices) such that $\varphi(D_1)=D_2$.*

Sunada's idea in the proofs is to employ the torus action $(z_1,\dots,z_n)\mapsto(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)$ on a bounded Reinhaldt domain to obtain an analogue of the Cartan

[1] A Reinhaldt domain D is an open set in $\mathbf{C}^n$ invariant under the transformation $(z_1,\dots,z_n)\mapsto(e^{i\theta_1}z_1,\dots,e^{i\theta_n}z_n)$ $(\theta_i\in\mathbf{R})$.

decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of the automorphism group and a root decomposition of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$, which allows him to determine the structure of $\mathfrak{g}$. Note that $\mathfrak{g}_{\mathbf{C}}$ is identified with a Lie algebra consisting of polynomial vector fields on $\mathbf{C}^n$. A key fact is that a system ξ of vector fields on a real subspace of $\mathbf{C}^n$ which consists of the restriction of vector fields in $\mathfrak{p}$ is integrable on an open dense subset. Using this fact, he could characterize D by solving the system of differential equations derived from ξ. Later on, his idea has been generalized by S. Shimizu, A. Kodama, D. E. Barret, T. Barton, N. G. Kruzhilin, J. P. Vigué and others for various classes of complex domains.

When Sunada was a research associate of NU (1974-1975) and UT (1975-1977), he focused his study on the family of holomorphic maps of a Kähler manifold into a *compact quotient of a symmetric bounded domain* ([S2], [S6]), which formed part of his doctoral dissertation submitted to UT. At that time, he had been looking for a suitable example related to the result in his MS thesis on non-linear elliptic equations.

He came across a general result due to A. Douady [20] which says that the set of holomorphic maps $\mathrm{Hol}(X, Y)$ of a compact complex manifold X into another complex manifold Y carries the structure of a complex space, and the evaluation map $\varphi_x : \mathrm{Hol}(X, Y) \longrightarrow Y$ defined by $\varphi_x(f) = f(x)$ is an analytic map. What Sunada conceived was that $\mathrm{Hol}(X, Y)$ should inherit some structures from the target manifold Y, if not always for every Y. For example, if Y is compact and *Kobayashi hyperbolic*, then so is each component of $\mathrm{Hol}(X, Y)$[2]. His question was: what about the case that Y is a compact quotient of a symmetric domain ? His answer is adequately provided in the following theorem.

Theorem 2.3. *Let X be a compact Kähler manifold, and Y be a compact quotient of a symmetric bounded domain $\Gamma\backslash\mathcal{D}$. We assume that Γ is torsion free. Then we have*

(1) *Each connected component of* $\mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$ *is a compact quotient of a symmetric bounded domain;*

(2) *Let $\Gamma_1\backslash\mathcal{D}_1$ be a connected component of* $\mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$*. Then the evaluation map φ_x is a totally geodesic immersion, and hence $\mathcal{D}_1$ is identified with a Hermitian symmetric subspace of $\mathcal{D}$ (in other words, the lifting of φ_x is a holomorphic embedding in the sense of Kuga-Satake* [41]*);*

(3) *If $f \in \Gamma_1\backslash\mathcal{D}_1$, then the group Γ_1 is identified with the centralizer of the image of the induced homomorphism $f_* : \pi_1(X) \longrightarrow \Gamma$;*

(4) *If $f, g \in$ $\mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$ are homotopic, then f and g are contained in the same component of* $\mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$.

A real analogue was also established for the space $\mathrm{Harm}(M, N)$ of *harmonic maps* of a compact Riemannian manifold M into a *compact quotient N of a symmetric space of non-positive curvature* ([S6]). This being the case, each component of $\mathrm{Harm}(M, N)$ is also a compact quotient of a symmetric space of non-positive curvature, and the evaluation map is totally geodesic. In his proof, he made use of the classical result due to Eells and Sampson [21] and the fact that the energy

[2]This is easy to see if one uses Brody's criterion on Kobayashi hyperbolicity "A compact complex manifold Y is Kobayashi hyperbolic if and only if there is no non-constant holomorphic map of $\mathbf{C}$ into Y".

functional on the mapping space $\mathrm{Map}(M, N)$ is convex (see also R. Schoen and S. T. Yau [43]).

Claims (3) and (4) in the theorem above tells, in particular, that, if the image of $f \in \mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$ is "topologically big", then f is rigid in the sense that, if $g \in \mathrm{Hol}(X, \Gamma\backslash\mathcal{D})$ is homotopic to f, then $g = f$.

In connection with this rigidity property, it is worthwhile to quote the following result [S2] which is a generalization of de Franchis' theorem on finiteness of non-constant holomorphic maps of an algebraic curve into an algebraic curve with genus greater than 1: *Let $\ell(\mathcal{D})$ be the maximum value of dimension of proper boundary components of $\mathcal{D}$* ([52]). *Then there are only a finite number of holomorphic mappings of M into $\Gamma\backslash\mathcal{D}$, each of rank greater than $\ell(\mathcal{D})$. In particular, the set of surjective holomorphic mappings is finite.*

The de Franchis theorem has several generalizations. For instance, Kobayashi and Ochiai [29] proved that there are only a finite number of dominant meromorphic maps onto a complex space of general type. Moreover, in his collaborative work [S11] with Junjiro Noguchi, a good friend of Sunada since their undergraduate years in TIT, they established the following theorem, a more algebraic-geometric result.

Theorem 2.4. *Let M be an algebraic variety and N be a smooth complete algebraic variety. We denote by $\mathrm{Rat}_\mu(M, N)$ the family of rational maps $f : M \longrightarrow N$ with* rank $f \geq \mu$. *If the μ-th exterior power $\wedge^\mu TN$ is negative, then $\mathrm{Rat}_\mu(M, N)$ is finite.*

We note that $\wedge^\mu T(\Gamma\backslash\mathcal{D})$ is negative if $\mu > \ell(\mathcal{D})$. Hence this theorem is a generalization of the result above.

After these work on complex analysis, Sunada's interest has shifted to geometric analysis, especially spectral geometry.

2.2. **Trace formulae.** Among Sunada's scientific papers, there are four papers whose titles include the term "trace formula" ([S7], [S10], [S12], [S33]). Two other papers [S5] and [S17] are also closely related to trace formulae. His intention in these papers was to use the trace formulae for the spectral study of Laplacians on general Riemannian manifolds.

Needless to say, the "trace formula philosophy" has its origin in the famous work by A. Selberg [42] who established a non-commutative version of the Poisson summation formula

$$\sum_{m\in\mathbf{Z}} f(m) = \sum_{n\in\mathbf{Z}} \widehat{f}(n), \tag{2.1}$$

where f is a rapidly decreasing function, and $\widehat{f}(\xi) = \displaystyle\int_{-\infty}^{\infty} f(x)e^{-2\pi\sqrt{-1}x\xi}dx$. The Selberg trace formula, which works effectively for a closed surface of constant negative curvature, gives rise to a precise relation between eigenvalues of the Laplacian and closed geodesics. The Poisson summation formula also leads to a relation between eigenvalues and closed geodesics; that is, eigenvalues of $-d^2/dx^2$ and closed geodesics $x \mapsto kx$ ($k \in \mathbf{Z}$) on $S^1 = \mathbf{R}/\mathbf{Z}$. This relation is more clearly understood if we rewrite (2.1) as

$$\sum_{k=0}^{\infty} \widehat{f}^c(\sqrt{\lambda_k}) = \sum_{k\in\mathbf{Z}} f(k), \tag{2.2}$$

where

$$\widehat{f}^c(s) = \int_{-\infty}^{\infty} f(t)\cos st\ dt,$$

and $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ are eigenvalues of $-d^2/dx^2$; that is, $\lambda_{2k-1} = \lambda_{2k} = 4\pi^2k^2$ $(k \geq 1)$. In the expression on the right-hand side of (2.2), the whole number $|k|$ $(k \in \mathbf{Z})$ is thought of as representing the length of a closed geodesic in S^1. If one defines the distribution $\Theta \in \mathcal{D}'(\mathbf{R})$ by

$$\langle \Theta, f\rangle = \sum_{k=0}^{\infty} \widehat{f}^c(\sqrt{\lambda_k}), \tag{2.3}$$

then the formula above is written as

$$\Theta = \sum_{k\in\mathbf{Z}} \delta_k \qquad (\delta_k \text{ is the Dirac delta with support } \{k\}), \tag{2.4}$$

which implies that the singular support of Θ coincides with the set of $\pm$ lengths of closed geodesics in S^1. This view was generalized by Chazarain [13] to a general Riemannian manifold M, who showed that the singular support of the distribution Θ defined in the same way as (2.3) is contained in the set

$$\{\pm\ell;\ \ell \text{ is the length of a closed geodesic in } M\}.$$

Sunada's view of the summation formula is a bit different, much akin to the original trace formula, and is explained in the following way. The integers k in (2.4) which are thought of as elements in the fundamental group $\pi_1(S^1) = \mathbf{Z}$ parameterize the connected components of the mapping space $\mathrm{Map}(S^1, S^1)$, and hence the right-hand side of (2.4) may be regarded as the sum of distributions over free homotopy classes of closed paths in S^1. In the case of a general M, the connected components of $\mathrm{Map}(S^1, M)$ are parameterized by conjugacy classes $[\sigma] \in [\pi_1(M)]$. Thus it is natural to expect that Θ is expressed as a sum of certain distributions $\Theta_{[\sigma]}$ closely related to homotopy class $[\sigma]$ of closed paths. He proved that this is actually the case, and presented his result below at the ICM Kyoto in 1990 ([S33]).

Theorem 2.5. (A generalized trace formula) *Let $\rho : \pi_1(M) \longrightarrow U(N)$ be a unitary representation, and $\lambda_0(\rho) \leq \lambda_1(\rho) \leq \cdots$ be the eigenvalues of the Laplacian Δ_ρ acting on sections of the flat vector bundle associated with ρ. Define $\Theta(\rho) \in \mathcal{D}'(\mathbf{R})$ by*

$$\langle \Theta(\rho), f\rangle = \sum_{k=0}^{\infty} \widehat{f}^c(\sqrt{\lambda_k(\rho)}).$$

Then with each conjugacy class $[\sigma] \in [\pi_1(M)]$, a distribution $\Theta_{[\sigma]} \in \mathcal{D}'(\mathbf{R})$ having the following properties is associated.

(1) $\Theta(\rho) = \sum_{[\sigma]} \mathrm{tr}(\sigma)\Theta_{[\sigma]}$*;*

(2) $\mathrm{supp}\ \Theta_{[\sigma]} \subset \{t \in \mathbf{R};\ |t| \geq \ell_{[\sigma]}\}$*, where $\ell_{[\sigma]}$ is the length of the shortest closed geodesics in M whose homotopy class is $[\sigma]$;*

(3) sing. supp $\Theta_{[\sigma]}$ *is contained in the set $\{\pm\ell;\ \ell$ is the length of a closed geodesic in M whose homotopy class is $[\sigma]\}$;*

(4) *Each* $\Theta_{[\sigma]}$ *extends to a generalized function on a space of test functions including the gaussian function*

$$f_\tau(t) = (4\pi\tau)^{-1/2}\exp\Big(-\frac{t^2}{4\tau}\Big) \quad (\tau > 0).$$

Moreover, for such a test function f*, the sum* $\sum_{[\sigma]}\langle\Theta_{[\sigma]}, f\rangle$ *converges absolutely and equals* $\langle\Theta, f\rangle$.

The distribution $\Theta_{[\sigma]}$ is explicitly defined as

$$\langle\Theta_{[\sigma]}, f\rangle = \int_{\Gamma_\sigma\backslash\tilde{M}} dx \int_{\mathbf{R}} f(t)U(t, \sigma x, x)dt, \tag{2.5}$$

where $\Gamma = \pi_1(M)$, Γ_σ is the centralizer of σ, $\pi : \tilde{M} \longrightarrow M$ is the universal covering map over M, and $U(t, x, y)$ is the kernel function of the operator $\cos(t\sqrt{\Delta_{\tilde{M}}})$, which is the fundamental solution of the wave equation on $\tilde{M}$

$$\Big(\frac{\partial^2}{\partial t^2} + \Delta\Big)u = 0,\ u(0, x) = \delta(x),\ u_t(0, x) = 0.$$

Claim (1) is proven in much the same manner as the classical trace formula. Indeed, if we denote by $U_M(t, x, y)$ the kernel function of the operator $\cos t\sqrt{\Delta_\rho}$ on M, then

$$U_M\big(t, \pi(x), \pi(y)\big) = \sum_{\sigma\in\Gamma}\rho(\sigma)U(t, \sigma x, y),$$

from which (1) follows immediately. The second claim (2) is a consequence of the finite propagation property of the wave equation. As for (3), one just follows the way in [13]. A technical difficulty caused by non-compactness of $\Gamma_\sigma\backslash\tilde{M}$ can be overcome again by using the finite propagation property.

If we take the gaussian function f_τ as a test function, then we have, for the trivial ρ,

$$\begin{aligned} \langle\Theta, f_\tau\rangle &= \int_M k_M(\tau, x, x)dx = \sum_{k=0}^{\infty} e^{-\tau\lambda_k}, \\ \langle\Theta_{[\sigma]}, f_\tau\rangle &= \int_{\Gamma_\sigma\backslash\tilde{M}} k_{\tilde{M}}(\tau, \sigma x, x)dx, \end{aligned} \tag{2.6}$$

where k_M and $k_{\tilde{M}}$ are the heat kernel on M and $\tilde{M}$, respectively.

The "path space feature" of trace formulae becomes more apparent if we use the modified *Wiener measure* μ_τ on the path space

$$\mathrm{Map}(S^1, M) = \coprod_{[\sigma]\in[\Gamma]} \mathrm{Map}_{[\sigma]}(S^1, M),$$

which is characterized by

$$\begin{aligned} &\int_{\mathrm{Map}(S^1,M)} f\big(c(t_1), \dots, c(t_N)\big)d\mu_\tau(c) \\ = &\int_{M\times\cdots\times M} k_M(\tau(t_2 - t_1), x_1, x_2) \times \cdots \times k_M(\tau(t_N - t_{N-1}), x_{N-1}, x_N) \\ &\quad \times k_M(\tau(1 + t_1 - t_N), x_N, x_1) f(x_1, \dots, x_N)\ dx_1 \cdots dx_N, \end{aligned}$$

where f is an arbitrary continuous function on the N-tuple product $M\times\cdots\times M$, and $0 \le t_1 < t_2 < \cdots < t_N < 1$ ([S12]). Indeed, one has $\mu_\tau(\mathrm{Map}(S^1, M)) = \langle\Theta, f_\tau\rangle$ and

$\mu_\tau(\mathrm{Map}_{[\sigma]}(S^1, M)) = \langle \Theta_{[\sigma]}, f_\tau\rangle$, and hence the generalized trace formula applied to the gaussian function reduces to the additivity of the Wiener measure.

Also, if one takes the functional F_τ on the path space defined by

$$F_\tau(c) = \exp\Big(-\tau \int_{S^1} q\big(c(s)\big)ds\Big),$$

where q is a smooth function on M, then, in view of the Feynmann-Kac formula, the summation formula

$$\int_{\mathrm{Map}(S^1,M)} F_\tau d\mu_\tau = \sum_{[\sigma]} \int_{\mathrm{Map}_{[\sigma]}(S^1,M)} F_\tau d\mu_\tau \tag{2.7}$$

is equivalent to the trace formula

$$\int_M h(t,x,x)dx = \sum_{[\sigma]} \int_{\Gamma_\sigma\backslash \tilde{M}} \tilde{h}(t,\sigma\tilde{x},\tilde{x})d\tilde{x},$$

where h (resp. $\tilde{h}$) is the fundamental solution of the equation

$$\Big(\frac{\partial}{\partial t} + \Delta + q\Big)u = 0$$

on M (resp. on $\tilde{M}$). In general, one may not expect to have an exact shape of the term $\displaystyle\int_{\mathrm{Map}_{[\sigma]}(S^1,M)} F_\tau\, d\mu_\tau$. However one may give an asymptotic expression instead ([S10]) provided that M is of non-positive curvature. More precisely, if, in addition, the function $\tilde{x} \to d(\tilde{x},\sigma\tilde{x})^2$ is non-degenerate (this is the case for symmetric spaces), then

$$\int_{\mathrm{Map}_{[\sigma]}(S^1,M)} F_\tau\, d\mu_\tau \sim (4\pi\tau)^{-\dim\, M_{[\sigma]}/2} e^{-\ell^2_{[\sigma]}/4\tau}(a_0 + a_1\tau + + a_2\tau^2 + \cdots)$$

as τ goes to zero. Here $M_{[\sigma]}$ denotes the set of closed geodesics in the homotopy class $[\sigma]$ (having the structure of a smooth connected manifold), and $\ell_{[\sigma]}$ is the length of $c \in M_{[\sigma]}$.

When $\tilde{M}$ is a symmetric space of non-positive curvature, so is $M_{[\sigma]}$ which is mapped into M by a totally geodesic immersion, and $\pi_1(M_{[\sigma]}) = \Gamma_\sigma$ as mentioned in the previous subsection. By choosing a suitable fibration map : $\Gamma_\sigma\backslash\tilde{M} \longrightarrow M_{[\sigma]}$, one can decompose the integration in (2.6) into integrations along the fiber and integration over $M_{[\sigma]}$. In a special case (for instance, $\tilde{M}$ is a hyperbolic space, and $q \equiv 0$), one may establish an exact trace formula in this way (see McKean [35]). The following example is due to Sunada [S5].

Example 2.1. *Let $M = \Gamma\backslash \mathbf{R}^n$ be a compact flat manifold. Then*

$$\sum_{i=0}^{\infty} e^{-\lambda_i(\rho)\tau} = \sum_{[\sigma]\in[\Gamma]} \mathrm{tr}\ \rho(\sigma)\alpha(\sigma)\mathrm{vol}(M_{[\sigma]})(4\pi\tau)^{-\dim\, M_{[\sigma]}/2}\exp(-\ell^2_{[\sigma]}/4\tau),$$

where, for $\sigma : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ $(A \in O(n))$, we put

$$\alpha(\sigma) = \big|\det\big(A - I|\mathrm{Image}(A-I)\big)\big|^{-1}.$$

(*Note that each $M_{[\sigma]}$ is also flat. Using this exact formula, one can prove that there are only finitely many isometry classes of flat manifolds with a given spectrum. One can also employ this formula to express the Ray-Singer zeta function in terms of the Hurwits zeta function and to compute the Reidemeister-Franz torsion* [S59].)

Let us go back to the simplest case $M = S^1$. We thus treat the eigenvalues $\{\lambda_k\}$ of *Hill's operator* $-d^2/dx^2 + q(x)$ ($q(x+1) = q(x)$). The following proposition, a special case of (2.7), yields a generalization of the *Jacobi inversion formula.*

Proposition 2.1. ([S7]) *Let μ be the Wiener measure on the loop space $\Omega = \{\omega : [0,1] \longrightarrow \mathbf{R};\ \omega(0) = \omega(1) = 0\}$. Then we have*

$$\sum_{k=0}^{\infty} e^{-\tau\lambda_k} = (4\pi\tau)^{-1/2} \sum_{n=-\infty}^{\infty} e^{-n^2/4\tau} f_n(q;\tau),$$

where

$$f_n(q;\tau) = \int_{S^1} dx \int_{\Omega} \exp\Big(-\tau \int_0^1 q\big(x + nt + \sqrt{\tau}\omega(t)\big)dt\Big) d\mu(\omega).$$

Using this integral expression, one can easily carry out the asymptotic expansion

$$f_n(q;\tau) \sim 1 + A_1^n(q)\tau + A_2^n(q)\tau^2 + \cdots \quad (\tau \downarrow 0),$$

and see that the coefficients $A_i^0(q)$ are expressed as

$$A_i^0(q) = \int_0^1 a_i(q(x), q'(x), \ldots)dx.$$

Here the a_i's are universal polynomials of $q, q', q'', \ldots$, which give us the well-known KdV-invariants (see McKean and Van Moerbeke [36]).

Sunada had interest in the coefficients $A_i^n(q)$ for $n \neq 0$ since they might give new invariants. But he found out, after tedious computations, that each $A_i^n(q)$ can be expressed as a polynomial of the $A_i^0(q)$'s.

It is interesting to point out that $f_n(q;\tau)$ is related to the *discriminant* $\Delta(\lambda)$, which is defined as the trace of the monodromy mapping $\varphi(x) \mapsto \varphi(x+1)$ acting on the solution space of the equation

$$\Big(-\frac{d^2}{dx^2} + q(x) - \lambda\Big)\varphi = 0.$$

Indeed, we have

Proposition 2.2. ([S7])

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda\tau}(4\pi\tau)^{-1/2}e^{-n^2/4\tau} f_n(q;\tau)d\tau \\ = \quad &\frac{-\Delta'(-\lambda)}{\sqrt{\Delta(-\lambda)^2 - 4}}\Big(\frac{\Delta(-\lambda) - \sqrt{\Delta(-\lambda)^2 - 4}}{2}\Big)^{|n|}. \end{aligned}$$

Using this proposition, one can prove that, if $q(x)$ is a *finite band potential*; that is, if the equation $\Delta(\lambda)^2 - 4 = 0$ has only finitely many simple roots, then $(4\pi\tau)^{-1/2} f_0(q;\tau)$ is a *hypergeometric function of Pochhammer's type.*

There is an abstract form of trace formulae which is regarded as a generalization of the *class formula* for a finite group Γ

$$1 = \sum_{[\sigma]\in[\Gamma]} |\Gamma_\sigma|^{-1}.$$

A straightforward generalization is the following proposition, which turns out to be useful in Sunada's work on isospectral manifolds as will be seen in the next subsection.

Proposition 2.3. [S17] *Let V be a Hilbert space on which a finite group Γ acts unitarily, and V^Γ be the subspace of V consisting of Γ-invariant vectors. If a non-positive operator $A : V \longrightarrow V$ of trace class is Γ-equivariant, then*

$$\mathrm{tr}(A|V^\Gamma) = \sum_{[\sigma]\in[\Gamma]} |\Gamma_\sigma|^{-1}\mathrm{tr}(\sigma A). \tag{2.8}$$

Even for a group Γ of infinite order, there are cases that (2.8) may still make sense, at least in a formal sense. The trace formula for the covering set-up is considered as such an example. To explain this, let $\pi : X \longrightarrow M$ be a regular covering map over a compact manifold M with covering transformation group Γ, and let A_0 be an integral operator on M with a lifting A on X. Then

$$\mathrm{tr}\ A_0 = \sum_{[\sigma]\in[\Gamma]} \mathrm{tr}_{\Gamma_\sigma}(\sigma A). \tag{2.9}$$

Here, in general, the Γ-*trace* $\mathrm{tr}_\Gamma(T)$ for a Γ-equivariant integral operator T on a manifold with a Γ-action is defined by $\int_{\mathcal{F}} t(x,x)dx$, where $\mathcal{F}$ is a fundamental domain for the Γ-action. If $\mathcal{F}$ is relatively compact, the Γ-trace is interpreted as a *von Neumann trace* (see [1]). The $\mathrm{tr}_{\Gamma_\sigma}$ in (2.9) needs to be handled carefully since a fundamental domain for Γ_σ-action is not relatively compact in general except for $\sigma = 1$. In many practical cases, (2.9) is justified. For instance, the summation formula $\langle\Theta, f\rangle = \sum_\sigma \langle\Theta_{[\sigma]}, f\rangle$ may be regarded as a disguised form of (2.9) for a special A.

2.3. **Density of states.** While Sunada was studying trace formulae, he became aware that a crude relation between an operator on a covering manifold and its base manifold must be useful for a justification of the notion of "(integrated) density of states", which was first introduced by physicists in the quantum theory of solids. To simplify physicist's explanation, we consider the Schrödinger operator H with a periodic potential on the Euclidean space. Restricting H to a bounded domain and imposing a boundary condition, we count the number of eigenvalues not exceeding λ. Then dividing this counting function by the volume of the domain, and blowing up the domain to fill the whole space, one gets the integrated density of states $\varphi(\lambda)$. The spectrum of the Schrödinger operator on the whole space is then characterized completely by $\varphi(\lambda)$. An interesting feature is that one has the same $\varphi(\lambda)$ whatever one chooses as a boundary condition (Dirichlet, Neumann or periodic boundary condition). The fact behind this feature is that the volume growth of boundaries is much less than that of the domains (see M. Shubin [46]).

A question here is whether the notion of the integrated density of states makes sense for a more general set-up. What Sunada considered is the case of a regular covering manifold X over a compact manifold M, which allows one to define the integrated density of states associated with a periodic boundary condition as follows: Suppose that the covering transformation group Γ has a family of normal subgroups $\{\Gamma_i\}_{i=1}^\infty$ of finite index such that $\Gamma_{i+1} \subset \Gamma_i$ and $\cap_{i=1}^\infty \Gamma_i = \{1\}$. We then have a tower of finite-fold covering maps of closed manifolds $\cdots \longrightarrow M_{i+1} \longrightarrow \Gamma_i\backslash X = M_i \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M$. We take a Γ-invariant function q. Let $\varphi_{M_i}(\lambda)$ denote the number of eigenvalues of $H_{M_i} = \Delta_{M_i} + q$ on the closed manifold M_i not exceeding λ.

Theorem 2.6. ([S33]) *Let*

$$H_X = \int \lambda dE(\lambda)$$

be the spectral resolution of H_X*, and put*

$$\varphi_\Gamma(\lambda) = \mathrm{tr}_\Gamma E(\lambda).$$

Then the measures $(\mathrm{vol}(M_i))^{-1} d\varphi_{M_i}$ *converge weakly to* $(\mathrm{vol}(M))^{-1} d\varphi_\Gamma$*.*

Thus the *spectral distribution function* $\mathrm{tr}_\Gamma E(\lambda)$ is essentially identified with the integrated density of states associated with the periodic boundary condition.

A new question arises. What happens when we replace the periodic boundary condition by the Dirichlet boundary condition ? A partial answer was given in his joint work [S38] with Toshiaki Adachi. To explain this, let $\{D_i\}_{i=1}^\infty$ be a family of bounded connected open sets in X with piecewise smooth boundaries satisfying $\overline{D}_i \subset D_{i+1}$, $\bigcup_{i=1}^\infty D_i = X$. Consider the Schrödinger operator $H_{D_i} = \Delta_{D_i} + q$ on each D_i with the Dirichlet boundary condition. We denote by $\varphi_{D_i}(\lambda)$ the number of eigenvalues of H_{D_i} not exceeding λ.

Theorem 2.7. *Define a "thick" boundary of a domain* D *by*

$$\partial_h D = \{x \in D;\ \mathrm{dist}(x, \partial D) \le h\}.$$

(1) *The group* Γ *is amenable if and only if there exists a family* $\{D_i\}_{i=1}^\infty$ *of bounded domains with piecewise smooth boundary satisfying the following property:*

$$\lim_{i\to\infty} \mathrm{vol}(\partial_h D_i)/\mathrm{vol}(D_i) = 0$$

for every $h > 0$.

(2) *If a family* $\{D_i\}_{i=1}^\infty$ *satisfies the property in* (1)*, then* $(\mathrm{vol}(D_i))^{-1} d\varphi_{D_i}$ *converges weakly to* $(\mathrm{vol}(M))^{-1} d\varphi_\Gamma$*.*

For the definition of *amenability*, refer to R. J. Zimmer [55]. One may see J. Dodziuk and V. Mathai [19] for a recent development.

The notion of Γ-trace which was effectively used in the study of density of states turns out to be also useful in the spectral study of covering manifolds. In his paper [S33], Sunada took up this notion to determine a criterion for a periodic Schrödinger operator on a manifold to have *band structure*. Here band structure means that the spectrum is a union of mutually disjoint, possibly degenerate closed intervals, such that any compact subset of $\mathbf{R}$ meets only finitely many. Subsequently he and Jochen Brüning [S35] generalized the result to the case of periodic elliptic operators.

In order to explain the criterion, we shall employ $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$, the tensor product of the *reduced group* C^**-algebra* of a discrete group Γ with the algebra $\mathcal{K}$ of compact operators on a separable Hilbert space of infinite dimension. The C^*-algebra $C^*_{\mathrm{red}}(\Gamma, \mathcal{K})$ has a canonical von Neumann trace tr_Γ. We then define the *Kadison constant* $C(\Gamma)$ by

$$C(\Gamma) = \inf\{\mathrm{tr}_\Gamma P; P \text{ is a non-zero projection in } C^*_{\mathrm{red}}(\Gamma, \mathcal{K})\}.$$

By definition, Γ is said to have the *Kadison property* if $C(\Gamma) > 0$. Examples of such Γ are abelian groups, free groups and surface groups (see [S32])[3]

We now let X be a Riemannian manifold of dimension n on which a discrete group Γ acts isometrically, effectively, and properly discontinuously. We assume

[3] It is a conjecture proposed by Kadison that, if Γ is torsion free, then $C(\Gamma) = 1$.

that the quotient space $\Gamma\backslash X$ (which may have singularities) is compact. Let E be a Γ-equivariant hermitian vector bundle over X, and $D : C^\infty(E) \longrightarrow C^\infty(E)$ a formally self-adjoint elliptic operator which commutes with the Γ-action (in short, such a D is called a Γ-periodic operator).

Theorem 2.8. [S35] (1) *If Γ has the Kadison property, then the spectrum of any Γ-periodic elliptic operator has band structure.*

(2) *Suppose that D is a Γ-periodic elliptic operator of order p, and is bounded from below. Let $N(\lambda)$ be the number of components of the spectrum of D which intersect the interval $(-\infty, \lambda]$. If Γ has the Kadison property, then*

$$\limsup_{\lambda\to\infty} N(\lambda)\lambda^{-n/p} \leq C(\Gamma)^{-1}\Gamma(1+n/p)\int_{\Gamma\backslash X} A(x)dx,$$

where the function $A(x)$ can be evaluated explicitly in terms of the principal symbol $\sigma D(x,\xi)$ of D as given below.

$$A(x) = (2\pi)^{-n-1}\int_{\mathbf{R}^n} d\xi \int_{-\infty-i\gamma}^{\infty-i\gamma} \mathrm{tr}\big(\sigma D(x,\xi) + i\tau\big)^{-1} e^{i\tau} d\tau$$

(γ *is an arbitrary real number*).

In the definition of band structure, we do not exclude the possible existence of eigenvalues. Indeed, one may construct a closed manifold with a free abelian fundamental group such that the Laplacian on the universal covering manifold possesses an eigenvalue ([S28]). On the other hand, the following theorem was proven in [S30].

Theorem 2.9. *Let X be the maximal abelian covering space over a closed Riemannian manifold M. Suppose that M admits a non-trivial S^1-action whose generating vector field is parallel. Then the Schrödinger operator on X with a smooth periodic potential has no eigenvalues.*

The proof of this generalization of the classical result due to L. E. Thomas [49] was carried out by using the idea of twisted operators (see Section 2.5).

2.4. **Isospectral manifolds.** When Sunada was a guest researcher of the SFB (Sonderforchungsbereich) "Theoretische Mathematik" program in Bonn University (1979-80), he revisited the problem of random walks on a Riemannian manifold, and studied the *spherical mean operator* which is considered as the transition operator of the random walk. He observed that, *if the radius is small enough, then the spherical mean operator is a Fourier integral operator of negative order so that it is a compact operator in the L^2 space* ([S9]). Since then, the problem of random walks, not only on a Riemannian manifold but also on a graph has been his steadfast interest.

Among his papers, the most cited one is "Riemannian coverings and isospectral manifolds" [S17] which appeared in Annals of Mathematics in 1985. The motivation behind this work came up also during his stay in Bonn. At that time, he wanted to understand the class field theory because it has been his dream to prove something related to this theory ever since he read Takagi's book on algebraic number theory. For this sake, he tried to find a geometric model of the class field theory. Thus the path he took is the reverse of what Hilbert had taken up to speculate a correct formulation of his "absolute" class field theory by looking at the theory of covering surfaces. Sunada observed that closed geodesics under covering maps behave like prime ideals under field extensions, and could soon formulate the class field theory in the Riemannian geometric setting[4]. He once said,"It is a simple toy model, but I enjoy playing with it very much."

Let us explain briefly his idea. A *prime cycle* is a 1-cycle represented by a prime closed geodesic. Given a finite-fold Riemannian covering map $\pi : M \longrightarrow M_0$, we say that a prime cycle $\mathfrak{P}$ in M *lies above* a prime cycle $\mathfrak{p}$ in M_0 if $\pi(\mathfrak{P}) = \mathfrak{p}^m (= m\mathfrak{p})$ with a positive integer m, and write $\mathfrak{P}|\mathfrak{p}$. The integer m is called the degree of $\mathfrak{P}$, and written as deg $\mathfrak{P}$. If π is n-fold, then there are at most n prime cycles lying above $\mathfrak{p}$, and

$$n = \sum_{\mathfrak{P}|\mathfrak{p}} \deg \mathfrak{P}.$$

This identity already gives rise to a flavor of algebraic number theory. This flavor becomes much stronger when we consider a regular covering map. Indeed, for a given $\mathfrak{p}$, the covering transformation group G acts transitively on the set $\{\mathfrak{P};\ \mathfrak{P}|\mathfrak{p}\}$, and if deg $\mathfrak{P} = f$, then there exists a unique $\sigma \in G$ such that $c(t + f^{-1}) = \sigma c(t)$, where c is a representative of $\mathfrak{P}$. The element σ depends only upon $\mathfrak{P}$. We write $(\mathfrak{P}|\pi)$ for σ and call it the *Frobenius transformation* associated with $\mathfrak{P}$. It is easily checked that $(\mathfrak{P}|\pi)$ is a generator of the stabilizer $G_{\mathfrak{P}} = \{\mu \in G;\ \mu\mathfrak{P} = \mathfrak{P}\}$ so that deg $\mathfrak{P}$ coincides with the order of $G_{\mathfrak{P}}$. We also find that $(\mu\mathfrak{P}|\pi) = \mu(\mathfrak{P}|\pi)\mu^{-1}$. Thus if π is an abelian covering map, i.e. G is abelian, then $(\mathfrak{P}|\pi)$ depends only upon $\mathfrak{p}$. We write $(\mathfrak{p}|\pi)$ instead of $(\mathfrak{P}|\pi)$.

Taking account of the Dedekind theorem, we define the counterpart of *ideal group* simply as the free abelian group generated by prime cycles, which we denote by I_M. An element $\mathfrak{a}$ in I_M, therefore, takes the form

$$\mathfrak{a} = {\mathfrak{p}_1}^{a_1} \cdots {\mathfrak{p}_k}^{a_k} \quad (a_i \in \mathbf{Z}),$$

which we call a geodesic cycle. A geodesic cycle $\mathfrak{a}$ is said to be *principal* if the homology class $[\mathfrak{a}]$ is zero. We denote by I_M^0 the subgroup of I_M consisting of principal geodesic cycles. An analogue of *ideal class group* is the quotient group $C_M = I_M/I_M^0$. As a matter of fact, C_M is nothing but the 1$^{\text{st}}$ integral homology group $H_1(M, \mathbf{Z})$.

Now for a covering map $\pi : M \longrightarrow M_0$ and for a prime cycle $\mathfrak{P}$ in M with $\mathfrak{P}|\mathfrak{p}$, define the norm $N_\pi(\mathfrak{P})$ to be $\mathfrak{p}^{\deg \mathfrak{P}}$, and extend it to the group homomorphism $I_M \longrightarrow I_{M_0}$. In the geometric context, the fundamental theorem in class field theory is stated as follows.

Proposition 2.4. [S62] (1) *For an n-fold covering map $\pi : M \longrightarrow M_0$, the index $[I_{M_0}, I_{M_0}^0 \cdot N_\pi(I_M)]$ is not greater than n. The equality holds if and only if π is abelian.*

(2) (Artin's reciprocity law) *If π is abelian, then the correspondence $\mathfrak{p} \mapsto (\mathfrak{p}|\pi)$ yields an isomorphism of the quotient group $I_{M_0}/\big(I_{M_0}^0 \cdot N_\pi(I_M)\big)$ onto G.*

(3) *For any subgroup H of finite index in I_{M_0} containing $I_{M_0}^0$, there exists an abelian covering map $\pi : M \longrightarrow M_0$ such that $H = I_{M_0}^0 \cdot N_\pi(I_M)$.*

In number theory, the quickest way to prove a similar inequality as in (1) is to make use of some properties of the L-functions, and the proofs of (2) and (3) are substantially sophisticated. In the geometric case, the proof makes use of elementary features of the Hurewicz homomorphism: $\pi_1(M_0) \longrightarrow H_1(M_0, \mathbf{Z})$.

[4]This was discussed in the unpublished paper [S62].

During his stay in Germany, Sunada happened to attend a conference (Arbeitstagung) held in Bonn, thereby having a chance to listen to a talk by M. F. Vignéras, a French mathematician. In that occasion, she explained her construction of isospectral Riemann surfaces by means of the Selberg trace formula ([51]). Here we quote the story disclosed by Sunada.

"Her method applies only to a very special class of manifolds; i.e., hyperbolic surfaces and spaces. After some time, I came to think that there should be a more general method which could apply to a broader class of manifolds. Actually I observed that Vignéras' construction somehow fits into my geometric model of number theory. In my geometric context, what I would like to look for is a pair of Riemannian manifolds with the same spectral zeta function. Soon I started to seek a similar statement in number theory, and finally found a book which gave me a satisfactory statement. It is the Proceedings edited by Cassel and Fröhlich [12] with exercise problems in the appendix. One of the problems is asking for a pair of number fields with the same Dedekind zeta function. This is it ! The nicest thing is that the answer to this problem is expressed in terms of Galois groups of field extensions. Since I already had a geometric model of number theory, it was almost immediate to write down the desired statement in the geometric setting. Of course, the proof in the geometric case should be different from that in number theory. But it is so convincing that I never have doubt about the validity of my statement. Actually a week later, I could give a proof."

The statement in number theory is the following

Proposition 2.5. *Let K be a finite Galois extension of $\mathbf{Q}$ with Galois group $G = G(K/\mathbf{Q})$, and let k_1 and k_2 be subfields of K corresponding to subgroups H_1 and H_2, respectively. Then the following two conditions are equivalent:*

(1) *Each conjugacy class of elements in G meets H_1 and H_2 in the same number of elements.*

(2) *The Dedekind zeta functions of k_1 and k_2 are the same.*

It is known that many examples of the triplet (G, H_1, H_2) arise from simple algebraic groups. If G is a reductive algebraic group and H_1, H_2 are nonconjugate but associate parabolic subgroups, then (G, H_1, H_2) satisfies condition (1).

In Proposition 2.5, Sunada replaced the Dedekind zeta function by the spectral zeta function

$$\zeta_M(s) = \sum_{k=1}^{\infty} {\lambda_k}^{-s},$$

where $0 < \lambda_1 \le \lambda_2 \le \cdots$ are non-zero eigenvalues of the Laplacian on a compact Riemannian manifold M. He then proved the following by using Proposition 2.3.

Theorem 2.10. *Let $\pi : M \longrightarrow M_0$ be a regular Riemannian covering map with covering transformation group G, and let $\pi_1 : M_1 \longrightarrow M_0$ and $\pi_2 : M_2 \longrightarrow M_0$ be the covering maps corresponding to subgroups H_1 and H_2, respectively. If the triplet (G, H_1, H_2) satisfies condition (1) in the proposition above, then the zeta functions $\zeta_{M_1}(s)$ and $\zeta_{M_2}(s)$ are identical.*

Since $\zeta_{M_1}(s) = \zeta_{M_2}(s)$ if and only if M_1 and M_2 are isospectral, this result gives a chance to construct many non-isometric isospectral manifolds. Actually, by choosing a suitable (G, H_1, H_2) and M_0, we may construct many isospectral pairs, especially isospectral surfaces of constant negative curvature whose genus is much smaller than Vignéras' example (see R. Brooks and R. Tse [7] for an example of genus three).

We should note that Vignéras' example also gave a negative answer to the question posed by I. M. Gel'fand [22], who asked if the induced representation of $PSL_2(\mathbf{R})$ on $L^2(\Gamma\backslash PSL_2(\mathbf{R}))$ determines a discrete subgroup Γ up to conjugation. Sunada's idea also allows a simple construction of a counterexample (see [S18]).

The excitement created by Sunada's beautiful construction of isospectral manifolds is seen in several quotations given by mathematicians. As B. Cipra wrote, in his review paper [14] on the famous problem "Can one hear the shape of a drum ?" proposed by M. Kac, "Sunada's method gives rise to a veritable cottage industry of examples." In the preface of his book [11], Peter Buser wrote "Sunada's construction of isospectral manifolds was fascinating, and I got hooked on constructing examples for quite a while. So time went on and the book kept growing.... The editor, for instance, was interested, and so was my family." Moreover, Robert Brooks, Sunada's late friend, wrote, towards the end of a historical remark on isospectral manifolds in his expository article [8], "The situation changed dramatically in [S17]. Here Sunada showed how the phenomenon of isospectral manifolds could be understood in a systematic way." See also H. Pesce's article [39] which contains an excellent survey of Sunada's construction.

It should be pointed out that there are many examples of isospectral manifolds which are not obtained by Sunada's method (see D. Schueth [44] for example). See R. Brooks [9] for a graph version of the isospectral problem.

As for Kac's problem, C. Gordon, D. Webb and S. Wolpert [23] gave a counterexample (thus one can not hear the shape of a drum in general) by using the transplantation technique developed by Buser [10] and P. Bérard [4] which is closely related to Sunada's method.

Sunada was once (around 1988) asked by his colleague what a geometric analogue of the *Riemann Hypothesis* is. His answer was: $\lambda_1(M) \geq \lambda_0(\tilde{M})$ ([S33])[5]. Here $\lambda_1(M)$ is the first positive eigenvalue of the Laplacian on a compact manifold, and $\lambda_0(\tilde{M})$ is the bottom of the spectrum of the Laplacian on the universal covering manifold $\tilde{M}$. He admits that this was merely a joke. But it turns out that this joke has led him to a serious business to know more about the spectra of covering manifolds.

2.5. **Twisted Laplacians.** It is quite natural for Sunada to proceed to the study of a geometric analogue of the analytic number theory. He wanted to establish an analogue of the *Dirichlet theorem for arithmetic progression* which asserts that, given positive integers a, d which are supposed to be coprime, there are infinitely many primes in the series $a, a+d, a+2d, \ldots$. More precisely, one has

$$|\{\text{prime } p \leq x;\ p = a + kd \text{ for some } k\}| \sim \varphi(d)^{-1}\frac{x}{\log x} \quad (x \uparrow \infty),$$

where $\varphi(d)$ is the Euler function. Remember that the proof relies on the properties of the L-function

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

[5]Actually, for a closed surface, this condition is equivalent to that the Selberg zeta function satisfies the Riemann Hypothesis.

where χ is a character of the multiplicative group $(\mathbf{Z}/d\mathbf{Z})^\times$, which is the Galois group of the cyclotomic field $\mathbf{Q}(e^{2\pi\sqrt{-1}/d})$.

A geometric analogue of the *prime number theorem* has been known as the asymptotic formula for counting function of prime closed geodesics in a closed surface M of constant negative curvature. Let $\pi(x)$ be the number of prime cycles $\mathfrak{p}$ in M with $\ell(\mathfrak{p}) \leq x$. Then

$$\pi(x) \sim \frac{e^x}{x} \quad (x \uparrow \infty).$$

This is proven in a similar way as the usual proof of the prime number theorem by using the Selberg zeta function.

Taking account of our convention to regard $H_1(M, \mathbf{Z})$ as the (absolute) ideal class group, Sunada thinks that a geometric analogue of the Dirichlet theorem should be an asymptotic formula for the counting function of prime cycles in a fixed homology class:

$$\pi(x, \alpha) = \left|\{\mathfrak{p};\ [\mathfrak{p}] = \alpha,\ \ell(\mathfrak{p}) \leq x\}\right|,$$

where $\alpha \in H_1(M, \mathbf{Z})$ and $[\mathfrak{p}]$ stands for the homology class of $\mathfrak{p}$. There is, however, a big discrepancy between the number theory and its geometric model. That is, the ideal class group $H_1(M, \mathbf{Z})$ is of infinite order, and so the conventional proof does not work[6]. But one may still show the following theorem, which was independently proven by Phillips and Sarnak [40].

Theorem 2.11. ([S24]) *Let M be a closed surface of constant negative curvature with genus g. Then*

$$\pi(x, \alpha) \sim (g-1)^g \frac{e^x}{x^{g+1}} \quad (x \uparrow \infty).$$

The key to its proof lies in the following properties of the geometric L-function

$$L(s, \chi) = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p}) e^{-s\ell(\mathfrak{p})})^{-1},$$

where χ is a unitary character of $H_1(M, \mathbf{R})$.

(1) $L(s, \chi)$ is analytically continued to the whole complex plane as a meromorphic function.

(2) If χ is not a trivial character $\mathbf{1}$, then $L(s, \chi)$ is holomorphic in Re $s \geq 1$.

(3) $L(s, \mathbf{1})$ has a simple unique pole at $s = 1$ in Re $s \geq 1$.

(4) $L(s, \chi)$ has no zeros in Re $s \geq 1$.

(5) The poles of $L(s, \chi)$ in Re $s > 0$ are

$$s = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda_i(\chi)} \quad (i = 0, 1, 2, \ldots),$$

where $\lambda_0(\chi) \leq \lambda_1(\chi) \leq \cdots$ are eigenvalues of the Laplacian (*twisted Laplacian*) Δ_χ acting on sections of the flat line bundle associated with χ.

These properties are also derived from those of the Selberg zeta function.

Crucial in the discussion is the fact that $\lambda(\mathbf{1}) = 0$ is simple, and that $\lambda(\chi) = 0$ if and only if $\chi = \mathbf{1}$. Thus $\lambda_0(\chi)$ is simple and depends smoothly on χ whenever χ is in a small neighborhood of $\mathbf{1}$. A perturbation technique allows us to compute the Hessian of $\lambda_0(\chi)$ at $\chi = \mathbf{1}$ as

[6] A straightforward analogue of Tchebotarev's density theorem, a generalization of the Dirichlet theorem to arbitrary finite Galois extensions, can be proven without any difficulty ([S62]).

$$(\mathrm{Hess}\lambda_0)(\omega,\omega) = \frac{8\pi^2}{\mathrm{vol}(M)}\int_M \|\omega\|^2, \tag{2.10}$$

where ω is a harmonic 1-form on M. Put $J(M) = H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$ which is identified with the group of unitary characters and equipped with the flat metric induced from L^2-norm on the space of harmonic 1-forms. Consider

$$F_\alpha(s) = -\int_{J(M)} \chi(-\alpha)\Big(-\frac{d}{ds}\Big)^g \frac{L'(s,\chi)}{L(s,\chi)}d\chi,$$

where $d\chi$ is the normalized Haar measure. On the one hand, we have, in view of orthogonal relations of characters,

$$F_\alpha(s) = \sum_{\substack{k=1\\ k[\mathfrak{p}]=\alpha}} \sum_{\mathfrak{p}} k^g \ell(\mathfrak{p})^{g+1}e^{-k\ell(\mathfrak{p})}.$$

On the other hand, if we put

$$f_0(\chi) = \frac{1}{2}\big(1+\sqrt{1-4\lambda_0(\chi)}\big),$$

then we find

$$F_\alpha(s) = -\int_V \chi(-\alpha)\Big(-\frac{d}{ds}\Big)^g\Big(-\frac{1}{s-f_0(\chi)}\Big)d\chi + h(s), \tag{2.11}$$

where V is a small neighborhood of $\mathbf{1}$, and $h(s)$ is holomorphic in Re $s \geq 1$. Applying the Morse Lemma to the function f_0, and estimating the integral in (2.11), we conclude that there is a locally integrable function $h(t)$ with

$$\Big|F_\alpha(s) - \frac{(g-1)^g}{s-1}\Big| \leq h(t) \quad (s = 1+\epsilon+\sqrt{-1}t)$$

for a small positive ϵ. Finally applying a version of Ikehara's Tauberian theorem, we obtain Theorem 2.11.

Sunada says that the key idea of the proof came up when he was discussing the problem with Atsushi Katsuda inside a bullet train (Shinkansen) on their way to the Taniguchi Symposium held at Katata in 1985. Just as Sunada explained what he had wanted for the function F_α, Katsuda said "It seems OK if the Hessian of $\lambda_0(\chi)$ at $\chi = \mathbf{1}$ is positive definite." This is actually the end of the proof !

After completing the paper [S24], Sunada and Katsuda extended the above result to the case of a manifold with variable negative curvature. They knew that, for this sake, it was indispensable to go over to the theory of dynamical systems. In fact, a prime closed geodesic is identified with a closed (periodic) orbit of the geodesic flow. Therefore it is natural to consider the counting problem of closed orbits of a general dynamical system. The most appropriate set-up is the *dynamical systems of Anosov type*. Indeed, a dynamical analogue of the prime number theorem has been known in such a set-up (W. Parry and M. Pollicott [38]).

The precise set-up is described as follows. Let $\{\varphi_t\}$ be a smooth, transitive, weakly mixing Anosov flow on a compact manifold X. Given a homology class $\alpha \in H_1(X,\mathbf{Z})$, we consider the counting function $\pi(x,\alpha)$ defined in the same way as before where, in this turn, $\ell(\mathfrak{p})$ is the period of a closed orbit $\mathfrak{p}$. We denote by h the topological entropy and by m the (unique) invariant measure on X with maximal entropy. Define the *winding cycle* Φ, a linear functional on $H^1(X, \mathbf{R})$, by

$$\Phi([\omega]) = \int_X \langle Z, \omega \rangle dm,$$

where ω is a closed 1-form, and Z is the vector field generating $\{\varphi_t\}$. We also introduce the covariance form δ, a positive semi-definite quadratic form on $H^1(X, \mathbf{R})$, by setting

$$\delta([\omega], [\omega]) = \lim_{t \to \infty} \int_X dm(x) \Big(\int_0^t \langle Z, \omega \rangle \varphi_\tau(x) d\tau - t\Phi([\omega]) \Big)^2.$$

It is checked that, if Φ vanishes (this is the case for geodesic flows), then δ is positive definite so that it induces a flat metric on $J(X) = H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$. We denote by $\mathrm{vol}\big(J(X)\big)$ the volume with respect to this metric.

The following is the main theorem that Sunada and Katsuda proved after an exchange of a bunch of letters between IHES and Okayama University (e-mail was not yet available at that time).

Theorem 2.12. [S27] *If Φ vanishes, then*

$$\pi(x, \alpha) \sim (2\pi)^{-b/2} \mathrm{vol}\big(J(X)\big)^{-1} h^{-1} \frac{e^{hx}}{x^{(b/2)+1}} \qquad (b = \mathrm{rank}\ H^1(X, \mathbf{Z})).$$

The idea of the proof is almost the same as the case of surfaces of constant negative curvature except for the use of twisted Laplacians. They made use of a twisted version of what is called the *Ruelle operator* instead of twisted Laplacians, which is defined in terms of symbolic dynamics. Refinements and generalizations have been developed by M. Pollicott, R. Sharp, N. Anantharaman and Motoko Kotani. (see [45] and [30] for instance).

In connection with the existence problem of closed geodesics in a homology class, it is worthwhile to point out that, *if there is no closed geodesics homologous to zero in an n-dimensional compact Riemannian manifold M, then M has the same homotopy type as the torus T^n.*

Another effective use of the twisted Laplacians Δ_χ was made in the work [S47], in which the long time behavior of the heat kernel was studied. It is known that the heat kernel $k(t, x, y)$ on a general Riemannian manifold X has the following short time asymptotics

$$\begin{aligned} k(t, x, y) \sim (4\pi t)^{-m/2} \exp\big(-\frac{d(x, y)^2}{4t}\big) \\ \times (a_0(x, y) + a_1(x, y)t + a_2(x, y)t^2 + \dots) \quad (t \downarrow 0), \end{aligned}$$

where $m = \dim X$ and $d(x, y)$ denotes the Riemannian distance between x and y (here $d(x, y)$ is supposed to be small enough). On the one hand, the coefficients $a_i(x, y)$ have "local" nature in the sense that they are described by quantities defined only on a neighborhood of the shortest geodesic joining x and y. On the other hand, the behavior of $k(t, x, y)$ as t goes to infinity have to be controlled by the global properties of the manifold. Sunada observed that this is the case for abelian covering manifolds over compact manifolds, and gave an explicit asymptotic formula in the joint work [S42] with Motoko Kotani.

Let $\pi : X \longrightarrow M$ be a regular covering map whose covering transformation group Γ is free abelian, associated with which we have a surjective homomorphism of $H_1(M,\mathbf{Z})$ onto Γ and its extension to a surjective linear map of $H_1(M,\mathbf{R}) = H_1(M,\mathbf{Z}) \otimes\mathbf{R}$ onto $\Gamma\otimes\mathbf{R}$. We introduce an inner product on $H^1(M,\mathbf{R})$ by identifying it with the space of harmonic 1-forms on M. The dual inner product is equipped on $H_1(M,\mathbf{R}) = \mathrm{Hom}(H^1(M,\mathbf{R}),\mathbf{R})$. As the inner product on $\Gamma\otimes\mathbf{R}$, we take up the quotient inner product derived from the inner product on $H_1(M,\mathbf{R})$.

To describe long time asymptotic behavior of the heat kernel on X, we need a distance function different from the Riemannian one. For this sake, define the map $\tilde{\Phi}_\Gamma : X \to \Gamma\otimes\mathbf{R}$ by using the paring of $\Gamma\otimes\mathbf{R}$ and $\mathrm{Hom}(\Gamma,\mathbf{R})$ as

$$\langle\tilde{\Phi}_\Gamma(x),\omega\rangle = \int_{x_0}^{x}\tilde{\omega},$$

where $\omega\in\mathrm{Hom}(\Gamma,\mathbf{R})\subset H^1(M,\mathbf{R})$, and $\tilde{\omega}$ denotes its lift to X. It should be pointed out that the integral in the right-hand side does not depend on the choice of a path joining x_0 and x. We then put $d_\Gamma(x,y) = \|\tilde{\Phi}_\Gamma(y) - \tilde{\Phi}_\Gamma(x)\|$, where $\|\cdot\|$ denotes the Euclidean norm on $\Gamma\otimes\mathbf{R}$ associated with the inner product defined above.

Consider the flat torus $\Gamma\otimes\mathbf{R}/\Gamma\otimes\mathbf{Z}$ with the flat metric induced from the inner product above. Since $\tilde{\Phi}_\Gamma(\sigma x) = \tilde{\Phi}_\Gamma(x) + \sigma\otimes 1$ for $\sigma\in\Gamma$, we obtain a map $\Phi_\Gamma : M \to \Gamma\otimes\mathbf{R}/\Gamma\otimes\mathbf{Z}$ whose lift to X is $\tilde{\Phi}_\Gamma$. We call the flat torus $\Gamma\otimes\mathbf{R}/\Gamma\otimes\mathbf{Z}$ the Γ-*Albanese torus* and denote it by Alb^Γ. We also call Φ_Γ the Γ-*Albanese map*. The map Φ_Γ is harmonic.

Let $r = \mathrm{rank}(\Gamma)\quad (>0)$ and

$$C(X) = \mathrm{vol}(M)^{r/2-1}\mathrm{vol}(\mathrm{Alb}^\Gamma).$$

The following theorem tells that a *homogenization* takes place not only for the heat kernel but also for the space itself as time t goes to infinity.

Theorem 2.13. [S47](Local Central Limit Theorem)

$$\lim_{t\uparrow\infty}\Big((4\pi t)^{r/2}k(t,x,y) - C(X)\exp\big(-\frac{\mathrm{vol}(M)}{4t}d_\Gamma(x,y)^2\big)\Big) = 0,$$

uniformly for all $x,y\in X$.

This also tells that, if $B(t)$ is the Brownian motion on X, then the process $\delta\tilde{\Phi}_\Gamma\big(B(\delta^{-1}\mathrm{vol}(M)t)\big)$ goes, in distribution, to the Brownian motion on the Euclidean space $\Gamma\otimes\mathbf{R}$ as δ goes to zero (see [33] for a probabilistic interpretation). It is worthwhile to mention that the stochastic process $\tilde{\Phi}_\Gamma\big(B(t)\big)$ on $\Gamma\otimes\mathbf{R}$ is a *martingale* since $\tilde{\Phi}_\Gamma$ is a harmonic map.

One may establish an asymptotic expansion of $k(t,x,y)$ as a byproduct.

Theorem 2.14. [S47]

$$k(t,x,y)\sim(4\pi t)^{-r/2}C(X)(1 + c_1(x,y)t^{-1} + c_2(x,y)t^{-2} + \dots)\quad (t\uparrow\infty),$$

$$c_l(x,y)\sim\frac{1}{l!}\big(-\frac{\mathrm{vol}(M)}{4}\big)^l d_\Gamma(x,y)^{2l}\quad (d(x,y)\uparrow\infty).$$

The explicit forms of the coefficients $c_i(x,y)$ in Theorem 2.14 are complicated in general. But one may still give the exact shape of $c_1(x,y)$ as follows: Let $\omega_1,\dots,\omega_r$ be an orthonormal basis of the space $\mathrm{Hom}(\Gamma,\mathbf{R})(\subset H^1(M.\mathbf{R}))$ and let $G : C^\infty(M)\to C^\infty(M)$ be the Green operator.

Theorem 2.15. [S47]

$$c_1(x,y) = -\frac{\mathrm{vol}(M)}{4} d_\Gamma(x,y)^2 - \frac{\mathrm{vol}(M)}{2}\big(G(\sum_{i=1}^r |\omega_i|^2)(\pi(x)) + G(\sum_{i=1}^r |\omega_i|^2)(\pi(y))\big)$$
$$+ \frac{\mathrm{vol}(M)}{4}\Big(\int_M G(\sum_{i=1}^r |\omega_i|^2)\sum_{i=1}^r |\omega_i|^2 + 2\sum_{i,j=1}^r \int_M G(\langle \omega_i,\omega_j\rangle)\langle\omega_i,\omega_j\rangle\Big).$$

This theorem is seemingly technical. However its discrete analogue turns out to be useful in Sunada's study of geometric crystallography as seen in Subsection 2.8.

For the proof of these theorems, the perturbation theory is again employed. Let L_χ be the flat line bundle associated with χ. One may take an orthonormal basis $\{s_{\chi,k}\}_{k=0}^\infty$ of $L^2(L_\chi)$ such that

(1) $\Delta_\chi s_{\chi,k} = \lambda_k(\chi) s_{\chi,k}$,
(2) $s_{\chi,k}$ is bounded and integrable with respect to χ,
(3) $s_{\chi,0}$ is smooth in χ around $\chi = \mathbf{1}$ and $s_{\mathbf{1},0} \equiv \mathrm{vol}(M)^{-1/2}$.

The core of the proof is in the fact that the heat kernel $k(t,x,y)$ on X is expressed as

$$k(t,x,y) = \sum_{k=0}^\infty \int_{\hat{\Gamma}} \exp(-\lambda_k(\chi)t)\tilde{s}_{\chi,k}(x)\overline{\tilde{s}_{\chi,k}(y)}d\chi,$$

where $\tilde{s}_{\chi,k}$ is the lift of $s_{\chi,k}$, and $d\chi$ is the normalized Haar measure of $\widehat{\Gamma}$. A careful analysis of this integral expression yields the theorems above (for Theorem 2.15, one needs to know the coefficients in the Taylor expansion of $\lambda_0(\chi)$ at $\chi = \mathbf{1}$ up to the 4th order).

The idea of twisted Laplacians has been developed further by Sunada for the spectral study of general covering manifolds. The starting point of his study was to understand a "mechanism" of the following result by R. Brooks [6]: *Let $\tilde{M}$ be the universal covering manifold of a compact Riemannian manifold M. The bottom of the spectrum $\lambda_0(\tilde{M})$ of the Laplacian $\Delta_{\tilde{M}}$ on $\tilde{M}$ is zero if and only if $\pi_1(M)$ is amenable.* Brooks' idea is to employ the theory of integral currents and Fϕlner's theorem on amenability.

The observation Sunada made is that $\Delta_{\tilde{M}}$ is unitarily equivalent to the twisted Laplacian associated with the regular representation of $\pi_1(M)$. To explain this in a more general set-up, we consider a regular covering map $\pi : X \longrightarrow M$ over a compact Riemannian manifold M with covering transformation group G. Given a unitary representation $\rho : G \longrightarrow U(\mathbf{H})$ on a Hilbert space $\mathbf{H}$, we set $\ell^2(\rho) = \{f; X \longrightarrow \mathbf{H};\ f(\sigma x) = \rho(\sigma)f(x),\ \int_M \|f\|^2 < \infty\}$, which has a natural Hilbert space structure. The twisted Laplacian Δ_ρ is defined to be the restriction to $\ell^2(\rho)$ of the Laplacian acting on $\mathbf{H}$-valued functions on X. Actually Δ_ρ is the Laplacian acting on sections of the flat vector bundle (possibly of infinite rank) associated with ρ. One can establish the following estimates for the bottom of the spectrum $\lambda_0(\rho) = \inf \sigma(\Delta_\rho)$ from below and above.

Theorem 2.16. ([S25])

$$c_1\delta(\rho,\mathbf{1})^2 \le \lambda_0(\rho) \le c_2\delta(\rho,\mathbf{1})^2. \tag{2.12}$$

Here c_1, c_2 are positive constants independent of ρ

The quantity $\delta(\rho, \mathbf{1})$ is the *Kazhdan distance* between ρ and the trivial representation $\mathbf{1}$ defined by $\delta(\rho, \mathbf{1}) = \inf\{\sup_{g\in A} \|\rho(g)v - v\|;\ v \in \mathbf{H},\ \|v\| = 1\}$, where A is a finite set of generators of G. The inequalities in (2.12), in particular, say that $\lambda_0(\rho) = 0$ if and only if $\delta(\rho, \mathbf{1}) = 0$. Applying this to the regular representation[7] $\rho_H : G \longrightarrow U\big(\ell^2(H\backslash G)\big)$ associated with a subgroup H of G, we conclude that $\lambda_0(\Delta_{H\backslash X}) = 0$ if and only if $\delta(\rho_H, \mathbf{1}) = 0$ since Δ_{ρ_H} is unitarily equivalent to the discrete Laplacian $\Delta_{H\backslash X}$ on the quotient manifold $H\backslash X$[8]. In particular, since $\delta(\rho_G, \mathbf{1}) = 0$ if and only if G is *amenable*[9], we find that $\lambda_0(\Delta_X) = 0$ if and only if G is amenable.

Sunada says that he came up with the estimates (2.12) as a natural generalization of (2.10). Indeed, the integral $\displaystyle\int_M \|\omega\|^2$ on the right-hand side of (2.10) gives rise to the distance between χ and the trivial character.

In connection with Theorem 2.16, Sunada also established a relationship between spectra and the notion of *weak containment*. As a corollary, he proved that, if $\pi : X \longrightarrow M$ is an amenable covering map, then $\sigma(\Delta_M) \subset \sigma(\Delta_X)$.

A discrete version of Theorem 2.16 was given in [S31]. The estimate from below in this case is related to a construction of *expanders*, a model of efficient communication networks (see [S57]).

Sunada was greatly impressed by the beautiful relation between spectra and group structure obtained by Robert Brooks in the paper mentioned above. He remembers that Brooks was the first to read through his preprint on isospectral manifolds. Sunada and Brooks had mutually influenced one another in many occasions thereafter, until Brooks passed away in 2002.

As for the development of the spectral study of covering spaces towards a different direction, one may refer to the paper by P. Kuchment and and Y. Pinchover [32].

The idea of "twisting" was used by M. A. Shubin and Sunada [S55] in their rigorous derivation of the classical T^3-law for *specific heat* of crystals, a typical thermodynamical quantity in solid state physics, which dates back to Einstein's pioneering work in 1907 and its subsequent refinement by Debye in 1912.

2.6. **Ihara Zeta functions.** In the mid-80s, Sunada started to study the zeta function associated with a cocompact torsion-free discrete subgroup of $PSL_2(\mathbf{Q}_p)$, which was originally introduced by Y. Ihara [28] in 1966 as an analogue of the Selberg zeta function. In the background of Ihara zeta functions, there is a consensus that the p-adic version of the real hyperbolic plane $GL_2(\mathbf{R})/(\mathbf{R}^\times \times O(2))$ is $GL_2(\mathbf{Q}_p)/(\mathbf{Q}_p^\times \times GL_2(\mathbf{Z}_p))$[10]. The latter space is, as a special case of Bruhat-Tits buildings, regarded as the set of vertices in the regular tree of degree $p+1$, and hence its compact quotient yields a finite regular graph. Thus it is natural to interpret the Ihara zeta functions in terms of finite regular graphs.

According to Sunada, his idea in this study is simple. He says that he just mimicked the geometric interpretation of Selberg zeta functions in terms of closed

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[7] ρ_H is the induced representation $\mathrm{Ind}_H^G(\mathbf{1})$ of the trivial representation.

[8] This implies that, if ρ_{H_1} and ρ_{H_2} are equivalent for two subgroups H_1, H_2, then $\Delta_{H_1\backslash X}$ and $\Delta_{H_2\backslash X}$ are unitary equivalent. This gives another proof of Theorem 2.10.

[9] This is equivalent to that G has an *invariant mean*.

[10] $GL_2(\mathbf{Z}_p)$ and $O(2)$ are maximal compact subgroups of $GL_2(\mathbf{Q}_p)$ and $GL_2(\mathbf{R})$, respectively.

geodesics. Since he already had a geometric model of number theory, it was a sort of exercise for him to carry out the idea.

Let P be the set of all *prime cycles* in a finite regular graph X of degree $q+1$. Here a prime cycle is an equivalence class of a closed path without backtracking (we call closed geodesic for simplicity) which is not a power of another one. Two closed paths are said to be equivalent if one is obtained by a cyclic permutation of edges in another. In 1985, Sunada [S22], following the suggestion stated in the preface of J.-P. Serre [47], expressed the Ihara zeta function as the Euler product

$$Z(u) = \prod_{\mathfrak{p}\in P}(1-u^{|\mathfrak{p}|})^{-1},$$

and gave a graph theoretic proof for the following determinant expression in terms of the adjacency operator $\mathcal{A}$

$$Z(u) = (1-u^2)^{(1-q)N/2}\det\left(I - u\mathcal{A} + qu^2 I\right)^{-1},$$

where N is the number of vertices. From this identity, it follows that *the Ihara zeta function of a regular graph X satisfies the Riemann Hypothesis if and only if every eigenvalue μ of $\mathcal{A}$ satisfies $|\mu| \le 2\sqrt{q}$ except for $\mu = \pm(q+1)$*. Graphs satisfying the Riemann Hypothesis was later called *Ramanujan graphs* by A. Lubotzky, R. Phillips and P. Sarnak [34].

Sunada's observation was immediately generalized by K. Hashimoto and A. Hori to the case of semi-regular graphs which correspond to p-adic semi-simple groups of rank one. H. Bass [3] noticed that "regularity" of graphs is not necessary for a determinant expression. Sunada in the collaborative work with Kotani [S48] used the idea of *discrete geodesic flows* to give another proof of Bass' result. Actually, they interpreted the Ihara zeta function as the dynamical zeta function associated with a symbolic dynamical system. An instructive proof together with a generalization to weighted graphs can be found in M. D. Horton, H. M. Stark and A. A. Terras [27]. See also [37].

There have been various attempts to introduce Ihara zeta functions for infinite graphs. See for instance D. Guido, T. Isola, and M. L. Lapidus [26], R. I. Grigorchuk and A. Zuk [24].

2.7. **Quantum ergodicity.** Mathematics is developed through interactions and communications among mathematicians. Sunada's work on quantum ergodicity is one of such examples. Around 1990, he read an article [53] of Steven Zelditch who has been an appreciative reader of Sunada's work. The subject of that paper is on asymptotic behavior of eigenfunctions of Laplacians.

Before explaining Zelditch's result and its generalization, let us recall an elementary fact ([S39]). Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ be eigenvalues of Δ on a compact Riemannian manifold M, and let $\{\varphi_k\}_{k=0}^{\infty}$ be an orthonormal basis of eigenfunctions with $\Delta\varphi_k = \lambda_k\varphi$. We then have the asymptotic formula

$$\begin{aligned} &\sum_{k=0}^{\infty} e^{-\lambda_k t}\int_M f(x)|\varphi_k(x)|^2 dx = \int_M f(x)k(t,x,x)dx \\ \sim\ &(4\pi t)^{-n/2}\int_M f(x)dx \qquad (n = \dim M) \end{aligned}$$

as $t \downarrow 0$. Applying the Hardy-Littlewood Tauberian theorem, we obtain

$$\lim_{\lambda\to\infty} \phi(\lambda)^{-1} \sum_{\lambda_k \le \lambda} \int_M f(x)|\varphi_k(x)|^2 dx = \mathrm{vol}(M)^{-1} \int_M f(x)dx,$$

where $\phi(\lambda) = |\{k;\ \lambda_k \le \lambda\}|$. This identity holds without any condition on M. The following tells that a dynamical property of the geodesic flow allows us to obtain a refined limit formula.

Proposition 2.6. (Quantum ergodicity [53], [15]) *If the geodesic flow is ergodic, then there exists a subsequence $\{\lambda_{k_j}\}$ of $\{\lambda_k\}$ of full density such that for every $f \in C^\infty(M)$,*

$$\lim_{j\to\infty} \int_M f(x)|\varphi_{k_j}(x)|^2 dx = \mathrm{vol}(M)^{-1} \int_M f(x)dx.$$

The meaning of "full density" is that

$$\lim_{\lambda\to\infty} \frac{|\{k_j;\ \lambda_{k_j} \le \lambda\}|}{|\{k;\ \lambda_k \le \lambda\}|} = 1.$$

Ergodicity in classical mechanical system means that the time average of an observable is equal to the space average. In his attempt to give a more natural flavor of ergodicity to the limit theorem above, Sunada introduces the concept of *quantum ergodicity at infinite energy level.*

To explain this in full generality, Sunada considers a positive self-adjoint elliptic pseudo-differential operator $\widehat{H}$ of order one with eigenvalues $0 \le e_1 \le e_2 \le \cdots$. Put $H = \sigma(\widehat{H})$, the principal symbol of $\widehat{H}$, and denote by φ_t the Hamiltonian flow on $T^*M\backslash 0$ generated by the Hamiltonian H. A *quantum observable* is defined to be a pseudo-differential operator A of order zero. The time evolution of A is given by ${U_t}^*AU_t$ where $U_t = \exp t\sqrt{-1}\widehat{H}$

To follow the definition of classical ergodicity, we define the *time average* up to time t by

$$\frac{1}{t}\int_0^t {U_s}^*AU_s ds,$$

which converges weakly to $\overline{A} = \sum_e P_eAP_e$. Here e runs over the spectrum of $\widehat{H}$, and P_e is the orthogonal projection onto the eigenspace $V_e = \{\varphi;\ \widehat{H}\varphi = e\varphi\}$. We also define the *space average* by

$$\langle A \rangle = \lim_{e\uparrow\infty} N(e)^{-1} \sum_{\substack{k \\ e_k \le e}} \langle A\varphi_k, \varphi_k\rangle$$

where $N(e) = |\{k;\ e_k \le e\}|$ and $\{\varphi_k\}$ is a complete orthonormal basis of eigenfunctions of $\widehat{H}$ with $\widehat{H}\varphi_k = e_k\varphi_k$. It is worthwhile to note that $\langle A\rangle$ is also expressed as

$$\langle A \rangle = \lim_{e\to\infty} \int_{S(V_{\le e})} \langle A\varphi, \varphi\rangle dS(\varphi),$$

where dS is the normalized uniform measure on the unit sphere $S(V_{\le e})$ of the space $V_{\le e} = \sum_{\lambda \le e} V_\lambda$.

Now the quantum dynamical system $\{U_t\}$ is said to be *quantum ergodic* (at infinite energy level) if, for every observable A, the space average $\langle \overline{A}^* \overline{A} \rangle$ exists, and

$$\langle \overline{A}^* \overline{A} \rangle = |\langle A \rangle|^2.$$

This is an analogue of the criterion for the classical ergodicity which can be stated as "$\langle |\overline{f}|^2 \rangle = |\langle f \rangle|^2$ for every classical observable f", where $\overline{f}$ is the time average, and $\langle f \rangle$ is the space average.

Sunada [S43] proved the followings.

Theorem 2.17. *The flow φ_t restricted to the hypersurface $\Sigma_1 = H^{-1}(1)$ is ergodic if and only if $\{U_t\}$ is quantum ergodic, and*

$$\lim_{t \to \infty} \langle {A_t}^* A_t \rangle = \langle \overline{A}^* \overline{A} \rangle.$$

Theorem 2.18. (1) *Quantum ergodicity is equivalent to the following near-diagonal asymptotic.*

$$\lim_{e \to \infty} N(e)^{-1} \sum_{\substack{i,j \\ e_i = e_j \le e}} \big| \langle A\varphi_i, \varphi_j \rangle \big|^2 = \Big| \int_{\Sigma_1} \sigma(A) d\omega_1 \Big|^2.$$

where ω_1 is the normalized Liouville measure on Σ_1.

(2) *The condition $\lim_{t \to \infty} \langle {A_t}^* A_t \rangle = \langle \overline{A}^* \overline{A} \rangle$ is equivalent to the following off-diagonal asymptotic.*

$$\lim_{\delta \downarrow 0} \limsup_{e \uparrow \infty} \sum_{\substack{i \\ e_i \le e}} \sum_{\substack{j \\ 0 < |e_i - e_j| \le \delta}} \big| \langle A\varphi_i, \varphi_j \rangle \big|^2 = 0.$$

One can prove that if $\{U_t\}$ is quantum ergodic (hence if φ_t is ergodic), then there exists a subsequence $\{k_j\}$ of full density such that, for every pseudo-differential operator A of order zero, the sequence $\langle A\varphi_{k_j}, \varphi_{k_j} \rangle = \int_M A\varphi_{k_j} {\varphi_{k_j}}^*$ converges to $\int_{\Sigma_1} \sigma(A) d\omega_1$, where φ^* is the complex conjugate of φ.

Note that, for $\widehat{H} = \Delta^{1/2}$, we have $H(x, \xi) = \|\xi\|$ so that, in this case, Σ_1 is the tangent unit sphere bundle, and φ_t is the geodesic flow.

The notion of quantum ergodicity *at finite energy level* was formulated by Tatsuya Tate [48]. See also S. Zelditch [54] for an abstract set-up.

Sunada's work of quantum ergodicity looks isolated from others. One can see, however, that he always demonstrates high capabilities common to all his work to push to the completion the original concepts.

2.8. **Discrete geometric analysis.** Sunada says that there are several sources available for him to start with the study of *discrete geometric analysis*, a field dealing with analysis on graphs by using geometric ideas. One is, as a matter of course, Ihara zeta functions, where adjacency operators play a role of Laplacians. Another is the symbolic dynamics used in the study of unstable dynamical systems in which Ruelle operator on oriented graphs is an indispensable tool. There is one more source, a bit quirky one. He wrote a book (in Japanese) on the elementary aspects of area and volume including Dehn's theorem on scissors congruence of tetrahedra ([S53]). While he was writing the part on the area theory, this question came to his mind: when a rectangle K is divided into finitely many squares, is the ratio of two sides rational ? Sunada asked Prof. Koji Shiga, his supervisor when he was in TIT, whether the answer is known. Shiga immediately told him that the problem was already solved by M. Dehn in 1902 affirmatively ([18]). Sunada tried in vain to understand the proof, and so he tried to find his own proof. After some trials, he found out that he could reduce the problem to a discrete version of Poisson's equation $\Delta f = g$ ([SB6]). This was not surprising to him since, as he was also told by Shiga, Dehn's theorem can be shown by an idea of electric circuits ([5]). Note that the fundamental laws of electric circuits are also described in terms of a discrete analogue of "the method of orthogonal projections" as H. Weyl already observed in 1920s (see [S57]).

A much serious reason came up when he learned of a strange phenomenon for the spectra of discrete magnetic Schrödinger operators defined on $\mathbf{Z}^2$:

$$\begin{aligned}(H_{\alpha_1,\alpha_2}\varphi)(m,n) &= \frac{1}{4}[e^{\sqrt{-1}\alpha_1 n}\varphi(m+1,n) + e^{-\sqrt{-1}\alpha_1 n}\varphi(m-1,n) \\ &\quad + e^{\sqrt{-1}\alpha_2 m}\varphi(m,n+1) + e^{-\sqrt{-1}\alpha_2 m}\varphi(m,n-1)],\end{aligned}$$

$(\alpha_1, \alpha_2 \in \mathbf{R})$. The operator $H_{\alpha_1,\alpha_2} : \ell^2(\mathbf{Z}^2) \longrightarrow \ell^2(\mathbf{Z}^2)$ is bounded and self-adjoint. The operator $I - H_{\alpha_1,\alpha_2}$ is regarded as a discretization of the following Schrödinger operator with the uniform magnetic field $B = (\alpha_2 - \alpha_1)dx \wedge dy$

$$-\big(\frac{\partial}{\partial x} + \sqrt{-1}\alpha_1 y\big)^2 - \big(\frac{\partial}{\partial y} + \sqrt{-1}\alpha_2 x\big)^2,$$

whose spectrum consists only of eigenvalues of infinite multiplicity provided that $\theta = \alpha_2 - \alpha_1$ is not zero; say

$$\{2|\theta|\big(k + \frac{1}{2}\big);\ k = 0, 1, 2, \ldots\}.$$

On the other hand, the spectrum $\sigma(H_{\alpha_1,\alpha_2})$, a closed subset of $[-1, 1]$, is quite sensitive to the parameter $\theta = \alpha_2 - \alpha_1$, which physicists call the *magnetic flux*. If $\theta/2\pi$ is rational, then $\sigma(H_{\alpha_1,\alpha_2})$ has band structure, while if $\theta/2\pi$ is a Liouville number, then $\sigma(H_{\alpha_1,\alpha_2})$ is a Cantor set.

Having this example in mind[11], Sunada formulates a discrete analogue of general Schrödinger operators with periodic magnetic fields ([S40]) and defines the notion of magnetic flux. For this end, he starts with a realization Φ of a graph X in a Riemannian manifold M (thus Φ is a piecewise smooth map of X into M). We denote by V the set of vertices, and by E the set of all oriented edges of X. Both X and M are supposed to have a free Γ-action, and Φ is supposed to be Γ-equivariant. A magnetic field on M is a closed 2-form $\mathbf{B}$. We assume that $\mathbf{B}$ is Γ-invariant, and that $\mathbf{B}$ has a global vector potential (1-form) $\mathbf{A}$ so that $\mathbf{B} = d\mathbf{A}$ ($\mathbf{A}$ is not necessarily Γ-invariant). We also assume $H^1(M, \mathbf{R}) = \{0\}$. Define function $f_{\mathbf{A}}$ on E by

$$f_{\mathbf{A}}(e) = \exp\Big(\sqrt{-1}\int_e \Phi^*\mathbf{A}\Big),$$

which is a $U(1)$-valued 1-cochain on X. Using the assumptions on $\mathbf{B}$ and M, we find a function φ_σ on M for each $\sigma \in \Gamma$ such that $d\varphi_\sigma = \sigma^*\mathbf{A} - \mathbf{A}$. If we define $s_\sigma : X \longrightarrow U(1)$ by $s_\sigma(x) = \exp(-\sqrt{-1}\varphi_\sigma(\Phi(x)))$, then

$$f_{\mathbf{A}}(\sigma e) = f_{\mathbf{A}}(e)\frac{s_\sigma(o(e))}{s_\sigma(t(e))},$$

[11]Sunada's interest in geometry and analysis of magnetic fields is already seen in [S37] and [S61].

where $o(e)$ (resp. $t(e)$) denotes the origin (resp. terminus) of e. This implies that the cohomology class $[f_{\mathbf{A}}] \in H^1(X, U(1))$ is Γ-invariant. We then define the operator $H_{f_{\mathbf{A}}} : \ell^2(X) \longrightarrow \ell^2(X)$ by

$$(H_{f_{\mathbf{A}}}\varphi)(x) = \frac{1}{\deg x} \sum_{e \in E_x} f_{\mathbf{A}}(e)\varphi(t(e)),$$

where $E_x = \{e \in E;\ o(e) = x\}$ and $\deg x = |E_x|$. The space $\ell^2(X)$ is the Hilbert space consisting of functions φ on V such that $\sum_{x \in V} |\varphi(x)|^2 \deg x < \infty$. The operator $I - H_{f_{\mathbf{A}}}$ is considered a discretization of the magnetic Schrödinger operator $\nabla^*_{\mathbf{A}}\nabla_{\mathbf{A}}$, where $\nabla_{\mathbf{A}} = d - \sqrt{-1}\mathbf{A}$. (see [31]). One may easily check that $H_{f_{\mathbf{A}}}$ coincides with H_{α_1,α_2} for the square lattice $X = \mathbf{Z}^2$ realized in $M = \mathbf{R}^2$, and for the vector potential $\mathbf{A} = \alpha_1 y dx + \alpha_2 x dy$ ($\mathbf{B} = d\mathbf{A} = (\alpha_2 - \alpha_1) dx \wedge dy$).

Taking account of the discussion above, Sunada introduces an abstract set-up. On the one hand, he begins with a regular covering graph $X = (V, E)$ over a finite graph X_0 with covering transformation group Γ, and a 1-cochain $f \in C^1(X, U(1))$ on X whose cohomology class $[f] \in H^1(X, U(1))$ is Γ-invariant. On the other hand, he considers a random walk on X with a Γ-invariant transition probability $p : E \longrightarrow \mathbf{R}$; namely $p(\sigma e) = p(e)$, $p(e) \geq 0$, $\sum_{e \in E_x} p(e) = 1$. Then he defines the operator $H_f : \ell^2(X) \longrightarrow \ell^2(X)$ by $(H_f\varphi)(x) = \sum_{e \in E_x} p(e) f(e)\varphi(t(e))$. If the random walk is symmetric (reversible) in the sense that there is a Γ-invariant positive-valued function m on V with $p(e)m(o(e)) = p(\overline{e})m(t(e))$ ($\overline{e}$ being the inverse edge of e), then H_f is bounded self-adjoint operator acting in the Hilbert space $\ell^2(X) = \{\varphi : V \longrightarrow \mathbf{C}; \sum_{x \in V} |\varphi(x)|^2 m(x) < \infty\}$. He calls H_f a *magnetic transition operator.*

The notion of magnetic flux is introduced in this set-up as follows. From the assumption that $[f]$ is Γ-invariant, it follows that there exists a $U(1)$-valued function s_σ on V such that $f(\sigma e) = f(e)s_\sigma(t(e))s_\sigma(o(e))^{-1}$. Put

$$\Theta_f(\sigma, \gamma) = \frac{s_\sigma(\gamma x)s_\gamma(x)}{s_{\sigma\gamma}(x)}.$$

It is observed that the right-hand side does not depend on x and $\Theta_f(\sigma, \gamma)$ is a group 2-cocycle of Γ with coefficients in $U(1)$; that is,

$$\Theta_f(\sigma_1, \sigma_2\sigma_3)\Theta_f(\sigma_2, \sigma_3) = \Theta_f(\sigma_1, \sigma_2)\Theta_f(\sigma_1\sigma_2, \sigma_3).$$

Thus we obtain $[\Theta_f] \in H^2(\Gamma, U(1))$. This is what Sunada calls the *magnetic flux class.*

The following is a partial generalization of what we have mentioned for H_{α_1,α_2}.

Theorem 2.19. [S40] *If Γ is abelian and $[\Theta_f] \in H^2(\Gamma, \mathbf{Q}/\mathbf{Z})$, then the spectrum of H_f has band structure.*

The magnetic flux class $[\Theta_f]$ depends only on $[f]$ so that we have a homomorphism $\Theta : H^1(X, U(1))^\Gamma \longrightarrow H^2(\Gamma, U(1))$. In connection with this, we have the following exact sequence

$$1 \to H^1(\Gamma, U(1)) \to H^1(X_0, U(1)) \xrightarrow{\pi^*} H^1(X, U(1))^\Gamma \xrightarrow{\Theta} H^2(\Gamma, U(1)) \to 1,$$

where $H^1(\Gamma, U(1)) \to H^1(X_0, U(1))$ is the homomorphism induced from the surjective homomorphism $\mu : \pi_1(X_0) \longrightarrow \Gamma$ associated with the covering map $\pi : X \longrightarrow X_0$. If X is the maximal abelian covering graph of X_0, we observe that $H^1(\Gamma, U(1)) \to \mathrm{H}^1(X_0, U(1))$ is an isomorphism so that $H^1(X, U(1))^\Gamma \xrightarrow{\Theta} H^2(\Gamma, U(1))$

is also an isomorphism. Therefore the magnetic flux class $[\Theta_f]$ determines the unitary equivalence class of H_f. This is not the case in general (an analogue of the *Aharonov-Bohm effect*).

The study of magnetic transition operators stimulated Sunada's interest in the theory of random walks itself. His interest was multiplied when he was asked by a Japanese probabilist about a geometric meaning of the constant which appears in the *local limit formula* for classical lattices in plane, based on a few explicit values computed in an ad hoc manner (see the table below). The local limit formula for the simple random walk on a lattice X claims that there exists a positive constant $C(X)$ such that[12]

$$\lim_{n\to\infty}(4\pi n)p(n,x,y)(\deg\, y)^{-1} = C(X),$$

where $p(n,x,y)$ is the n-step transition probability[13]. The constant $C(X)$, which, by definition, depends only on the graph structure of X, is surely an interesting quantity in its own right.

X	hexagonal	triangular	quadrilateral	kagome
$C(X)$	$2\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

The paper [S44] written jointly with Kotani and Tomoyuki Shirai gave an answer to the question. They treated the local limit formula for a symmetric random walk on abelian covering graphs X over finite graphs (which they call *crystal lattices*), and gave an expression of $C(X)$ in terms of a discrete analogue of Albanese tori (see Theorem 2.22 below for another expression and [S45], [S60]).

In the article [S47] which we have mentioned in Subsection 2.5, Sunada and Kotani established more precise asymptotics for $p(n,x,y)$; namely, a local central limit theorem and an asymptotic expansion. Actually the idea for the heat kernel can be applied without almost any change to the discrete case. To state the asymptotics, what we need is a discrete version of harmonic theory described below[14].

For simplicity, let us restrict ourselves to the case of *simple* random walks on crystal lattices; that is, the case that $p(e) = (\deg\, o(e))^{-1}$. Let X be an abelian covering graph over a finite graph X_0 whose covering transformation group is Γ. We assume that Γ is free abelian. As in the case of manifolds, we have a surjective linear map of $H_1(X_0,\mathbf{R})$ onto $\Gamma\otimes\mathbf{R}$ and its transpose $\mathrm{Hom}(\Gamma,\mathbf{R}) \longrightarrow H^1(X_0,\mathbf{R})$. Recall that the first cohomology group $H^1(X_0,\mathbf{R})$ is defined as $C^1(X_0,\mathbf{R})/\mathrm{Image}\, d$, where $d : C^0(X_0,\mathbf{R}) \longrightarrow C^1(X_0,\mathbf{R})$ is the coboundary map of the cochain groups of X_0 as a CW-complex.

[12]For simplicity, X is assumed to be non-bipartite from now on. In bipartite case, a minor modification of statement is required.

[13]If we consider a d-dimensional lattice, we should replace $4\pi n$ by $(4\pi n)^{d/2}$ in the limit formula.

[14]A partial generalization to "nilpotent crystal lattice" was established by Satoshi Ishiwata.

Define the inner products on $C^0(X_0,\mathbf{R})$ and $C^1(X_0,\mathbf{R})$ respectively by

$$\langle f_1, f_2\rangle = \sum_{x\in X_0} f_1(x)f_2(x)\deg x, \tag{2.13}$$

$$\langle \omega_1, \omega_2\rangle = \frac{1}{2}\sum_{e\in E_0} \omega_1(e)\omega_2(e), \tag{2.14}$$

and let $d^* : C^1(X_0,\mathbf{R}) \longrightarrow C^0(X_0,\mathbf{R})$ be the adjoint operator of the coboundary operator d with respect to these inner products. Put

$$\mathbf{H}^1(X_0) = \{\omega \in C^1(X_0,\mathbf{R});\ d^*\omega = -(\deg x)^{-1}\sum_{e\in(E_0)_x}\omega(e) = 0\}.$$

This is a discrete analogue of the space of "harmonic 1-forms" on X_0. We now identify $H^1(X_0,\mathbf{R})$ with $\mathbf{H}^1(X_0)$ via the canonical isomorphism $\mathbf{H}^1(X_0)$ onto $H^1(X_0,\mathbf{R})$ (the discrete Hodge-Kodaira theorem). Through this identification, we obtain an inner product on $H^1(X_0,\mathbf{R})$ and an inner product g_0 on $\Gamma\otimes\mathbf{R}$.

Next define $\Phi : X \longrightarrow \Gamma\otimes\mathbf{R}$ by

$$\langle \Phi(x), \omega\rangle = \sum_{i=1}^n \tilde{\omega}(e_i), \quad (\omega \in \mathrm{Hom}(\Gamma,\mathbf{R}) \subset H^1(X_0,\mathbf{R})),$$

where $c = (e_1,\dots,e_n)$ is a path in X joining a reference point x_0 and x, and $\tilde{\omega}$ is the pull-back of ω to X. We readily check that the piecewise linear map Φ interpolated by line segments is a periodic realization of X in the sense that $\Phi(\sigma x) = \Phi(x) + \sigma$. We call $\Phi : X \longrightarrow (\Gamma\otimes\mathbf{R}, g_0)$ the *standard realization* of X.

Now the materials are ready to state the asymptotics for $p(n,x,y)$. Indeed, instead of the Laplacian, we only have to treat the discrete Laplacian defined by $I - H_1$ (remember that H_1 is just the ordinary transition operator for the random walk). But we are not going to repeat almost the same statements here. Instead we state a consequence of the results and a peculiar feature of the standard realization Φ.

Theorem 2.20. [S50] *Let $\|\cdot\|$ be the norm associated with the inner product g_0. Then*

$$\|\Phi(x) - \Phi(y)\|^2 = \lim_{n\to\infty} 2n\Big\{\frac{p(n,x,x)}{p(n,y,x)} + \frac{p(n,y,y)}{p(n,x,y)} - 2\Big\}.$$

This theorem, shown by means of the graph versions of Theorem 2.14 and 2.15, says that the left-hand side does not depend on the choice of the "periodic lattice" Γ since the right-hand side depends only on the graph structure. Furthermore, we may conclude that the standard realization Φ has *maximal symmetry* in the following sense.

Theorem 2.21. [S50] *Let $\mathrm{Aut}(X)$ be the automorphism group of the crystal lattice X as an abstract graph, and let M_{g_0} be the group of congruent transformations of $(\Gamma\otimes\mathbf{R}, g_0)$. Then there exists a homomorphism $\rho : \mathrm{Aut}(X) \longrightarrow M_{g_0}$ such that $\Phi(gx) = \rho(g)\Phi(x)$ for $g \in \mathrm{Aut}(X)$.*

In [S56], Sunada observed that there is a remarkable relation between the constant $C(X)$ appearing in the local limit formula and a certain energy of crystals which characterizes the standard realization by a minimal principle.

We think of a crystal (a periodically realized crystal lattice) in $\mathbf{R}^d$ as a system of harmonic oscillators, that is, each edge represents a harmonic oscillator whose energy is the square of its length. We shall define the energy of a crystal "per a unit cell" in the following way. Given a bounded domain D in $\mathbf{R}^d$, denote by $\mathcal{E}(D)$ the sum of the energy of harmonic oscillators whose endpoints are in D, and normalize it in such a way as

$$\mathcal{E}_0(D) = \frac{\mathcal{E}(D)}{\deg(D)^{1-2/d}\mathrm{vol}(D)^{2/d}},$$

where $\deg(D)$ is the sum of degree (valency) of vertices in D. Take an increasing sequence of bounded domains $\{D_i\}_{i=1}^{\infty}$ with $\cup_{i=1}^{\infty} D_i = \mathbf{R}^d$ (for example, a family of concentric balls). The energy of the crystal (per a unit cell) is defined as the limit

$$E_{ner} = \lim_{i\to\infty} \mathcal{E}_0(D_i).$$

Indeed the limit exists under a mild condition on $\{D_i\}_{i=1}^{\infty}$, and E_{ner} does not depend on choices of $\{D_i\}_{i=1}^{\infty}$. We write $E_{ner}(\Phi)$ for the energy when the crystal is given by a periodic realization Φ. It is easy to observe that $E_{ner}(T\circ\Phi) = E_{ner}(\Phi)$ for every homothetic transformation T.

Theorem 2.22. [S60] *For every periodic realization Φ of X in $\mathbf{R}^d$, we have*[15]

$$E_{ner}(\Phi) \geq dC(X)^{-2/d}.$$

The equality holds if and only if Φ is standard.

The proof of this remarkable inequality, available at present, is not carried out by finding a direct link between the two quantities, but is based upon a canonical expression of the standard realization, an analogue of *Albanese maps* in algebraic geometry ([S50], [S60]).

Example 2.2. *Figure* 1 *exhibits three kinds of periodic realizations of the hexagonal lattice* (*the maximal abelian covering graph over the graph with two vertices joined by three parallel edges*). *The figure* (*honeycomb*) *on the right side is its standard realization.*

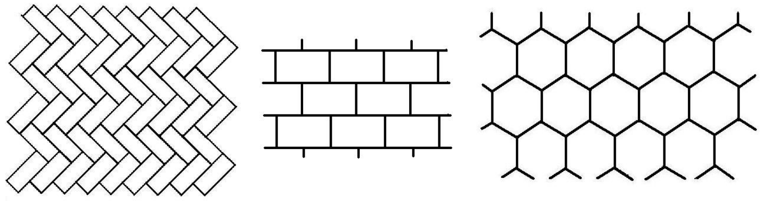

FIGURE 1. Various periodic realizations of the hexagonal lattice

We also observe that the standard realizations of the triangular lattice and the quadrilateral lattice are the regular triangular lattice and the regular square lattice, respectively.

[15]This inequality is for non-bipartite crystal lattices. In bipartite case, the right-hand side should be replaced by $d(C(X)/2)^{-2/d}$.

These examples tell that "a random walker on a crystal lattice X can detect the most natural way for X to sit in space".

The notions of Γ-Albanese tori and Γ-Albanese maps are defined in the same manner as the continuous case. For the maximal abelian covering graphs, the Γ-Albanese maps are closely related to the Abel-Jacobi maps introduced by R. Bacher, P. De La Harpe, and T. Nagnibeda [2] (see [S60]).

The paper [S54] jointly written with Kotani handled *large deviations* of random walks on crystal lattices which exhibit another aspect of asymptotics. The theory of large deviations, in general, concerns the asymptotic behavior of remote tails of sequences of probability distributions. For the simple random walk[16] the problem is to find an asymptotic behavior of the transition probability $p(n,x,y)$ while y is kept near the boundary $\partial B_n(x) = \{y \in V;\ d(x,y) = n\}$, where $d(x,y)$ is the graph distance between x and y. Note that $B_n(x) = \{y \in V;\ d(x,y) \leq n\}$ is the reachable region of the random walker starting from x.

What they did first was to look at the shape of $B_n(x)$ in space when n goes to infinity. More precisely, by fixing a periodic realization $\Phi : X \longrightarrow \Gamma \otimes \mathbf{R}$ with $\Phi(x) = 0$, and considering the scaling-limit figure

$$D = \lim_{n\to\infty} \frac{1}{n}\Phi\big(B_n(x)\big),$$

they observed that this coincides with the unit ball for the norm on $\Gamma \otimes \mathbf{R}$ defined by

$$\|\xi\|_1 = \inf\{\|\mathbf{x}\|_1 \mid \mathbf{x} \in H_1(X_0,\mathbf{R}),\ \mu_{\mathbf{R}}(\mathbf{x}) = \xi\},$$

where $\|\mathbf{x}\|_1$ denotes the ℓ^1-norm of $\mathbf{x}$ as a 1-chain of the finite graph X_0. To be exact, the norm $\|\cdot\|_1$ on $C_1(X_0,\mathbf{R})$ is defined by $\left\|\sum_{e\in E^o} a_e e\right\|_1 = \sum_{e\in E^o} |a_e|$ ($E^o \subset E$ is an orientation, i.e., $E^o \cap \overline{E^o} = \emptyset$ and $E^o \cup \overline{E^o} = E$). Thus it is concluded that D is a convex polyhedron. The detailed combinatorial structure of the convex polyhedron D in the case of the maximal abelian covering graph was discussed in their paper.

The metric space $\Gamma \otimes \mathbf{R}$ with the distance $d_\infty(\xi_1,\xi_2) = \|\xi_1 - \xi_2\|_1$ turns out to coincide with the *Gromov-Hausdorff limit* of $(X, n^{-1}d)$ as n goes to infinity (M. Gromov [25]).

Now what about the large deviation asymptotic of $p(n,x,y)$? Their answer is the following. *There exists a convex function H on $\Gamma \otimes \mathbf{R}$ (possibly assuming $+\infty$) such that, for ξ in the interior of D, and $\{y_n\}_{n=1}^\infty$ in V such that $\{\Phi(y_n) - n\xi\}$ is bounded, we have*

$$\lim_{n\to\infty} \frac{1}{n}\log p(n,x,y_n) = -H(\xi),$$

and $D = \{\xi;\ H(\xi) < \infty\}$. The convex function H is explicitly described in terms of the maximal positive eigenvalues $\mu_0(\omega)$ of the twisted transition operators[17] L_ω defined by $L_\omega f(x) = \sum_{e\in E_{0x}} p(e)e^{\omega(e)} f\big(t(e)\big)$ ($\omega \in C^1(X_0,\mathbf{R})$). More precisely, if we think of μ_0 as a function on $H^1(X_0,\mathbf{R})$ using the fact that $L_{\omega+df} = e^{-f}L_\omega e^f$

[16]With a suitable modification, one may handle general symmetric and non-symmetric random walks.

[17]Actually, L_ω is linearly equivalent to the twisted operator associated with a real character of $H_1(X_0,\mathbf{Z})$.

($f \in C^0(X_0, \mathbf{R})$), then

$$H(\xi) = \sup_{\mathbf{x} \in \mathrm{Hom}(\Gamma, \mathbf{R})} \big[(\xi, \mathbf{x}) - \log \mu_0(\mathbf{x})\big]$$

(recall that $\mathrm{Hom}(\Gamma, \mathbf{R})$, the dual of $\Gamma \otimes \mathbf{R}$, is regarded as a subspace of $H^1(X_0, \mathbf{R})$).

See [S57] for an overview of discrete geometric analysis including Sunada's own work, which is based on his lectures given at Gregynog Hall, University of Wales in 2007, as an activity of the Project "Analysis on graphs and its applications" in the Isaac Newton Institute.

2.9. **Strongly isotropic crystals (a diamond twin).** A mathematician who declares himself to be a geometer surely would like to follow the tradition of ancient Greek mathematics; namely, the wish to classify beautiful figures as the Greek mathematicians did for regular polyhedra. This has been Sunada's desire since he was studying the standard realizations of crystal lattices.

The crystal he first looked at is the *diamond crystal* since the diamond as a real crystal is not only beautiful, but also has many interesting physical properties. He had thought that there should be some peculiar feature in its microscopic structure, and eventually found out that it possesses the following three properties:

(1) The diamond crystal has *maximal symmetry* in the sense that every automorphism of the diamond lattice as an abstract graph extends to a congruent transformation.

(2) It is *vertex-transitive* in the sense that the automorphism group acts transitively on the set of vertices.

(3) It has the *strong isotropic property* in the sense that every permutation of the edges with the same endpoint extends to an automorphism (thus extends to a congruent transformation which leaves the crystal invariant).

Sunada's desire was to list all crystals having these properties. His answer is stated in the following theorem.

Theorem 2.23. [S56] *A crystal having the properties* (1), (2), (3) *must be either the diamond crystal or the standard realization of the maximal abelian covering graph over the complete graph K_4.*

It is worthy to note that the diamond crystal is the standard realization of the maximal abelian covering graph over the graph with two vertices joined by 4 parallel edges, and thus (1) is a consequence of Theorem 2.21. The crystal mentioned in the above theorem, entitled to be called a *diamond twin* as a hypothetical crystal, is what Sunada called the K_4 *crystal*[18]. At the time when he was writing the paper [S56], he did not know that it has been known in crystallography. Actually, there is an interesting history about (re)discovery of this crystal.

It is believed that the crystallographer who discovered this crystal structure for the first time is Laves (1923). In 1954, A. F. Wells mentioned the structure and called it "Net 1f" in his article "The geometrical basis of crystal chemistry" appeared in Acta Crystallography. It is not definite whether he knew of Laves' work. In his book entitled "Three Dimensional Nets and Polyhedra" published in 1977, Wells renamed it "(10,3)-a". H. S. M. Coxeter [17] called it "Laves' graph of girth ten". M. O'Keeffe and his colleagues discussed this structure in some details

[18] The K_4 crystal has *chirality*.

and renamed it "srs" due to its chemical relevance in Acta Cryst. **A59** (2003), 22. Ludwig Danzer discovered the structure once again around 1994. John H. Conway calls this crystal the "triamond" (see the book [16] published recently).

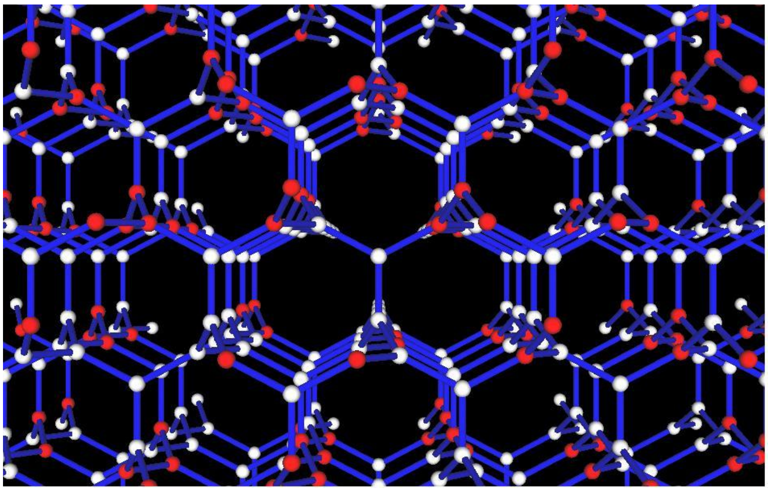

FIGURE 2. K_4 crystal (the image created by Hisashi Naito)

The remarkable feature of the K_4 crystal[19] has recently attracted the attention of researchers in the material science who are now evaluating its various physical properties by first principles calculations ([S58], [S63]). For instance, their calculations show that, as an sp^2 carbon crystal[20], the K_4 crystal has a meta-stable state, and therefore suggest that there is possible pressure which may induce structural phase transition from graphite to K_4. It is also observed that valence electrons are mainly localized along the bonding, which gives rise to the metallic property of this carbon crystal.

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From our description of Sunada's work, one can get some picture of how new concepts and ideas were formulated and how elegant techniques of proofs were generated, which therefore allows one to have a sense of the depth and beauty of his findings. More importantly, we notice that, though the topics Sunada has been concerned with are diverse, one can still find a consistent story in his study. His new awareness of mathematical issues has always been emerging from his previous work. This is one main reason why Sunada's work is considered highly original.

To this day, it is indeed more than a dream comes true for Sunada. For, his numerous contributions have had and will continue to make a significant impact on discrete geometric analysis, spectral geometry, dynamical systems, probability, and others in mathematics.

[19] Sunada's observation was reviewed by three magazines; namely, *Nature Materials*, *Science*, and *Scientific American*.

[20] The diamond is an sp^3 carbon crystal.

Once again, we congratulate Professor Sunada on his 60th birthday. May he have many more good years to come.

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Department of Mathematics, Okayama University, Okayama, Japan

Institute of Mathematics, University of the Philippines, Diliman, Philippines