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QUASI-SOCLE IDEALS AND GOTO NUMBERS OF PARAMETERS

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ABSTRACT. Goto numbers $\mathrm{g}(Q) = \max\{q \in \mathbb{Z} \mid Q : \mathfrak{m}^q \text{ is integral over } Q\}$ for certain parameter ideals Q in a Noetherian local ring $(A, \mathfrak{m})$ with Gorenstein associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ are explored. As an application, the structure of quasi-socle ideals $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ in a one-dimensional local complete intersection and the question of when the graded rings $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ are Cohen-Macaulay are studied in the case where the ideals I are integral over Q.

CONTENTS

1. Introduction and the main results

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A and let $q > 0$ be an integer. We put $I = Q : \mathfrak{m}^q$ and refer to those ideals as quasi-socle ideals in A. In this paper we are interested in the following question about quasi-socle ideals I, which are also the main subject of the researches [GMT, GKM, GKMP].

Question 1.1.

(1) Find the conditions under which $I \subseteq \overline{Q}$, where $\overline{Q}$ stands for the integral closure of Q.

(2) When $I \subseteq \overline{Q}$, estimate or describe the reduction number

$$\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} = QI^n\}$$

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of I with respect to Q in terms of some invariants of Q or A.

(3) Clarify what kind of ring-theoretic properties of the graded rings

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n, \ \mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}, \ \text{and } \ \mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$$

associated to the ideal I enjoy.

The present research is a continuation of [GMT, GKM, GKMP] and aims mainly at the analysis of the case where A is a complete intersection with $\dim A = 1$. Following W. Heinzer and I. Swanson [HS], for each parameter ideal Q in a Noetherian local ring $(A, \mathfrak{m})$ we define

$$\mathrm{g}(Q) = \max\{q \in \mathbb{Z} \mid Q : \mathfrak{m}^q \subseteq \overline{Q}\}$$

and call it the Goto number of Q. In the present paper we are also interested in computing Goto numbers $\mathrm{g}(Q)$ of parameter ideals. In [HS] one finds, among many interesting results, that if the base local ring $(A, \mathfrak{m})$ has dimension one, then there exists an integer $k \gg 0$ such that the Goto number $\mathrm{g}(Q)$ is constant for every parameter ideal Q contained in $\mathfrak{m}^k$. We will show that this is no more true, unless $\dim A = 1$, explicitly computing Goto numbers $\mathrm{g}(Q)$ for certain parameter ideals Q in a Noetherian local ring $(A, \mathfrak{m})$ with Gorenstein associated graded ring $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. However, before entering details, let us briefly explain the reasons why we are interested in Goto numbers and quasi-socle ideals as well.

The study of *socle* ideals $Q : \mathfrak{m}$ dates back to the research of L. Burch [B], where she explored certain socle ideals of finite projective dimension and gave a beautiful characterization of regular local rings (cf. [GH, Theorem 1.1]). More recently, A. Corso and C. Polini [CP1, CP2] studied, with interaction to the linkage theory of ideals, the socle ideals $I = Q : \mathfrak{m}$ of parameter ideals Q in a Cohen-Macaulay local ring $(A, \mathfrak{m})$ and showed that $I^2 = QI$, once A is *not* a regular local ring. Consequently the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ and the fiber cone $\mathrm{F}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$ are Cohen-Macaulay and so is the ring $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$, if $\dim A \geq 2$. The first author and H. Sakurai [GSa1, GSa2, GSa3] explored also the case where the base ring is not necessarily Cohen-Macaulay but Buchsbaum, and showed that the equality $I^2 = QI$ (here $I = Q : \mathfrak{m}$) holds true for numerous parameter ideals Q in a given Buchsbaum local ring $(A, \mathfrak{m})$, whence $\mathrm{G}(I)$ is a Buchsbaum ring, provided that $\dim A \geq 2$ or that $\dim A = 1$ but the multiplicity $\mathrm{e}(A)$ of A is not less than 2. Thus socle ideals $Q : \mathfrak{m}$ still enjoy very good properties even in the case where the base local rings are *not* Cohen-Macaulay.

However a more important fact is the following. If J is an equimultiple Cohen-Macaulay ideal of reduction number one in a Cohen-Macaulay local ring, the associated graded ring $\mathrm{G}(J) = \bigoplus_{n \geq 0} J^n/J^{n+1}$ of J is a Cohen-Macaulay ring and, so is the Rees algebra $\mathcal{R}(J) = \bigoplus_{n \geq 0} J^n$ of J, provided $\mathrm{ht}_A J \geq 2$. One knows the number and degrees of defining equations of $\mathcal{R}(J)$ also, which makes the process of desingularization of $\operatorname{Spec} A$ along the subscheme $\mathrm{V}(J)$ fairly explicit to understand. This observation motivated the ingenious research of C. Polini and B. Ulrich [PU], where they posed, among many important results, the following conjecture.

Conjecture 1.2 ([PU])**.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A \geq 2$. Assume that $\dim A \geq 3$ when A is regular. Let $q \geq 2$ be an integer and let Q be a parameter ideal in A such that $Q \subseteq \mathfrak{m}^q$. Then*

$$Q : \mathfrak{m}^q \subseteq \mathfrak{m}^q.$$

This conjecture was settled by H.-J. Wang [Wan], whose theorem says:

Theorem 1.3 ([Wan])**.** *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim A \geq 2$. Let $q \geq 1$ be an integer and Q a parameter ideal in A. Assume that $Q \subseteq \mathfrak{m}^q$ and put $I = Q : \mathfrak{m}^q$. Then*

$$I \subseteq \mathfrak{m}^q, \quad \mathfrak{m}^q I = \mathfrak{m}^q Q, \quad \text{and} \quad I^2 = QI,$$

provided that A is not regular if $d \geq 2$ and that $q \geq 2$ if $d \geq 3$.

The research of the first author, N. Matsuoka, and R. Takahashi [GMT] reported a different approach to the Polini-Ulrich conjecture. They proved the following.

Theorem 1.4 ([GMT])**.** *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$ and $\mathrm{e}(A) \geq 3$, where $\mathrm{e}(A)$ denotes the multiplicity of A. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^2$. Then $\mathfrak{m}^2 I = \mathfrak{m}^2 Q$, $I^3 = QI^2$, and $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring, so that $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ is also a Cohen-Macaulay ring, provided $d \geq 3$.*

The researches [Wan] and [GMT] are performed independently and their methods of proof are totally different from each other's. The technique of [GMT] can not go beyond the restrictions that A is a Gorenstein ring, $q = 2$, and $\mathrm{e}(A) \geq 3$. However, despite these restrictions, the result [GMT, Theorem 1.1] holds true even in the case where $\dim A = 1$, while Wang's result says nothing about the case where $\dim A = 1$. As is suggested in [GMT], the one-dimensional case is substantially different from higher-dimensional cases and more complicated to control. This observation has led S. Goto, S. Kimura, N. Matsuoka, and T. T. Phuong to the researches [GKM] (resp. [GKMP]), where they have explored quasi-socle ideals in Gorenstein numerical semigroup rings over fields (resp. the case where $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a Gorenstein ring and $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ $(a_i \geq 1)$ are diagonal parameter ideals in A, that is $x_1, x_2, \cdots, x_d$ is a system of parameters in A which generates a reduction of the maximal ideal $\mathfrak{m}$). The present research is a continuation of [GMT, GKM, GKMP] and the main purpose is to go beyond the restriction in [GKMP] that the parameter ideals $Q = (x_1^{a_1}, x_2^{a_2}, \cdots, x_d^{a_d})$ are *diagonal* and the assumption in [GKM] that the parameter ideals are *monomial*.

To state the main results of the present paper, let us fix some notation. Let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1 \leq i \leq d}$ be positive integers and let $\{x_i\}_{1 \leq i \leq d}$ be elements of A with $x_i \in \mathfrak{m}^{a_i}$ for each $1 \leq i \leq d$ such that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1 \leq i \leq d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Hence $\mathfrak{m}^\ell = \sum_{i=1}^d x_i \mathfrak{m}^{\ell - a_i}$ for $\ell \gg 0$, so that $Q = (x_1, x_2, \cdots, x_d)$ is a parameter ideal in A. Let $q \in \mathbb{Z}$, $I = Q : \mathfrak{m}^q$,

$$\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^d a_i, \quad \text{and} \quad \ell = \rho + 1 - q,$$

where $\mathrm{a}(*)$ denote the a-invariants of graded rings ([GW, (3.1.4)]). We put

$$\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\} \quad \text{and} \quad \ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}.$$

With this notation our main result is sated as follows.

Theorem 1.5. *Suppose that* $\mathrm{G}(\mathfrak{m}) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ *is a Cohen-Macaulay ring and consider the following four conditions:*

(1) $\ell_1 \geq a_i$ *for all* $1 \leq i \leq d$.

(2) $I \subseteq \overline{Q}$.

(3) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(4) $\ell_2 \geq a_i$ *for all* $1 \leq i \leq d$.

Then one has the implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). *If* $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring, then one has the equality* $I = Q + \mathfrak{m}^\ell$, *so that* $\ell_1 \leq \ell \leq \ell_2$, *whence conditions* (1), (2), (3), *and* (4) *are equivalent to the following*:

(5) $\ell \geq a_i$ *for all* $1 \leq i \leq d$.

Consequently, the Goto number $\mathrm{g}(Q)$ *of* Q *is given by the formula*

$$\mathrm{g}(Q) = \left\lceil \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1 \right\rceil - \max\{a_i \mid 1 \leq i \leq d\},$$

provided $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring; in particular* $\mathrm{g}(Q) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1$, *if* $d = 1$.

Let $R = k[R_1]$ be a homogeneous ring over a filed k with $d = \dim R > 0$. We choose a homogeneous system $f_1, f_2, \cdots, f_d$ of parameters of R and put $\mathfrak{q} = (f_1, f_2, \cdots, f_d)$. Let $M = R_+$. Then, applying Theorem 1.5 to the local ring $A = R_M$, we readily get the following, where $\mathrm{g}(\mathfrak{q}) = \max\{n \in \mathbb{Z} \mid \mathfrak{q} : M^n \text{ is integral over } \mathfrak{q}\}$.

Corollary 1.6. *Suppose that* R *is a Gorenstein ring. Then*

$$\mathrm{g}(\mathfrak{q}) = \left\lceil \mathrm{a}(R) + \sum_{i=1}^{d} \deg f_i + 1 \right\rceil - \max\{\deg f_i \mid 1 \leq i \leq d\}.$$

Hence $\mathrm{g}(\mathfrak{q}) = \mathrm{a}(R) + 1$, *if* $d = 1$.

Corollary 1.7. *With the same notation as is in* Theorem 1.5 *let* $d = 1$ *and put* $a = a_1$. *Assume that* $\mathrm{G}(\mathfrak{m})$ *is a reduced ring. Then the following conditions are equivalent to each other.*

(1) $I \subseteq \overline{Q}$.

(2) $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) $I \subseteq \mathfrak{m}^a$.

(4) $\ell_2 \geq a$.

Later we will give some applications of these results. So, we are now in a position to explain how this paper is organized. Theorem 1.5 will be proven in Section 2. Once we have proven Theorem 1.5, exactly the same technique as is developed by [GKMP] works to get a complete answer to Question 1.1 in the case where $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring and Q is a parameter ideal given in Theorem 1.5, which we shall briefly discuss in Section 2.

Sections 3 and 4 are devoted to the analysis of quasi-socle ideals in the ring A of the form $A = B/yB$, where y is subsystem of parameters in a Cohen-Macaulay local ring $(B, \mathfrak{n})$ of dimension 2. Here we notice that this class of local rings contains all the local complete intersections of dimension one. In Section 3 (resp. Section 4) we focus our attention on the case where B is *not* a regular local ring (resp. B is a regular local ring), and our results are summarized into Theorems 3.1 and 4.1. The proofs given in Sections 3 and 4 are based on the beautiful method developed by Wang [Wan] in higher dimensional cases and similar to each other, but the techniques are substantially different, depending on the assumptions that B is a regular local ring or not. In Sections 3 and 4 we shall give a careful description of the reason why such a difference should occur. In the final Section 5 we explore, in order to see how effectively our theorems work in the analysis of concrete examples, the numerical semigroup rings $A = k[[t^{6n+5}, t^{6n+8}, t^{9n+12}]]$ $(\subseteq k[[t]])$, where $n \geq 0$ are integers and $k[[t]]$ is the formal power series ring over a field k. Here we note

$$A \cong k[[X, Y, Z]]/(Y^3 - Z^2,\ X^{3n+4} - Y^{3n+1}Z) \text{ and}$$

$$\mathrm{G}(\mathfrak{m}) \cong k[X, Y, Z]/(Y^{3n+4}, Y^{3n+1}Z, Z^2),$$

where $k[[X, Y, Z]]$ denotes the formal powers series ring over the field k. Hence A is a local complete intersection with $\dim A = 1$, whose associated graded ring $\mathrm{G}(\mathfrak{m})$ is not a Gorenstein ring but Cohen-Macaulay.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be Noetherian local ring with $d = \dim A > 0$. We denote by $\mathrm{e}(A) = \mathrm{e}^0_{\mathfrak{m}}(A)$ the multiplicity of A with respect to the maximal ideal $\mathfrak{m}$. Let $J \subseteq K$ $(\subsetneq A)$ be ideals in A. We denote by $\overline{J}$ the integral closure of J. When $K \subseteq \overline{J}$, let

$$\mathrm{r}_J(K) = \min\,\{n \in \mathbb{Z} \mid K^{n+1} = JK^n\}$$

denote the reduction number of K with respect to J. For each finitely generated A-module M let $\mu_A(M)$ and $\ell_A(M)$ be the number of elements in a minimal system of generators for M and the length of M, respectively. We denote by $\mathrm{v}(A) = \ell_A(\mathfrak{m}/\mathfrak{m}^2)$ the embedding dimension of A.

2. The case where $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring

The purpose of this section is to prove Theorem 1.5. Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\{a_i\}_{1 \le i \le d}$ be positive integers and let $\{x_i\}_{1 \le i \le d}$ be elements of A such that $x_i \in \mathfrak{m}^{a_i}$ for each $1 \le i \le d$. Assume that the initial forms $\{x_i \bmod \mathfrak{m}^{a_i+1}\}_{1 \le i \le d}$ constitute a homogeneous system of parameters in $\mathrm{G}(\mathfrak{m})$. Let $q \in \mathbb{Z}$ and $Q = (x_1, x_2, \cdots, x_d)$. We put $I = Q : \mathfrak{m}^q$.

Let us begin with the following.

Proposition 2.1. *Let $\ell_3 \in \mathbb{Z}$ and suppose that $\mathfrak{m}^{\ell_3} \subseteq \overline{Q}$. Then $\ell_3 \ge a_i$ for all $1 \le i \le d$.*

Proof. Assume that $\mathfrak{m}^{\ell_3} \subseteq \overline{Q}$ with $\ell_3 \in \mathbb{Z}$. Then $\ell_3 > 0$. We want to show $\ell_3 \ge \max\{a_i \mid 1 \le i \le d\}$. Assume the contrary and let x be an arbitrary element of $\mathfrak{m}$ and put $y = x^{\ell_3}$. Then since y is integral over Q, we have an equation

$$y^n + c_1 y^{n-1} + \cdots + c_n = 0$$

with $n > 0$ and $c_i \in Q^i$ for all $1 \le i \le n$. We put $a = \max\{a_i \mid 1 \le i \le d\}$ (hence $\ell_3 < a$) and let $a = a_u$ with $1 \le u \le d$. Let $B = A/(x_i \mid 1 \le i \le d, i \ne u)$ and $\mathfrak{n} = \mathfrak{m}B$. Let $\overline{*}$ denote the image in B. Then

$$\overline{y}^n + \overline{c_1}\overline{y}^{n-1} + \cdots + \overline{c_n} = 0$$

in B. Therefore, because $i\ell_3 < ia$ and $\overline{c_i} \in Q^i B = \overline{x_u^i} B \subseteq \mathfrak{n}^{ia}$, we get $\overline{c_i} \in \mathfrak{n}^{i\ell_3+1}$ for all $1 \le i \le n$. Consequently, $\overline{c_i}\ \overline{y}^{n-i} \in \mathfrak{n}^{i\ell_3+1}\mathfrak{n}^{(n-i)\ell_3} = \mathfrak{n}^{n\ell_3+1}$, so that we have $\overline{y}^n = \overline{x^{n\ell_3}} \in \mathfrak{n}^{n\ell_3+1}$. Hence, for every $z \in \mathfrak{n}$, the initial form $z \bmod \mathfrak{n}^2$ of z is nilpotent in the associated graded ring $\mathrm{G}(\mathfrak{n}) = \bigoplus_{n \ge 0} \mathfrak{n}^n/\mathfrak{n}^{n+1}$, which is impossible, because $\dim \mathrm{G}(\mathfrak{n}) = \dim B = 1$. Thus $\ell_3 \ge a_i$ for all $1 \le i \le d$. $\square$

We put $\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i$ (cf. [GW, (3.1.6)]) and $\ell = \rho + 1 - q$. Let $\ell_1 = \inf\{n \in \mathbb{Z} \mid \mathfrak{m}^n \subseteq I\}$ and $\ell_2 = \sup\{n \in \mathbb{Z} \mid I \subseteq Q + \mathfrak{m}^n\}$.

We are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. (4) ⇒ (3) We may assume $\ell_2 < \infty$. Then, since $I \subseteq Q + \mathfrak{m}^{\ell_2}$, we have $\mathfrak{m}^q I \subseteq \mathfrak{m}^q Q + \mathfrak{m}^{q+\ell_2}$, whence $\mathfrak{m}^q I = \mathfrak{m}^q Q + [Q \cap \mathfrak{m}^{q+\ell_2}]$. Notice that

$$Q \cap \mathfrak{m}^{q+\ell_2} = \sum_{i=1}^{d} x_i \mathfrak{m}^{q+\ell_2-a_i},$$

because the initial forms $\{x_i \mod \mathfrak{m}^{a_i+1}\}_{1\leq i\leq d}$ constitute a homogeneous system of parameters in the Cohen-Macaulay ring $\mathrm{G}(\mathfrak{m})$, and we have $\mathfrak{m}^{q+\ell_2-a_i} \subseteq \mathfrak{m}^q$, since $\ell_2 \geq a_i$ for all $1 \leq i \leq d$. Thus $\mathfrak{m}^q I = \mathfrak{m}^q Q$.

(3) ⇒ (2) See [NR, Section 7, Theorem 2].

(2) ⇒ (1) This follows from Proposition 2.1.

We now assume that $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring. Then $I = Q + \mathfrak{m}^\ell$ by [Wat] (see [O, Theorem 1.6] also), whence $\ell_1 \leq \ell \leq \ell_2$, so that the implication (1) ⇒ (4) follows. Therefore, $I \subseteq \overline{Q}$ if and only if $\ell = \rho + 1 - q \geq a_i$ for all $1 \leq i \leq d$, or equivalently

$$q \leq \left\lceil \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1 \right\rceil - \max\{a_i \mid 1 \leq i \leq d\}.$$

Thus $\mathrm{g}(Q) = \left\lceil \mathrm{a}(\mathrm{G}(\mathfrak{m})) + \sum_{i=1}^{d} a_i + 1 \right\rceil - \max\{a_i \mid 1 \leq i \leq d\}$, so that

$$\mathrm{g}(Q) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1,$$

if $d = 1$. □

Remark 2.2 (cf. Example 5.3)**.** Unless $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring, the implication (1) ⇒ (4) in Theorem 1.5 does not hold true in general, even though A is a complete intersection and $\mathrm{G}(\mathfrak{m})$ is a Cohen-Macaulay ring. For example, let $V = k[[t]]$ be the formal power series ring over a field k and look at the numerical semigroup ring $A = k[[t^5, t^8, t^{12}]] \subseteq V$. Then $A \cong k[[X, Y, Z]]/(Y^3 - Z^2, X^4 - YZ)$, while $\mathrm{G}(\mathfrak{m}) \cong k[X, Y, Z]/(Y^4, YZ, Z^2)$, whence $\mathrm{G}(\mathfrak{m})$ is a Cohen-Macaulay ring but not a Gorenstein ring. Let $Q = (t^{20})$ in A and let $I = Q : \mathfrak{m}^3$; hence $a_1 = 4$ and $q = 3$. Then $I = (t^{20}, t^{22}, t^{23}, t^{26}, t^{29}) \subseteq \mathfrak{m}^3$ and $I^3 = QI^2$, so that $I \subseteq \overline{Q}$, while $I^2 = QI + (t^{44}) \subseteq Q$ but $t^{44} \notin QI$, since $t^{24} \notin I$. Thus $I^2 = Q \cap I^2 \neq QI$, so that $\mathrm{r}_Q(I) = 2$ and the ring $\mathrm{G}(I)$ is not Cohen-Macaulay. It is direct to check that $\mathfrak{m}^4 \subseteq I$, $\mathfrak{m}^3 \not\subseteq I$, and $I \not\subseteq Q + \mathfrak{m}^4 = \mathfrak{m}^4$ since $t^{22} \in I$ but $t^{22} \notin \mathfrak{m}^4$. Thus $\ell_1 = 4$ and $\ell_2 = 3$.

Proof of Corollary 1.7. Since $Q \subseteq \mathfrak{m}^a$, we readily get the equivalence (3) $\Leftrightarrow$ (4). We also have $\overline{\mathfrak{m}^a} = \mathfrak{m}^a$, because the ring $\mathrm{G}(\mathfrak{m})$ is reduced. Hence $\overline{Q} \subseteq \mathfrak{m}^a$. Therefore $I \subseteq \mathfrak{m}^a$, if $I \subseteq \overline{Q}$. Thus all conditions (1), (2), (3), and (4) are, by Theorem 1.5, equivalent to each other. $\square$

Thanks to Theorem 1.5, similarly as in [GKMP] we have the following complete answer to Question 1.1 for the parameter ideals $Q = (x_1, x_2, \cdots, x_d)$. We later need it in the present paper. Let us note a brief proof.

Theorem 2.3. *With the same notation as is in* Theorem1.5 *assume that* $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring. Suppose that* $\ell \geq a_i$ *for all* $1 \leq i \leq d$. *Then the following assertions hold true.*

(1) $\mathrm{G}(I)$ *is a Cohen-Macaulay ring,* $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$, *and* $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$, *where* $\lceil \frac{q}{\ell} \rceil = \min\{n \in \mathbb{Z} \mid \frac{q}{\ell} \leq n\}$.
(2) $\mathrm{F}(I)$ *is a Cohen-Macaulay ring.*
(3) $\mathcal{R}(I)$ *is a Cohen-Macaulay ring if and only if* $q \leq (d-1)\ell$.
(4) *Suppose that* $q > 0$. *Then* $\mathrm{G}(I)$ *is a Gorenstein ring if and only if* $\ell \mid q$.
(5) *Suppose that* $q > 0$. *Then* $\mathcal{R}(I)$ *is a Gorenstein ring if and only if* $q = (d-2)\ell$.

To prove Proposition 2.3 we need the following. We skip the proof, since one can prove it exactly in the same way as is given in [GKMP, Lemma 2.2].

Lemma 2.4 (cf. [GKMP, Lemma 2.2])**.** *With the same notation as is in* Theorem1.5 *assume that* $\mathrm{G}(\mathfrak{m})$ *is a Gorenstein ring. If* $\ell \geq a_i$ *for all* $1 \leq i \leq d$, *then*

$$Q \cap \mathfrak{m}^{(n+1)\ell+m} \subseteq \mathfrak{m}^m Q I^n$$

for all integers $m, n \geq 0$.

Proof of Theorem 2.3. (1) Let $n \geq 0$ be an integer. Then, since $I = Q + \mathfrak{m}^\ell$, we get $I^{n+1} = QI^n + \mathfrak{m}^{(n+1)\ell}$, so that

$$Q \cap I^{n+1} = QI^n + [Q \cap \mathfrak{m}^{(n+1)\ell}] \subseteq QI^n,$$

because $Q \cap \mathfrak{m}^{(n+1)\ell} \subseteq QI^n$ by Lemma 2.4. Therefore $Q \cap I^{n+1} = QI^n$ for all $n \geq 0$, so that $\mathrm{G}(I)$ is a Cohen-Macaulay ring and $\mathrm{r}_Q(I) = \min\{n \in \mathbb{Z} \mid I^{n+1} \subseteq Q\}$. Let $n \in \mathbb{Z}$

and suppose that $I^{n+1} \subseteq Q$. Then $\mathfrak{m}^{(n+1)\ell} \subseteq Q$, whence $(n+1)\ell \geq \rho + 1$ (recall that $\rho = \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q))$. Therefore

$$n + 1 \geq \frac{\rho + 1}{\ell} = \frac{q + \ell}{\ell} = \frac{q}{\ell} + 1,$$

so that $n \geq \frac{q}{\ell}$. Conversely, if $n \geq \frac{q}{\ell}$, then $(n+1)\ell \geq (\frac{q}{\ell} + 1)\ell = q + \ell = \rho + 1$, whence $\mathfrak{m}^{(n+1)\ell} \subseteq Q$, so that $I^{n+1} \subseteq Q$. Thus $\mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

Let Y_i's be the initial forms of x_i's with respect to I. Then $Y_1, Y_2, \cdots, Y_d$ is a homogeneous system of parameters of $\mathrm{G}(I)$, whence it constitutes a regular sequence in $\mathrm{G}(I)$. Therefore

$$\mathrm{G}(\overline{I}) \cong \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d)$$

as graded A-algebras ([VV]), where $\overline{I} = I/Q$. Hence $\mathrm{a}(\mathrm{G}(\overline{I})) = \mathrm{a}(\mathrm{G}(I)) + d$ (cf. [GW, (3.1.6)]). Thus $\mathrm{a}(\mathrm{G}(I)) = \lceil \frac{q}{\ell} \rceil - d$, since $\mathrm{a}(\mathrm{G}(\overline{I})) = \mathrm{r}_Q(I) = \lceil \frac{q}{\ell} \rceil$.

(2) By Lemma 2.4

$$\begin{aligned} Q \cap \mathfrak{m}I^{n+1} &= Q \cap [\mathfrak{m}QI^n + \mathfrak{m}^{(n+1)\ell+1}] \\ &= \mathfrak{m}QI^n + [Q \cap \mathfrak{m}^{(n+1)\ell+1}] \\ &\subseteq \mathfrak{m}QI^n. \end{aligned}$$

Hence $Q \cap \mathfrak{m}I^{n+1} = \mathfrak{m}QI^n$ for all $n \geq 0$. Thus $\mathrm{F}(I)$ is a Cohen-Macaulay ring (cf. e.g., [CGPU, CZ]; recall that $\mathrm{G}(I)$ is a Cohen-Macaulay ring).

(3) The Rees algebra $\mathcal{R}(I)$ of I is a Cohen-Macaulay ring if and only if $\mathrm{G}(I)$ is a Cohen-Macaulay ring and $a(\mathrm{G}(I)) < 0$ ([GSh, Remark (3.10)], [TI]). By assertion (1) the latter condition is equivalent to saying that $\lceil \frac{q}{\ell} \rceil < d$, or equivalently $q \leq (d-1)\ell$.

(4) Notice that $\mathrm{G}(I)$ is a Gorenstein ring if and only if so is the graded ring

$$\mathrm{G}(\overline{I}) = \mathrm{G}(I)/(Y_1, Y_2, \cdots, Y_d).$$

Let $r = \mathrm{r}_Q(I)$ $(= \lceil \frac{q}{\ell} \rceil)$. Then $\mathrm{G}(\overline{I})$ is a Gorenstein ring if and only if $(0) : \overline{I}^i = \overline{I}^{r+1-i}$ for all $i \in \mathbb{Z}$ (cf. [O, Theorem 1.6]). Therefore, if $\mathrm{G}(I)$ is a Gorenstein ring, we have $(0) : \overline{I} = \overline{I}^r = \overline{\mathfrak{m}}^{r\ell}$, where $\overline{\mathfrak{m}} = \mathfrak{m}/Q$. On the other hand, since $\overline{I} = \overline{\mathfrak{m}}^\ell$ and $q = \rho + 1 - \ell$, we get

$$(0) : \overline{I} = (0) : \overline{\mathfrak{m}}^\ell = \overline{\mathfrak{m}}^q$$

by [Wat] (see [O, Theorem 1.6] also). Hence $q = r\ell$, because $\overline{\mathfrak{m}}^{r\ell} = \overline{\mathfrak{m}}^{q} \neq (0)$ and $q > 0$. Thus $\ell \mid q$ and $r = \frac{q}{\ell}$. Conversely, suppose that $\ell \mid q$; hence $r = \frac{q}{\ell}$. Let $i \in \mathbb{Z}$. Then since $\overline{I} = \overline{\mathfrak{m}}^{\ell}$, we get $\overline{I}^{r+1-i} = \overline{\mathfrak{m}}^{(r+1-i)\ell}$, while

$$(0) : \overline{I}^{i} = (0) : \overline{\mathfrak{m}}^{i\ell} = \overline{\mathfrak{m}}^{\rho+1-i\ell}$$

by [O, Theorem 1.6]. Hence $(0) : \overline{I}^{i} = \overline{I}^{r+1-i}$ for all $i \in \mathbb{Z}$, because

$$(r+1-i)\ell = q + \ell - i\ell = \rho + 1 - i\ell.$$

Thus $\mathrm{G}(\overline{I})$ is a Gorenstein ring, whence so is $\mathrm{G}(I)$.

(5) The Rees algebra $\mathcal{R}(I)$ of I is a Gorenstein ring if and only if $\mathrm{G}(I)$ is a Gorenstein ring and $a(\mathrm{G}(I)) = -2$, provided $d \geq 2$ ([I, Corollary (3.7)]). Suppose that $\mathcal{R}(I)$ is a Gorenstein ring. Then $d \geq 2$ by assertion (2) (recall that $q > 0$). Since $a(\mathrm{G}(I)) = \mathrm{r}_Q(I) - d = -2$, thanks to assertions (1) and (4), we have $\frac{q}{\ell} = \mathrm{r}_Q(I) = d - 2$, whence $q = (d-2)\ell$. Conversely, suppose that $q = (d-2)\ell$. Then $d \geq 3$, since $q > 0$. By assertions (1) and (4), $\mathrm{G}(I)$ is a Gorenstein ring with $\mathrm{r}_Q(I) = \frac{q}{\ell} = d - 2$, whence $a(\mathrm{G}(I)) = (d-2) - d = -2$. Thus $\mathcal{R}(I)$ is a Gorenstein ring. □

We now discuss Goto numbers. For each Noetherian local ring A let

$$\mathcal{G}(A) = \{\mathrm{g}(Q) \mid Q \text{ is a parameter ideal in } A\}.$$

We explore the value $\min \mathcal{G}(A)$ in the setting of Theorem 1.5 with $\dim A = 1$. For the purpose the following result is fundamental.

Theorem 2.5 ([HS, Theorem 3.1])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension one. Then there exists an integer $k \gg 0$ such that $g(Q) = \min \mathcal{G}(A)$ for every parameter ideal Q of A contained in $\mathfrak{m}^k$.*

Thanks to Theorem 1.5 and Theorem 2.5, we then have the following.

Corollary 2.6. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = 1$. Then $\min \mathcal{G}(A) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1$, if $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring.*

We close this section with the following.

Proposition 2.7. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim A = 1$. Then $\mathrm{v}(A) \leq 2$ if and only if $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$.*

Proof. Suppose that $\mathrm{v}(A) \leq 2$. Then $\mathrm{G}(\mathfrak{m})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathfrak{m})) = \mathrm{e}(A) - 2$. Hence $\min \mathcal{G}(A) = \mathrm{a}(\mathrm{G}(\mathfrak{m})) + 1 = \mathrm{e}(A) - 1$ by Corollary 2.6. Conversely, assume that $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$. To prove the assertion, enlarging the field $A/\mathfrak{m}$ if necessary, we may assume that the field $A/\mathfrak{m}$ is infinite (use Theorem 2.5). Let $x \in \mathfrak{m}$ and assume that $Q = (x)$ is a reduction of $\mathfrak{m}$. We put $e = \mathrm{e}(A)$ and $q = \mathrm{g}(Q)$. Then $q \geq e - 1$. Let $B = A/Q$ and $\mathfrak{n} = \mathfrak{m}/Q$. Then $Q : \mathfrak{m}^q \subseteq \overline{Q} \subsetneq A$. Hence $\mathfrak{n}^q \neq (0)$, so that $\mathfrak{n}^i \neq \mathfrak{n}^{i+1}$ for any $0 \leq i \leq q$. Consequently, because $q + 1 \geq e$ and

$$e = \ell_A(A/Q) = \sum_{i \geq 0} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq \sum_{i=0}^{q} \ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) \geq q + 1,$$

we get $\mathfrak{n}^{q+1} = (0)$ and $\ell_A(\mathfrak{n}^i/\mathfrak{n}^{i+1}) = 1$ for all $0 \leq i \leq q$. Hence $\ell_A(\mathfrak{n}/\mathfrak{n}^2) \leq 1$, so that $\mathrm{v}(A) \leq 2$. $\square$

3. The case where $A = B/yB$ and B is not a regular local ring

Let us now explore quasi-socle ideals in the ring A of the form $A = B/yB$, where $(B, \mathfrak{n})$ is a Cohen-Macaulay local ring of dimension 2 and y is a subsystem of parameters in B. Recall that this class of local rings contains all the local complete intersections of dimension one.

In this section we assume that B is *not* a regular local ring and our goal is the following.

Theorem 3.1. *Let $(B, \mathfrak{n})$ be a Cohen-Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let n, q be integers such that $n \geq q > 0$. Let $y \in \mathfrak{n}^n$ and assume that y is regular in B. We put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true, where $m = n - q$.*

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q, I \subseteq \overline{Q}$, *and* $Q \cap I^2 = QI$. *Hence* $\mathrm{g}(Q) \geq n$.

(2) $I^2 = QI$, *if one of the following conditions is satisfied.*

- (i) $m \geq q - 1$;
- (ii) $m < q - 1$ *and* $Q \subseteq \mathfrak{m}^{q-m}$;
- (iii) $m > 0$ *and* $Q \subseteq \mathfrak{m}^{q-1}$.

(3) *Suppose that B is a Gorenstein ring. Then $I^3 = QI^2$ and $\mathrm{G}(I)$ is a Cohen-Macaulay ring, if one of the following conditions is satisfied.*
(i) $m < q-1$ *and* $Q \subseteq \mathfrak{m}^{q-(m+1)}$;
(ii) $Q \subseteq \mathfrak{m}^{q-1}$.

We begin with the following.

Lemma 3.2. *Let $(B, \mathfrak{n})$ be a Cohen-Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let q, ℓ, and m be integers such that $q \geq \ell > 0$ and $m \geq 0$. Let $x \in \mathfrak{n}^\ell$ and $y_i \in \mathfrak{n}$ $(1 \leq i \leq q+m)$ and assume that for all $1 \leq i \leq q+m$, the sequence x, y_i is B-regular. Then we have*

$$(x, \prod_{i=1}^{q+m} y_i) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}.$$

Proof. Let $\alpha \in (x, \prod_{i=1}^{q+m} y_i) : \mathfrak{n}^q$ and write $\alpha \cdot \prod_{i=1}^{q} y_i = ux + v \cdot \prod_{i=1}^{q+m} y_i$ with $u, v \in B$. Then, since

$$(\alpha - v \cdot \prod_{i=q+1}^{q+m} y_i) \cdot \prod_{i=1}^{q} y_i \in (x)$$

and since $x, \prod_{i=1}^{q} y_i$ is a B-regular sequence, we get $\alpha - v \cdot \prod_{i=q+1}^{q+m} y_i \in (x)$. Let us write

$$\alpha = wx + v \cdot \prod_{i=q+1}^{q+m} y_i$$

with $w \in B$. We want to show $v \in \mathfrak{n}^\ell$. Let $z \in \mathfrak{n}^\ell$ and write

$$\alpha z \cdot \prod_{i=1}^{q-\ell} y_i = u'x + v' \cdot \prod_{i=1}^{q+m} y_i$$

with $u', v' \in B$. Then, since

$$\alpha z \cdot \prod_{i=1}^{q-\ell} y_i = wxz \cdot \prod_{i=1}^{q-\ell} y_i + vz \cdot \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i,$$

we have

$$(vz - v' \cdot \prod_{i=q-\ell+1}^{q} y_i) \cdot \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i \in (x).$$

Therefore, since the sequence $x, \prod_{i=1}^{q-\ell} y_i \cdot \prod_{i=q+1}^{q+m} y_i$ is B-regular, we see $vz \in (x, \prod_{i=q-\ell+1}^{q} y_i)$, so that $v \in (x, \prod_{i=q-\ell+1}^{q} y_i) : \mathfrak{n}^\ell$, because z is an arbitrary element in $\mathfrak{n}^\ell$. We now notice that $\mathfrak{q} = (x, \prod_{i=q-\ell+1}^{q} y_i)$ is a parameter ideal in B such that

$\mathfrak{q} \subseteq \mathfrak{n}^\ell$. Then, since B is not a regular local ring, we have $\mathfrak{q} : \mathfrak{n}^\ell \subseteq \mathfrak{n}^\ell$, thanks to [Wan, Theorem 1.1]. Thus $v \in \mathfrak{n}^\ell$, whence $\alpha \in (x) + \mathfrak{n}^{\ell+m}$. □

Proposition 3.3. *Let $(B, \mathfrak{n})$ be a Cohen-Macaulay local ring of dimension 2 and assume that B is not a regular local ring. Let q, ℓ, and m be integers such that $q \geq \ell > 0$ and $m \geq 0$. Let $x, y \in B$ be a system of parameters of B and assume that $x \in \mathfrak{n}^\ell$ and $y \in \mathfrak{n}^{q+m}$. Then*

(1) $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$.

(2) $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$.

Proof. (1) We notice that the ideal $\mathfrak{n}^k$ is, for each integer $k > 0$, generated by the set

$$F_k = \{\prod_{i=1}^{k} z_i \mid z_i \in \mathfrak{n} \text{ and } x, z_i \text{ is a system of parameters of } B \text{ for all } 1 \leq i \leq k\}.$$

Let $\alpha \in (x, y) : \mathfrak{n}^q$. Let $z \in F_{q+m}$ and $z' \in F_q$ and write

$$z\alpha = ux + vy,$$

$$z'\alpha = u'x + v'y$$

with $u, v, u', v' \in B$. Then $z'z\alpha = z'ux + z'vy = zu'x + zv'y$, whence $y(z'v - zv') \in (x)$, so that $z'v \in (x, z)$, because the sequence x, y is B-regular. Since z' is an arbitrary element of F_k which generates the ideal $\mathfrak{n}^q$, we have

$$v \in (x, z) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$$

by Lemma 3.2. Hence $z\alpha = ux + vy \in (x) + \mathfrak{n}^{\ell+m}y$, so that

$$\alpha \in [(x) + \mathfrak{n}^{\ell+m}y] : \mathfrak{n}^{q+m},$$

because z is an arbitrary element of F_{q+m}. Since $y \in \mathfrak{n}^{q+m}$, we then have

$$y\alpha = \rho x + \tau y$$

with $\rho \in B$ and $\tau \in \mathfrak{n}^{\ell+m}$. Therefore $\alpha - \tau \in (x)$, so that $\alpha \in (x) + \mathfrak{n}^{\ell+m}$. Thus $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m}$.

(2) The ideal $\mathfrak{n}^q$ is generated by the set

$$F = \{z \in \mathfrak{n}^q \mid y, z \text{ is a system of parameters in } B\}.$$

Let $\alpha \in (x,y) : \mathfrak{n}^q$ and $z, z' \in F$. We write $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$ with $u, v, u', v' \in B$. We want to show $ux \in \mathfrak{n}^q x$. Since $z'z\alpha = z'ux + z'vy = zu'x + zv'y$, we have $x(z'u - zu') \in (y)$, whence $z'u \in (z, y)$. Therefore $u \in (z,y) : \mathfrak{n}^q$, whence $u \in (z) + \mathfrak{n}^{q+m}$, because $(z,y) : \mathfrak{n}^q \subseteq (z) + \mathfrak{n}^{q+m}$ by assertion (1) (take $x = z$, and $\ell = q$). Thus $ux \in (zx) + \mathfrak{n}^{q+m}x \subseteq \mathfrak{n}^q x$, whence $\mathfrak{n}^q \cdot [(x,y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$. $\square$

We need also the following result to prove Theorem 3.1.

Proposition 3.4. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with $d = \dim A > 0$. Let Q be a parameter ideal in A and $q > 0$ an integer. We put $I = Q : \mathfrak{m}^q$. Then $I^3 = QI^2$ and $\mathrm{G}(I)$ is a Cohen-Macaulay ring, if $I \subseteq Q + \mathfrak{m}^{q-1}$ and $\mathfrak{m}^q I = \mathfrak{m}^q Q$.*

Proof. We have $\mathfrak{m}^q I^i = \mathfrak{m}^q Q^i$ and $Q^i \cap I^{i+1} = Q^i I$ for all $i \geq 1$ (cf. [GMT, Corollary 2.3]). Therefore, since $Q \cap I^2 = QI$, we may assume that $I^2 \not\subseteq Q$. Notice that $\mathfrak{m}I^2 = \mathfrak{m}I{\cdot}I \subseteq (Q + \mathfrak{m}^q){\cdot}I \subseteq Q$ and we have $I^2 \subseteq Q : \mathfrak{m}$. Hence $Q : \mathfrak{m} = Q + I^2$, because A is a Gorenstein ring. We similarly have $\mathfrak{m}I^3 \subseteq \mathfrak{m}I{\cdot}I^2 \subseteq (\mathfrak{m}Q + \mathfrak{m}^q){\cdot}I^2 = \mathfrak{m}I^2{\cdot}Q + \mathfrak{m}^q I^2 \subseteq Q^2$, so that $I^3 \subseteq Q^2 : \mathfrak{m} = Q{\cdot}[Q : \mathfrak{m}] = Q^2 + QI^2$. Therefore $I^3 = [Q^2 + QI^2] \cap I^3 = [Q^2 \cap I^3] + QI^2 = Q^2 I + QI^2 = QI^2$. Hence $I^3 = QI^2$, which implies, because $Q \cap I^2 = QI$, that $\mathrm{G}(I)$ is a Cohen-Macaulay ring. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{x}$ denotes the image of x in A. We put $J = (x,y) : \mathfrak{n}^q$; hence $I = JA$. We have by Proposition 3.3 that $J \subseteq (x) + \mathfrak{n}^{m+1}$ and $\mathfrak{n}^q J \subseteq \mathfrak{n}^q x + (y)$ (take $\ell = 1$). Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$, so that $I \subseteq \overline{Q}$ (cf. [NR]). Let $\alpha \in Q \cap I^2$ and write $\alpha = \overline{x}\beta$ with $\beta \in A$. Then, for all $\gamma \in \mathfrak{m}^q$, we have $\alpha\gamma = \overline{x}{\cdot}\beta\gamma \in \mathfrak{m}^q I^2 \subseteq Q^2 = (\overline{x}^2)$, so that $\beta\gamma \in (\overline{x}) = Q$. Therefore $\beta \in Q : \mathfrak{m}^q = I$, whence $\alpha = \overline{x}\beta \in QI$. Thus $Q \cap I^2 = QI$, which proves assertion (1).

If $m \geq q - 1$, we have $J \subseteq (x) + \mathfrak{n}^{m+1} \subseteq (x) + \mathfrak{n}^q$, whence $I \subseteq Q + \mathfrak{m}^q$. Therefore $I^2 \subseteq Q$, so that $I^2 = QI$ by assertion (1). Suppose that $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m}$. We choose the element x so that $x \in \mathfrak{n}^{q-m}$. Then, taking $\ell = q - m$, by Proposition 3.3 (1) we get $J = (x,y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^q$. Hence $I \subseteq Q + \mathfrak{m}^q$. Thus $I^2 = QI$. Suppose now that $m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$. To show $I^2 = QI$, we may assume by condition (ii) that

$m < q-1$. Then $Q \subseteq \mathfrak{m}^{q-m}$, since $Q \subseteq \mathfrak{m}^{q-1}$ and $m > 0$. Hence $I^2 = QI$. This proves assertion (2).

Let us consider assertion (3). Suppose that B is a Gorenstein ring and assume that condition (i) is satisfied. We choose the element x so that $x \in \mathfrak{n}^{q-(m+1)}$. Then $J = (x,y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{q-1}$ (take $\ell = q-(m+1)$), whence $I \subseteq Q + \mathfrak{m}^{q-1}$, so that the result follows from Proposition 3.4. Assume that condition (ii) is satisfied. By assertion (2) we may assume that $m < q-1$. Then, since $\mathfrak{m}^{q-1} \subseteq \mathfrak{m}^{q-(m+1)}$, we have $Q \subseteq \mathfrak{m}^{q-(m+1)}$, so that condition (i) is satisfied, whence the result follows. This completes the proof of Theorem 3.1. $\square$

4. The case where $A = B/yB$ and B is a regular local ring

Similarly as in Section 3, we explore quasi-socle ideals in the ring A of the form $A = B/yB$, where $(B, \mathfrak{n})$ is a regular local ring of dimension 2 and y is a subsystem of parameters in B; hence $\mathrm{v}(A) \le 2$ and $\min \mathcal{G}(A) = \mathrm{e}(A) - 1$ (Proposition 2.7).

Our goal of this time is the following.

Theorem 4.1. *Let $(B, \mathfrak{n})$ be a regular local ring of dimension 2. Let n, q be integers such that $n > q > 0$ and put $m = n - q$. Let $0 \ne y \in \mathfrak{n}^n$ and put $A = B/yB$ and $\mathfrak{m} = \mathfrak{n}/yB$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.*

(1) *$\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, and $Q \cap I^2 = QI$.*
(2) *$I^2 = QI$, if one of the following conditions is satisfied.*
  (i) *$m \ge q$;*
  (ii) *$m < q$ and $Q \subseteq \mathfrak{m}^{q-(m-1)}$.*
(3) *$I^3 = QI^2$ and the ring $\mathrm{G}(I)$ is Cohen-Macaulay, if one of the following conditions is satisfied.*
  (i) *$m < q$ and $Q \subseteq \mathfrak{m}^{q-m}$;*
  (ii) *$Q \subseteq \mathfrak{m}^{q-1}$.*

Our proof of Theorem 4.1 is, this time, based on the following.

Proposition 4.2. *Let $(B, \mathfrak{n})$ be a regular local ring of dimension 2 and let x, y be a system of parameters of B. Let $q, \ell > 0$ and $m \geq 0$ be integers such that $q+1 \geq \ell$ and assume that $x \in \mathfrak{n}^\ell$ and $y \in \mathfrak{n}^{q+m}$. Then the following assertions hold true.*

(1) $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m-1}$.

(2) *Suppose that $m > 0$. Then* $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$.

Proof. (1) Enlarging the field $B/\mathfrak{n}$ if necessary, we may assume that the field $B/\mathfrak{n}$ is infinite. Let $\mathrm{G}(\mathfrak{n}) = \bigoplus_{n \geq 0} \mathfrak{n}^n/\mathfrak{n}^{n+1}$ denote the associated graded ring of B. Then $\mathrm{G}(\mathfrak{n})$ is the polynomial ring with two indeterminates over $B/\mathfrak{n}$. For each element $0 \neq f \in B$ let $\mathrm{o}_\mathfrak{n}(f) = \max\{n \in \mathbb{Z} \mid y \in \mathfrak{n}^n\}$ and let $f^* = f \bmod \mathfrak{n}^{\mathrm{o}_\mathfrak{n}(f)+1}$ be the initial form of f; hence f^* is $\mathrm{G}(\mathfrak{n})$-regular. For each integer $k > 0$, the ideal $\mathfrak{n}^k$ is generated by the set

$$F_k = \{z \in \mathfrak{n}^k \mid z \in \mathfrak{n}^k \backslash \mathfrak{n}^{k+1} \text{ and } x^*, z^* \text{ is a homogeneous system of parameters in } \mathrm{G}(\mathfrak{n})\}.$$

Now let $\alpha \in (x, y) : \mathfrak{n}^q$, $z \in F_{q+m}$, and $z' \in F_q$. Then $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$. for some $u, v, u', v' \in B$. Hence, because the sequence x, y is B-regular, comparing two expressions of $z'z\alpha$, we get $z'v \in (x, z)$, whence $v \in (x, z) : \mathfrak{n}^q$. Recall now that $(x, z) : \mathfrak{n}^q = (x, z) + \mathfrak{n}^{\ell'}$ with

$$\begin{aligned}
\ell' &= [\mathrm{a}(\mathrm{G}(\mathfrak{n}/(x, z))) + 1] - q \\
&= [\mathrm{a}(\mathrm{G}(\mathfrak{n})/(x^*, z^*)) + 1] - q \\
&= [\mathrm{a}(\mathrm{G}(\mathfrak{n})) + \mathrm{o}_\mathfrak{n}(x) + \mathrm{o}_\mathfrak{n}(z)) + 1] - q \\
&\geq [(-2) + \ell + (q+m) + 1] - q = \ell + m - 1
\end{aligned}$$

(cf. [Wat]; see [O, Theorem 1.6] also), where $\mathrm{a}(*)$ denotes the a-invariant of the corresponding graded ring ([GW, (3.1.4)]). Therefore

$$z\alpha = ux + vy \in (x) + (zy) + \mathfrak{n}^{\ell'} y \subseteq (x) + \mathfrak{n}^{\ell+m-1} y,$$

because $\ell' \geq \ell + m - 1$ and $z \in \mathfrak{n}^{q+m}$ with $q \geq \ell - 1$. Hence $\alpha \in [(x) + \mathfrak{n}^{\ell+m-1} y] : \mathfrak{n}^{q+m}$, so that $\alpha y \in (x) + \mathfrak{n}^{\ell+m-1} y$, whence $\alpha \in (x) + \mathfrak{n}^{\ell+m-1}$, since the sequence x, y is B-regular. Thus $(x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{\ell+m-1}$.

(2) The ideal $\mathfrak{n}^q$ is generated by the set $F = \{z \in \mathfrak{n}^q \mid y, z \text{ is a } B\text{-regular sequence}\}$. Let $\alpha \in (x, y) : \mathfrak{n}^q$ and $z, z' \in F$. Then $z\alpha = ux + vy$ and $z'\alpha = u'x + v'y$ for some

$u, v, u', v' \in B$. We want to show that $z\alpha \in \mathfrak{n}^q x + (y)$. Because the sequence y, x is B-regular, comparing two expressions of $z'z\alpha$, we get $z'u \in (z, y)$, whence $u \in (z, y) : \mathfrak{n}^q$. Notice now that $(z, y) : \mathfrak{n}^q \subseteq (z) + \mathfrak{n}^{q+m-1}$ by assertion (1) (take $x = z$ and $q = \ell$). Then

$$z\alpha = ux + vy \in (zx) + \mathfrak{n}^{q+m-1}x + (y) \subseteq \mathfrak{n}^q x + (y),$$

since $m > 0$, whence we have $\mathfrak{n}^q \cdot [(x, y) : \mathfrak{n}^q] \subseteq \mathfrak{n}^q x + (y)$. $\square$

Our proof of Theorem 4.1 is now similar to that of Theorem 3.1. We briefly note it.

Proof of Theorem 4.1. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{x}$ denotes the image of x in A. Let $J = (x, y) : \mathfrak{n}^q$. Then by Proposition 4.2 that $J \subseteq (x) + \mathfrak{n}^m$ and $\mathfrak{n}^q J \subseteq \mathfrak{n}^q x + (y)$ (take $\ell = 1$). Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$, so that $I \subseteq \overline{Q}$. We have $Q \cap I^2 = QI$ exactly for the same reason as is in Proof of Theorem 3.1.

To see assertion (2), suppose that $m \geq q$. Then $J \subseteq (x) + \mathfrak{n}^q$, whence $I \subseteq Q + \mathfrak{m}^q$. Therefore $I^2 \subseteq Q$, so that $I^2 = QI$ by assertion (1). Suppose that $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m+1}$. We choose the element x so that $x \in \mathfrak{n}^{q-m+1}$. Then, taking $\ell = q - m + 1$, by Proposition 4.2 (1) we get $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^q$. Hence $I \subseteq Q + \mathfrak{m}^q$, so that $I^2 \subseteq Q$, whence $I^2 = QI$.

Suppose that condition (i) in assertion (3) is satisfied. We choose the element x so that $x \in \mathfrak{n}^{q-m}$. Then $J = (x, y) : \mathfrak{n}^q \subseteq (x) + \mathfrak{n}^{q-1}$ (take $\ell = q - m$), whence $I \subseteq Q + \mathfrak{m}^{q-1}$, so that the result follows from Proposition 3.4. Suppose that condition (ii) in assertion (3) is satisfied but $m < q$. Then $Q \subseteq \mathfrak{m}^{q-m}$, since $Q \subseteq \mathfrak{m}^{q-1}$ and $m > 0$. Hence the result follows. $\square$

Let us give a consequence of Theorem 4.1.

Corollary 4.3. *Let* $(A, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $\dim A = 1$ *and* $\mathrm{v}(A) = 2$. *Let* $q > 0$ *be an integer such that* $\mathrm{e}(A) > q > 0$ *and put* $m = \mathrm{e}(A) - q$. *Then if* $m \geq q - 2$, *for every parameter ideal* Q *in* A *the following assertions hold true, where* $I = Q : \mathfrak{m}^q$.

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$ *and* $\mathrm{r}_Q(I) \leq 3$.
(2) $q = 3$ *and* Q *is a reduction of* $\mathfrak{m}$, *if* $\mathrm{r}_Q(I) = 3$.
(3) $\mathrm{G}(I)$ *is a Cohen-Macaulay ring.*

Proof. Let $e = \mathrm{e}(A)$. Passing to the $\mathfrak{m}$-adic completion of A, we may assume that $A = B/yB$, where $(B, \mathfrak{n})$ is a regular local ring of dimension 2 and $0 \neq y \in \mathfrak{n}^e$. Hence $\mathfrak{m}^q I = \mathfrak{m}^q Q$ by Theorem 4.1 (1). We must show that $\mathrm{r}_Q(I) \leq 3$ and $\mathrm{G}(I)$ is a Cohen-Macaulay ring. Thanks to Theorem 4.1 (2), we may assume $m < q$ and $Q \not\subseteq \mathfrak{m}^{q-m}$. Hence $m = q-2$ or $m = q-1$. Let $Q = (\overline{x})$ with $x \in \mathfrak{n}$, where $\overline{*}$ denotes the image in A. Then $q-m \neq 1$ since $x \notin \mathfrak{n}^{q-m}$, whence $m = q-2$, that is $e = 2q-2$. Let $\mathfrak{n} = (x, z)$ with $z \in B$ and let $D = B/xB$. Then D is a DVR. Let us write $yD = z^\ell D$ with $\ell \geq e > q$ and we have $(x, y) : \mathfrak{n}^q = (x) + \mathfrak{n}^{\ell - q}$. If $\ell > e$, then $I = Q + \mathfrak{m}^{\ell - q} \subseteq Q + \mathfrak{m}^{e+1-q} = Q + \mathfrak{m}^{q-1}$, so that $I^2 = QI$ by Proposition 3.4. Assume that $\ell = e$. Then x^*, y^* is a homogeneous system of parameters in $\mathrm{G}(\mathfrak{n})$ with $\deg x^* = 1$ and $\deg y^* = e$, so that Q is a reduction of $\mathfrak{m}$ and $I = Q + \mathfrak{m}^{\ell'}$ by [Wat], where

$$\begin{aligned}
\ell' &= \mathrm{a}(\mathrm{G}(\mathfrak{m}/Q)) + 1 - q \\
&= [\mathrm{a}(\mathrm{G}(\mathfrak{n})/(x^*, y^*)) + 1] - q \\
&= [(-2) + (1 + e)] + 1 - q \\
&= e - q \\
&= m.
\end{aligned}$$

Therefore $\mathrm{r}_Q(I) = \lceil \frac{q}{m} \rceil = \lceil \frac{q}{q-2} \rceil$, thanks to Theorem 2.3 (1). Hence, if $\mathrm{r}_Q(I) \geq 4$, then $\frac{q}{q-2} > 3$, so that $q < 3$. This is impossible, since $m = q - 2 > 0$. Thus $\mathrm{r}_Q(I) \leq 3$. We similarly have $q = 3$, if $\mathrm{r}_Q(I) = 3$. $\square$

Let $4 \leq a < b$ be integers such that $\mathrm{GCD}(a, b) = 1$ and let

$$H = \langle a, b \rangle := \{a\alpha + b\beta \mid 0 \leq \alpha, \beta \in \mathbb{Z}\}$$

be the numerical semigroup generated by a, b. Let $A = k[[t^a, t^b]]$ $(\subseteq k[[t]])$ be the numerical semigroup ring of H and $\mathfrak{m} = (t^a, t^b)$ the maximal ideal in A, where $k[[t]]$ is the formal power series ring over a field k. Then

$$A \cong k[[X, Y]]/(X^b - Y^a),$$

where $B = k[[X, Y]]$ denotes the formal power series ring. Hence, applying Corollaries 2.7 and 4.3, we get the following.

Corollary 4.4. *The following assertions hold true.*

(1) $\min \mathcal{G}(A) = a - 1 \geq 3$.

(2) *Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^3$. Then $I^4 = QI^3$ and $\mathrm{G}(I)$ is a Cohen-Macaulay ring.*

5. Examples and remarks

Let $n \geq 0$ be an integer and put $a = 6n+5$, $b = 6n+8$, and $c = 9n+12$. Then $0 < a < b < c$ and $\mathrm{GCD}(a,b,c) = 1$. Let $A = k[[t^a, t^b, t^c]] \subseteq k[[t]]$, where $k[[t]]$ denotes the formal power series ring over a field k. Then

$$A \cong k[[X,Y,Z]]/(Y^3 - Z^2,\ X^{3n+4} - Y^{3n+1}Z),$$

where $k[[X,Y,Z]]$ denotes the formal powers series ring. Let $\mathfrak{m}$ be the maximal ideal in A. Then

$$\mathrm{G}(\mathfrak{m}) \cong k[X,Y,Z]/(Y^{3n+4}, Y^{3n+1}Z, Z^2).$$

Hence A is a complete intersection with $\dim A = 1$, whose associated graded ring $\mathrm{G}(\mathfrak{m})$ is not a Gorenstein ring but Cohen-Macaulay. We put

$$B = k[[X,Y,Z]]/(Y^3 - Z^2)$$

and let y denote the image of $X^{3n+4} - Y^{3n+1}Z$ in B. Let $\mathfrak{n} = (X,Y,Z)B$ be the maximal ideal in B. Then B is not a regular local ring and $A = B/yB$. We have $y \in \mathfrak{n}^{3n+2}$ and y is a subsystem of parameters of B. Therefore by Theorem 3.1 (1), (2), and (3) we have the following.

Example 5.1. Let $0 < q \leq 3n+2$ be an integer and put $m = (3n+2) - q$. Let Q be a parameter ideal in A and put $I = Q : \mathfrak{m}^q$. Then the following assertions hold true.

(1) $\mathfrak{m}^q I = \mathfrak{m}^q Q$, $I \subseteq \overline{Q}$, and $Q \cap I^2 = QI$. Hence $\mathrm{g}(Q) \geq 3n+2$.

(2) $I^2 = QI$, if one of the following conditions is satisfied.

  (i) $m \geq q - 1$;

  (ii) $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-m}$;

  (iii) $m > 0$ and $Q \subseteq \mathfrak{m}^{q-1}$.

(3) $I^3 = QI^2$ and the ring $\mathrm{G}(I)$ is Cohen-Macaulay, if one of the following conditions is satisfied.

  (i) $m < q - 1$ and $Q \subseteq \mathfrak{m}^{q-(m+1)}$;

(ii) $Q \subseteq \mathfrak{m}^{q-1}$.

Remark 5.2. In Example 5.1 (3) the equality $I^2 = QI$ does not necessarily hold true. For example, let $n = 0$; hence $A = k[[t^5, t^8, t^{12}]]$. Let $Q = (t^5)$ in A and $I = Q : \mathfrak{m}^2$. Then $I = (t^5, t^{12}, t^{16}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.

The assumption $y \in \mathfrak{n}^q$ in Theorem 3.1 is crucial in order to control quasi-socle ideals $I = Q : \mathfrak{m}^q$.

Example 5.3. In Example 5.1 take $n = 0$ and look at the local ring $A = k[[t^5, t^8, t^{12}]]$. Hence

$$A \cong k[[X, Y, Z]]/(Y^3 - Z^2, X^4 - YZ).$$

Let $0 < s \in \langle 5, 8, 12 \rangle := \{5\alpha + 8\beta + 12\gamma \mid 0 \leq \alpha,\ \beta,\ \gamma \in \mathbb{Z}\}$ and $Q = (t^s)$ in A, *monomial* parameters. Let us consider the quasi-socle ideal $I = Q : \mathfrak{m}^3$. Then we always have $I \subseteq \overline{Q}$, but $\mathrm{G}(I)$ is Cohen-Macaulay (resp. the equality $\mathfrak{m}^3 I = \mathfrak{m}^3 Q$ holds true) if and only if $s \in \{5, 10, 12, 15, 17\}$ (resp. $s \in \{5, 12, 17\}$), or equivalently $Q \cap I^2 = QI$. Thus the Cohen-Macaulayness in $\mathrm{G}(I)$ is rather wild, as we summarize in the following table.

s	I	$\mathfrak{m}^3 I = \mathfrak{m}^3 Q$	$\mathrm{G}(I)$ is CM	$\mathrm{r}_Q(I)$
5	$\mathfrak{m} = (t^5, t^8, t^{12})$	Yes	Yes	3
8	(t^8, t^{10}, t^{17})	No	No	3
10	$(t^{10}, t^{12}, t^{13}, t^{16})$	No	Yes	2
12	$(t^{12}, t^{15}, t^{18}, t^{21})$	Yes	Yes	1
13	$(t^{13}, t^{15}, t^{16}, t^{22})$	No	No	2
15	$(t^{15}, t^{17}, t^{18}, t^{21}, t^{24})$	No	Yes	2
16	$(t^{16}, t^{18}, t^{22}, t^{25})$	No	No	2
17	$(t^{17}, t^{20}, t^{23}, t^{24}, t^{26})$	Yes	Yes	1
18	$(t^{18}, t^{20}, t^{21}, t^{24}, t^{27})$	No	No	2
≥ 20	$(t^s, t^{s+2}, t^{s+3}, t^{s+6}, t^{s+9})$	No	No	2

Remark 5.4. To see that the results of Theorem 4.1 are sharp, the reader may consult [GKM, GKMP] for examples of monomial parameter ideals $Q = (t^s)$ $(0 < s \in H)$ in numerical semigroup rings $A = k[[H]]$. See [GKMP, Proposition 10] for the case where $H = \langle a, b \rangle$ with $\mathrm{GCD}(a, b) = 1$. Here let us pick up the simplest ones.

(1) The equality $I^2 = QI$ does not necessarily hold true. Let $A = k[[t^3, t^4]]$, $Q = (t^3)$, and $I = Q : \mathfrak{m}^2$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 2$.

(2) The reduction number $\mathrm{r}_Q(I)$ could be not less than 3. Let $A = k[[t^4, t^5]]$, $Q = (t^4)$, and $I = Q : \mathfrak{m}^3$. Then $I = \mathfrak{m} \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$.

(3) The ring $\mathrm{G}(I)$ is not necessarily Cohen-Macaulay. Let $A = k[[t^5, t^6]]$, $Q = (t^{11})$, and $I = Q : \mathfrak{m}^4$. Then $I = (t^{11}, t^{12}, t^{15}) \subseteq \overline{Q}$ and $\mathrm{r}_Q(I) = 3$. However, since $t^{36} \in Q \cap I^3$ but $t^{36} \notin QI^2$, we have $Q \cap I^3 \neq QI^2$, so that $\mathrm{G}(I)$ is not a Cohen-Macaulay ring.

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