Heat convection equation with nonhomogeneous boundary condition

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Abstract.
We consider the stationary heat convection equations and the time periodic heat convection equations (Boussinesq approximation) with non-homogeneous boundary condition, and obtain the existence result similar to the Navier-Stokes equations’ case.

The boundary value for the fluid velocity should satisfy so-called general outflow condition (GOC). For the 2 or 3 dimensional bounded domain, the existence of the solution can be shown if the boundary condition satisfies the stringent outflow condition (SOC). Similarly to the Navier-Stokes equations, we obtain the existence result for the 2 dimensional symmetric domain and symmetric data with the boundary value satisfying only (GOC).

Key words.
Stationary Boussinesq flow with non-homogeneous boundary condition, time periodic Boussinesq flow with non-homogeneous boundary condition, 2 or 3 dimensional bounded domain with multiply connected boundary, general outflow condition, 2 dimensional symmetric flow

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n = 2, 3$). The boundary $\partial \Omega$ consists of $N + 1$ smooth connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$ where $N \geq 1$, $\Omega$ being inside of $\Gamma_0$.

Firstly, we consider the stationary motion of heat convection of incompressible viscous fluid governed by the Boussinesq approximation.

\[
\begin{align*}
\nu \Delta v - (v \cdot \nabla)v - \nabla p + \eta g \theta + f_1 &= 0 & \text{in } \Omega \\
\operatorname{div} v &= 0 & \text{in } \Omega \\
\kappa \Delta \theta - (v \cdot \nabla) \theta + f_2 &= 0 & \text{in } \Omega
\end{align*}
\]
From now on, we call this equation “Boussinesq equation” in short. The boundary condition is:

\[(1.2) \ v(x) = \beta_0(x) \text{ and } \theta(x) = \gamma_0(x) \text{ on } \partial \Omega.\]

Here \(v = v(x)\) is the fluid velocity, \(p = p(x)\) is the pressure, \(\theta = \theta(x)\) is the temperature. \(f_i = f_i(x) (i = 1, 2)\) are external forces, \(g\) is the gravitational constant vector, and \(\nu\) (kinematic viscosity), \(\eta\) (coefficient of volume expansion), \(\kappa\) (thermal conductivity) are positive constants. \(\beta_0 = \beta_0(x)\) and \(\gamma_0 = \gamma_0(x)\) are given functions defined on \(\partial \Omega\).

We consider also time dependent problem. Let \(T > 0\).

\[
\begin{cases}
\frac{\partial v}{\partial t} = \nu \Delta v - (v \cdot \nabla)v - \nabla p + \eta g \theta + f_1 & \text{in } \Omega \times (0, T) \\
\text{div } v = 0 & \text{in } \Omega \times (0, T) \\
\frac{\partial \theta}{\partial t} = \kappa \Delta \theta - (v \cdot \nabla) \theta + f_2 & \text{in } \Omega \times (0, T)
\end{cases}
\]

The boundary condition is:

\[(1.4) \ v(x, t) = \beta_0(x, t) \text{ and } \theta(x, t) = \gamma_0(x, t) \text{ on } \partial \Omega \times (0, T).\]

Periodicity condition is:

\[(1.5) \ v(x, 0) = v(x, T) \text{ and } \theta(x, 0) = \theta(x, T) \text{ in } \Omega.\]

Here \(v = v(x, t)\) is the fluid velocity, \(p = p(x, t)\) is the pressure, \(\theta = \theta(x, t)\) is the temperature. \(f_i = f_i(x, t) (i = 1, 2)\) are external forces. \(g, \nu, \eta, \kappa\) are as above. \(\beta_0 = \beta_0(x, t)\) and \(\gamma_0 = \gamma_0(x, t)\) are given functions defined on \(\partial \Omega \times [0, T]\).

According to the Gauss Theorem, the boundary value \(\beta_0\) should satisfy

\[(GOC) \ \int_{\partial \Omega} \beta_0 \cdot nd\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_0 \cdot nd\sigma = 0\]

If \(N \geq 1\), the condition

\[(SOC) \ \int_{\Gamma_k} \beta_0 \cdot nd\sigma = 0 \quad (\forall k = 0, 1, \cdots, N)\]

is stronger than (GOC).

It is well known that if \(\beta_0\) enjoys the condition (SOC), then the existence of stationary solution to the Navier-Stokes equations can be shown. See
Hopf[2], [3], Ladyzhenskaya[10], Fujita[4], Galdi[6]. If the boundary value satisfies only (GOC), the existence results for stationary Navier-Stokes problem are partly known (Amick[1], Fujita[5], Morimoto[15]). In order to solve the nonstationary problem, we need no condition about the boundary value except (GOC) because we can use the Gronwall inequality. But, for the time periodic problem, we can not apply the Gronwall inequality and it is known only partial answer (Morimoto[16]).

In this note, we report that the similar results are obtained for the Boussinesq equations, without smallness condition for the data.

There are several results concerning the Boussinesq equation. As for the stationary problem, in our previous works [12] and [13], we treated the problem where the boundary of the domain is simply connected, the boundary condition for the velocity is Dirichlet zero, and that for the temperature is mixed one. As for the periodic problem, for cylindrical domain \(2 \leq n \leq 4\), \(f_1 = 0, f_2 = 0\) and the Dirichlet zero boundary condition for \(v\) but Dirichlet-Neumann condition for \(\theta\), Morimoto[14] showed the existence of weak periodic solutions under smallness condition for \(\eta\). For the noncylindrical domain, Ōeda[17] showed the existence of strong periodic solutions for \(n = 3, f_1 = 0, f_2 = 0\) under some smallness condition for the data, and Inoue-Ōtani[8] also obtained strong periodic solutions for \(n = 2,3\) under some smallness condition for the data.

2 Notation and results

We assume the following for the domain.

\(C^\infty_0(\Omega)\) and \(L^2(\Omega)\) are as usual. \(H^1(\Omega)\) is a usual Sobolev space.

The inner product and the norm of \(L^2(\Omega)^n\) are denoted by \((\cdot, \cdot)\) and \(\| \cdot \|\).

\(C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n; \text{div} \ u = 0 \text{ in } \Omega\}\)

\(H = H(\Omega)\) is the closure of \(C^\infty_{0,\sigma}(\Omega)\) in \(L^2(\Omega)^n\) and \(H^1_\sigma(\Omega) = \{u \in H^1(\Omega)^n; \text{div} \ u = 0 \text{ in } \Omega\}\)

\(V = V(\Omega)\) is the closure of \(C^\infty_{0,\sigma}(\Omega)\) in \(H^1(\Omega)^n\).

\(V'\) is the dual space of \(V\).
Since $\Omega$ is bounded, we use the Dirichlet norm $\|\nabla u\|$ for $u \in V$, which is equivalent to the $(H^1)^n$ norm.

**Theorem 2.1** Suppose (A0) and
(A1) $\beta_0(x) \in H^{1/2}(\partial \Omega)$ satisfies

$$(SOC) \int_{\Gamma_k} \beta_0 \cdot n \sigma = 0 \quad (0 \leq k \leq N)$$

where $n$ is the unit outward normal vector to $\partial \Omega$.

(A2) $\beta_0 \in H^{1/2}(\partial \Omega)$

(A3) $f_1 \in V', \quad f_2 \in H^{-1}(\Omega)$.

Then, there exist $v, \theta$ such that

$$v - b_0 \in V(\Omega), \quad \theta - \vartheta_0 \in H^1_0(\Omega)$$

for some extension $b_0, \vartheta_0$ of $\beta_0, \gamma_0$ and satisfy the equations

$$
\nu(\nabla v, \nabla \varphi) + ((v \cdot \nabla)v, \varphi) - \eta(g \theta, \varphi) - \nu' < f_1, \varphi >_V = 0 \quad (\forall \varphi \in V(\Omega))
$$

$$
\kappa(\nabla \theta, \nabla \psi) + ((v \cdot \nabla)\theta, \psi) - \nu' < f_2, \psi >_{H^0_1} = 0 \quad (\forall \psi \in H^1_0(\Omega)).
$$

Next we consider the (GOC) case. We obtain an existence result for two dimensional case assuming the symmetry.

(A0)* Let $\Omega \subset \mathbb{R}^2$ be a bounded domain symmetric with respect to the $x_2$-axis. The boundary $\partial \Omega$ consists of $N + 1$ connected components $\Gamma_k (0 \leq k \leq N)$ and every $\Gamma_k$ intersects the $x_2$-axis.

A two dimensional vector function $u = (u_1, u_2)$ is called symmetric with respect to the $x_2$-axis or symmetric in short if and only if $u_1(x_1, x_2)$ is odd in $x_1$ and $u_2(x_1, x_2)$ is even in $x_1$, that is,

$$u_1(-x_1, x_2) = -u_1(x_1, x_2), \quad u_2(-x_1, x_2) = u_2(x_1, x_2).$$

Symmetric function spaces:

$$H^s = H^s(\Omega) = \{ u \in H(\Omega) ; u \text{ is symmetric with respect to the } x_2\text{-axis} \}$$

$$V^s = V^s(\Omega) = \{ u \in V(\Omega) ; u \text{ is symmetric with respect to the } x_2\text{-axis} \}$$

$$L^{2,e} = L^{2,e}(\Omega) = \{ \theta \in L^2(\Omega) ; \theta(x_1, x_2) \text{ is an even scalar function of } x_1 \}$$

$$H^{1,e} = H^{1,e}(\Omega) = H^1(\Omega) \cap L^{2,e}(\Omega)$$

$$H^1_0 = H^1_0(\Omega) = H^1_0(\Omega) \cap L^{2,e}(\Omega)$$

Now we state our results with symmetry.
Theorem 2.2 Suppose $(A0)^{\pi}$ and
$(A1)^{\pi} \beta_0 \in H^{1/2}(\partial \Omega)$ is symmetric and satisfies
\[
(GOC) \int_{\partial \Omega} \beta_0 \cdot nd\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_0 \cdot nd\sigma = 0
\]
where $n$ is the unit outward normal vector to $\partial \Omega$.

$(A2)^{\pi} \gamma_0 \in H^{1/2}(\partial \Omega)$ is an even function of $x_1$.

$(A3)^{\pi} f_1 \in (V)^{\prime}, \ f_2 \in (H_0^{1,\pi}(\Omega))^{\prime}$

$(A4)^{\pi} g$ is a symmetric constant vector.

Then, there exist $v, \theta$ such that

\[
v - b_0 \in V^s(\Omega), \ \theta - \vartheta_0 \in H_0^{1,\pi}(\Omega)
\]

for some extension $b_0, \vartheta_0$ of $\beta_0, \gamma_0$ and satisfy the equations

\[
(2.3) \ \nu(\nabla v, \nabla \varphi) + ((v \cdot \nabla) v, \varphi) - \eta(\theta, \varphi) - (v, \varphi)_V < f_1, \varphi >_V = 0 \quad (\forall \varphi \in V^s(\Omega))
\]

\[
(2.4) \ \kappa(\nabla \theta, \nabla \psi) + ((v \cdot \nabla) \theta, \psi) - (H_0^{1,\pi}, \psi)_V < f_2, \psi >_{H_0^{1,\pi}} = 0 \quad (\forall \psi \in H_0^{1,\pi}(\Omega)).
\]

Now we show the results for periodic problems. We use the following notation.

\[
L^p(0,T;X) = \{u : [0,T] \to X; \int_0^T \|u(t)\|^p_X dt < \infty\} (1 \leq p < \infty)
\]

\[
L^\infty(0,T;X) = \{u : [0,T] \to X; \text{ess sup}_{t \in [0,T]} \|u(t)\|_X < \infty\}
\]

\[
C^1([0,T];X) = \{u : [0,T] \to X; \text{continuous}\}
\]

\[
C^1_{\pi}([0,T];X) = \{u \in C^1([0,T];X); u(\cdot,0) = u(\cdot,T)\}
\]

where $X$ is a Banach space.

Theorem 2.3 Suppose $(A0)$ and
$(A1)_{\pi} \beta_0(x,t) \in C^1_{\pi}([0,T];H^{1/2}(\partial \Omega))$ satisfies
\[
(SOC) \int_{\Gamma_k} \beta_0 \cdot nd\sigma = 0 \quad (0 \leq k \leq N)
\]
where \( n \) is the unit outward normal vector to \( \partial \Omega \)

(A2) \( \gamma_0 \in C^1_\pi([0, T]; H^{1/2}(\partial \Omega)) \)

(A3) \( f_1 \in L^2(0, T; V'), \quad f_2 \in L^2(0, T; H^{-1}(\Omega)) \).

Then, there exist periodic functions \( v \) and \( \theta \) of period \( T \) such that

\[
v - b_0 \in L^2(0, T; V) \cap L^\infty(0, T; H)
\]

\[
\theta - \vartheta_0 \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))
\]

for some extensions \( b_0, \vartheta_0 \) of the boundary values \( \beta_0, \gamma_0 \), and satisfying the equations

\[
\frac{dv}{dt}(v, \varphi) = -\nu(\nabla v, \nabla \varphi) - ((v \cdot \nabla) v, \varphi) + \eta(g\theta, \varphi) + v' < f_1, \varphi >_V \\
(\forall \varphi \in V),
\]

\[
\frac{d\theta}{dt}(\theta, \psi) = -\kappa(\nabla \theta, \nabla \psi) - ((v \cdot \nabla) \theta, \psi) + f_2, \psi >_{H^{-1}_{\theta}} (\forall \psi \in H^1(\Omega)).
\]

As for the uniqueness, we obtain the following result.

**Theorem 2.4** If the periodic solution is small, then it is unique.

If the boundary value \( \beta_0 \) satisfies only (GOC), we assume the symmetry and obtain the existence of periodic solutions.

**Theorem 2.5** Suppose \((A0)^s\) and

(A1) \( \beta_0 \in C^1_\pi([0, T]; H^{1/2}(\partial \Omega)) \) is symmetric and satisfies

\[
(GOC) \int_{\partial \Omega} \beta_0 \cdot nd\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_0 \cdot nd\sigma = 0
\]

where \( n \) is the unit outward normal vector to \( \partial \Omega \)

(A2) \( \gamma_0 \in C^1_\pi([0, T]; H^{1/2}(\partial \Omega)) \) is an even function of \( x_1 \)

(A3) \( f_1 \in L^2(0, T; (V^*)'), \quad f_2 \in L^2(0, T; (H^1_{\theta}(\Omega))') \)

(A4) \( g \) is a symmetric constant vector.

Then, there exist periodic functions \( v \) and \( \theta \) of period \( T \) such that

\[
v - b_0 \in L^2(0, T; V^*) \cap L^\infty(0, T; H^*)
\]

\[
\theta - \vartheta_0 \in L^2(0, T; H^1_{\theta}(\Omega)) \cap L^\infty(0, T; H^1_{\theta}(\Omega))
\]
for some extensions $b_0, \vartheta_0$ of the boundary values $\beta_0, \gamma_0$, and satisfying the equations

\[
(2.7) \quad \frac{d}{dt}(v, \varphi) = -\nu(\nabla v, \nabla \varphi) - ((v \cdot \nabla)v, \varphi) + \eta(g\vartheta, \varphi) + \langle f_1, \varphi \rangle_{V^*} \quad (\forall \varphi \in V^*),
\]

\[
(2.8) \quad \frac{d}{dt}(\theta, \psi) = -\kappa(\nabla \theta, \nabla \psi) - ((v \cdot \nabla)\theta, \psi) + \langle H_{1,e}^1, e_0 \rangle_{H_0^1} \langle f_2, \psi \rangle_{H_0^1, e_0} \quad (\forall \psi \in H_0^1,e_0(\Omega)).
\]

3 Preliminary

We need extension of the boundary values $\beta_0$. It is classical if $\beta_0$ does not depend on $t$ and satisfies (SOC). See, e.g., Fujita[5]. For (GOC) case, see Fujita[4]. The following lemmas are time depending case, and due to Kobayashi[9]

**Lemma 3.1** Suppose $\beta_0$ satisfies $(A1)_{\pi}$. Then for every $\varepsilon > 0$, there exists a function $b_0 \in C^1_{\pi}([0,T]; H_0^1(\Omega))$ such that

\[
b_0(x, t) = \beta_0(x, t) \quad (\forall x \in \partial \Omega, \forall t \in [0,T])
\]

\[
(L) \quad |((u \cdot \nabla)b_0, u)| \leq \varepsilon ||\nabla u||^2 \quad (\forall u \in V, \forall t \in [0,T]).
\]

**Lemma 3.2** Assume $(A0)^*$ holds and $\beta_0$ satisfies $(A1)_{\pi}^*$. Then, for every $\varepsilon > 0$, there exists a symmetric function $b_0 \in C^1_{\pi}([0,T]; H_0^1(\Omega))$ such that

\[
b_0(x, t) = \beta_0(x, t) \quad (\forall x \in \partial \Omega, \forall t \in [0,T])
\]

\[
(LF) \quad |((u \cdot \nabla)b_0, u)| \leq \varepsilon ||\nabla u||^2 \quad (\forall u \in V^*, \forall t \in [0,T]).
\]

As for $\gamma_0$, we have the following results easily.

**Lemma 3.3** Suppose $\gamma_0$ satisfies $(A2)_{\pi}$. Then, for every $\varepsilon > 0$, there exists $\vartheta_0 \in C^1_{\pi}([0,T]; H^1(\Omega))$ satisfying

\[
\vartheta_0(x, t) = \gamma_0(x, t) \quad (\forall x \in \partial \Omega, \forall t \in [0,T])
\]

\[
|((u \cdot \nabla)\vartheta_0, \vartheta)| \leq \varepsilon ||\nabla u|| ||\nabla \vartheta|| \quad (\forall u \in V, \forall \vartheta \in H_0^1, \forall t \in [0,T]).
\]
Lemma 3.4 Suppose \((A0)^*\) holds and \(\gamma_0\) satisfies \((A2)^*_\)\(\). Then, for every \(\varepsilon > 0\), there exists \(\vartheta_0 \in C^1_\pi([0, T]; H^{1,\varepsilon}(\Omega))\) satisfying
\[
\vartheta_0(x, t) = \gamma_0(x, t) \quad (\forall x \in \partial \Omega, \ \forall t \in [0, T])
\]
\[
|((u \cdot \nabla)\vartheta_0, \vartheta)| \leq \varepsilon \|\nabla u\| \|\nabla \vartheta\| \quad (\forall \vartheta \in V^*, \ \forall \vartheta \in H^{1,\varepsilon}_0, \ \forall t \in [0, T]).
\]

Lemma 3.5 (For the proof, see, e.g., Temam[19]) Let
\[
u \in L^2(0, T; V) \cap L^\infty(0, T; H), \quad \vartheta \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).
\]
Then
\[
(u \cdot \nabla)u \in L^p(0, T; V'), \quad (u \cdot \nabla)\vartheta \in L^p(0, T; H^{-1}(\Omega))
\]
where \(p = 2\) for \(n = 2\) and \(p = 4/3\) for \(n = 3\). More precisely,
\[
\|((u \cdot \nabla)u\|_{L^2(0, T; V')} \leq c \|\nabla u\|_{L^\infty(0, T; H)} \|u\|_{L^2(0, T; V)},
\]
\[
\|((u \cdot \nabla)\vartheta\|_{L^2(0, T; H^{-1})} \leq \{\|u\|_{L^\infty(0, T; H)} \|u\|_{L^2(0, T; V)} \|\vartheta\|_{L^\infty(0, T; L^2)} \|\vartheta\|_{L^2(0, T; H^1)}\}^{1/2}
\]
for \(n = 2\), and
\[
\|((u \cdot \nabla)u\|_{L^{4/3}(0, T; V')} \leq \|u\|^{1/2}_{L^\infty(0, T; H)} \|u\|^{3/2}_{L^2(0, T; V)}
\]
\[
\|((u \cdot \nabla)\vartheta\|_{L^{4/3}(0, T; H^{-1})} \leq \|u\|^{1/4}_{L^\infty(0, T; H)} \|u\|^{3/4}_{L^2(0, T; V)} \|\vartheta\|^{1/4}_{L^\infty(0, T; L^2)} \|\vartheta\|^{3/4}_{L^2(0, T; H^1)}
\]
for \(n = 3\).

Lemma 3.6 (Theorem 5.1 [11] p.58) Let \(B_0, B, B_1\) be three Banach spaces, \(B_0, B_1\) reflexive, \(B_0 \subset B \subset B_1\) with injection bounded and canonical injection \(B_0 \subset B\) compact. Put
\[
W = \{v \in L^{p_0}(0, T; B_0); v' \in L^{p_1}(0, T; B_1)\} \quad (1 < p_0, p_1 < \infty).
\]
Then the injection \(W \subset L^{p_0}(0, T; B)\) is compact.
4 Proof of Theorem 2.1

Let \( \varepsilon_1, \varepsilon_2 \) be arbitrary positive number and choose \( b_0 \) (resp. \( \vartheta_0 \)) as in Lemma 3.1 (resp. Lemma 3.3) for \( \varepsilon = \varepsilon_1 \) (resp. \( \varepsilon = \varepsilon_2 \)) respectively. Put \( \nu = u + b_0 \), \( \theta = \vartheta + \vartheta_0 \). Then the equations (2.1) and (2.2) are rewritten as follows.

\[
\nu(\nabla u, \nabla \varphi) + ((u \cdot \nabla) u, \varphi) + ((b_0 \cdot \nabla) u, \varphi) - \eta (g \vartheta, \varphi) = L_1(\varphi) \quad (\forall \varphi \in V(\Omega))
\]

\[
\kappa(\nabla \vartheta, \nabla \psi) + ((u \cdot \nabla) \vartheta, \psi) + ((b_0 \cdot \nabla) \vartheta_0, \psi) = L_2(\psi) \quad (\forall \psi \in H_0^1(\Omega)).
\]

where \( L_1(\varphi) \) (resp. \( L_2(\psi) \)) is defined for \( \varphi \in V \) (resp. \( \psi \in H_0^1 \)) as follows.

\[
L_1(\varphi) = -\nu(\nabla b_0, \nabla \varphi) - ((b_0 \cdot \nabla) b_0, \varphi) + \eta (g \vartheta_0, \varphi) + \nu < f_1, \varphi > \nu
\]

\[
L_2(\psi) = -\kappa(\nabla \vartheta_0, \nabla \psi) - ((b_0 \cdot \nabla) \vartheta_0, \psi) + H^{-1} < f_2, \psi > \Omega.
\]

Let \( \varphi = u \) in (4.1), \( \psi = \vartheta \) in (4.2).

\[
\nu \|\nabla u\|^2 + ((u \cdot \nabla) b_0, u) - \eta (g \vartheta, u) = L_1(u)
\]

\[
\kappa \|\nabla \vartheta\|^2 + ((u \cdot \nabla) \vartheta_0, \vartheta) = L_2(\vartheta)
\]

Since \( L_1 \) (resp. \( L_2 \)) is a continuous linear functional on \( V \) (resp. \( H_0^1 \)), we obtain, using Lemma 3.1, Lemma 3.3,

\[
(\nu - \varepsilon_1)\|\nabla u\|^2 \le \eta |g| \|u\| \|\vartheta\| + \|L_1\| \|\nabla u\|
\]

\[
\kappa \|\nabla \vartheta\|^2 \le \varepsilon_2 \|\nabla u\| \|\nabla \vartheta\| + \|L_2\| \|\nabla \vartheta\|
\]

If we choose \( \varepsilon_1, \varepsilon_2 \) sufficiently small, we can find some positive constants \( M_1, M_2 \) such that

\[
\|\nabla u\| \le M_1, \quad \|\nabla \vartheta\| \le M_2
\]

hold and we get the existence of \( u \in V(\Omega) \) and \( \vartheta \in H_0^1(\Omega) \) satisfying (4.1), (4.2) and Theorem 2.1 is proved.
5 Proof of Theorem 2.2

Lemma 3.2 is crucial for the proof of Theorem 2.2. Let \( \varepsilon_1, \varepsilon_2 \) be arbitrary positive numbers and choose \( b_0(\text{resp. } \vartheta_0) \) as in Lemma 3.2 (resp. Lemma 3.4) for \( \varepsilon = \varepsilon_1 \) (resp. \( \varepsilon = \varepsilon_2 \)) respectively. Then, we have only to repeat the process in the preceding section, in the symmetric spaces \( V^s(\Omega) \) and \( H^{1,\varepsilon}_0(\Omega) \) using Lemma 3.2, Lemma 3.4 and omit the details.

6 Proof of Theorem 2.3

Let \( \varepsilon_1, \varepsilon_2 \) be arbitrary positive numbers and choose \( b_0(\text{resp. } \vartheta_0) \) as in Lemma 3.1 (resp. Lemma 3.3) for \( \varepsilon = \varepsilon_1 \) (resp. \( \varepsilon = \varepsilon_2 \)) respectively.

Suppose \( v \) and \( \theta \) satisfy (2.5), (2.6), and \( v-b_0 \in L^2(0, T; V) \cap L^\infty(0, T; H), \theta - \vartheta_0 \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega)). \)

Put \( v = u + b_0 \) and \( \theta = \vartheta + \vartheta_0 \). Then, \( u \) and \( \vartheta \) satisfy

\[
\frac{d}{dt}(u, \varphi) = -\nu(\nabla u, \nabla \varphi) - ((u \cdot \nabla)u, \varphi) - ((u \cdot \nabla)b_0, \varphi)
-((b_0 \cdot \nabla)u, \varphi) + \eta(g\vartheta, \varphi) + L_1(\varphi) \quad (\forall \varphi \in V(\Omega))
\]

\[
\frac{d}{dt}(\vartheta, \psi) = -\kappa(\nabla \vartheta, \nabla \psi) - ((u \cdot \nabla)\vartheta, \psi) - ((u \cdot \nabla)\vartheta_0, \psi)
-((b_0 \cdot \nabla)\vartheta, \psi) + L_2(\psi) \quad (\forall \psi \in H^1_0(\Omega))
\]

where

\[
L_1(\varphi) = -\nu(\nabla b_0, \nabla \varphi) - ((b_0 \cdot \nabla)b_0, \varphi) + \eta(g\vartheta_0, \varphi) - (\partial_t b_0, \varphi) + \nu < f_1, \varphi >_V
\]

\[
L_2(\psi) = -\kappa(\nabla \vartheta_0, \nabla \psi) - ((b_0 \cdot \nabla)\vartheta_0, \psi) - (\partial_t \vartheta_0, \psi) + \nu < f_2, \psi >_{H^1_0}
\]

We look for periodic solutions \( u \) and \( \vartheta \) to (6.1) and (6.2).

Let \( \{\varphi_m\}_{m=1}^\infty \subset V(\Omega) \) be eigenfunctions of the Stokes operator, that is,

\[
(\nabla \varphi_m, \nabla w) = \lambda_m(\varphi_m, w) \quad (\forall w \in V(\Omega)).
\]

After orthonormalizing them and using the same symbol, we can consider they are a complete orthonormal basis of \( H(\Omega) \).

Let \( \{\psi_m\}_{m=1}^\infty \subset H^1_0(\Omega) \) be eigenfunctions of the Laplace operator,

\[
(\nabla \psi_m, \nabla \psi) = \mu_m(\psi_m, \psi) \quad (\forall \psi \in H^1_0(\Omega)).
\]
They can be considered as a complete ortho-normal basis of $L^2(\Omega)$. We use Galerkin method.

Let us look for functions

$$u^{(m)}(x, t) = \sum_{k=1}^{m} f_k^{(m)}(t) \varphi_k(x) \quad \text{and} \quad \psi^{(m)}(x, t) = \sum_{k=1}^{m} g_k^{(m)}(t) \psi_k(x),$$

satisfying the following system of ordinary differential equations:

\begin{align}
\frac{d}{dt}(u^{(m)}, \varphi_j) &= -\nu(\nabla u^{(m)}, \nabla \varphi_j) - ((u^{(m)} \cdot \nabla) u^{(m)}, \varphi_j) - ((u^{(m)} \cdot \nabla) b_0, \varphi_j) \\
&\quad - ((b_0 \cdot \nabla) u^{(m)}, \varphi_j) + \eta(g \psi^{(m)}, \varphi_j) + \mathcal{L}_1(\varphi_j) \quad (1 \leq j \leq m) \\
\frac{d}{dt}(\psi^{(m)}, \psi_j) &= -\kappa(\nabla \psi^{(m)}, \nabla \psi_j) - ((u^{(m)} \cdot \nabla) \psi^{(m)}, \psi_j) - ((u^{(m)} \cdot \nabla) \vartheta_0, \psi_j) \\
&\quad - ((b_0 \cdot \nabla) \psi^{(m)}, \psi_j) + \mathcal{L}_2(\psi_j) \quad (1 \leq j \leq m)
\end{align}

and the initial condition

$$u^{(m)}(0) = u_{m0} \in [\varphi_1, \varphi_2, \ldots, \varphi_m], \quad \psi^{(m)}(0) = \psi_{m0} \in [\psi_1, \psi_2, \ldots, \psi_m].$$

It is easy to show the local existence in time of solution to (6.3), (6.4) with the initial condition (6.5). Next we show an a priori estimate of the solutions.

\begin{align}
\frac{1}{2} \frac{d}{dt} \|u^{(m)}\|^2 &= -\nu \|\nabla u^{(m)}(t)\|^2 - ((u^{(m)} \cdot \nabla) b_0, u^{(m)}) \\
&\quad + \eta(g \psi^{(m)}, u^{(m)}) + \mathcal{L}_1(u^{(m)})
\end{align}

\begin{align}
\frac{1}{2} \frac{d}{dt} \|\psi^{(m)}\|^2 &= -\kappa \|\nabla \psi^{(m)}(t)\|^2 - ((u^{(m)} \cdot \nabla) \vartheta_0, \psi^{(m)}) + \mathcal{L}_2(\psi^{(m)})
\end{align}

According to our choice, $b_0$ and $\vartheta_0$ satisfy the following inequalities

\begin{align}
|((u \cdot \nabla)b_0, u)| &\leq \varepsilon_1 \|\nabla u\|^2 \quad (\forall u \in V(\Omega), \forall t \in [0, T]) \\
|((u \cdot \nabla)\vartheta_0, \vartheta)| &\leq \varepsilon_2 \|\nabla u\| \|\nabla \vartheta\| \quad (\forall u \in V(\Omega), \forall \vartheta \in H^1_0(\Omega), \forall t \in [0, T]).
\end{align}

Use Poincaré’s inequality:

$$\|u\| \leq c_0 \|\nabla u\| \quad (u \in H^1_0(\Omega))$$
and Hölder’s inequality to estimate (6.6), and we obtain

\[
(6.10) \quad \frac{d}{dt} \|u^{(m)}\|^2 + 2(\nu - C\varepsilon_1)\|\nabla u^{(m)}\|^2 \leq \frac{\eta^2|g|^2}{2\varepsilon_1} \|\vartheta^{(m)}\|^2 + C_1(t),
\]

where \(|g|\) is the length of the vector \(g\), \(C\) is a constant depending only on \(\Omega\) and

\[
C_1(t) = \frac{1}{2\varepsilon_1} \left\{ \nu^2\|\nabla b_0\|^2 + \|b_0\|_{L^4(\Omega)}^4 + \eta^2|g|^2\|\vartheta_0\|^2 + \|\partial_t b_0\|^2 + \|f_1\|^2_{V'} \right\}
\]

We estimate (6.7) similarly and obtain,

\[
(6.11) \quad \frac{d}{dt} \|\vartheta^{(m)}\|^2 + 2(\kappa - C\varepsilon_2)\|\nabla \vartheta^{(m)}\|^2 \leq \varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t)
\]

where \(C\) is a constant depending only on \(\Omega\) and

\[
C_2(t) = \frac{1}{2\varepsilon_2} \left\{ \|b_0\|_{L^4}^2, \|\vartheta_0\|_{L^4}^2 + \kappa^2\|\nabla \vartheta_0\|^2 + \|\partial_t \vartheta_0\|^2 + \|f_2\|^2_{H^{-1}} \right\}.
\]

Choose \(\varepsilon_1 > 0, \varepsilon_2 > 0\) so small that \(\nu - C\varepsilon_1 > 0\) and \(\kappa - C\varepsilon_2 > 0\) hold true. We fix \(\varepsilon_1\). Put

\[
\alpha = 2(\nu - C\varepsilon_1), \quad \beta = \frac{\eta^2|g|^2}{2\varepsilon_1}, \quad \gamma = 2(\kappa - C\varepsilon_2)
\]

Then \(\alpha, \beta, \gamma\) are positive constants and the inequalities

\[
(6.12) \quad \frac{d}{dt} \|u^{(m)}\|^2 + \alpha\|\nabla u^{(m)}\|^2 \leq \beta\|\vartheta^{(m)}\|^2 + C_1(t),
\]

\[
(6.13) \quad \frac{d}{dt} \|\vartheta^{(m)}\|^2 + \gamma\|\nabla \vartheta^{(m)}\|^2 \leq \varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t)
\]

hold true. According to our assumptions, \(C_1\) and \(C_2\) belong to \(L^1(0, T)\) and are independent of \(m\). Integrating (6.12), we have

\[
(6.14) \quad \|u^{(m)}(t)\|^2 + \alpha \int_0^t \|\nabla u^{(m)}(s)\|^2 ds \\
\leq \|u_{m0}\|^2 + \beta \int_0^t \|\vartheta^{(m)}(s)\|^2 ds + \int_0^t C_1(s) ds
\]
Therefore
\[
\int_0^t \| \nabla u^{(m)}(s) \|^2 ds \leq \frac{1}{\alpha} \left\{ \| u_{m0} \|^2 + \beta \int_0^t \| \vartheta'(m)(s) \|^2 ds + \int_0^t C_1(s) ds \right\}
\]
\[
\leq \frac{1}{\alpha} \left\{ \| u_{m0} \|^2 + \beta c_0^2 \int_0^t \| \nabla \vartheta(m)(s) \|^2 ds + \int_0^t C_1(s) ds \right\}
\]

After integrating (6.13), we use (6.15) and obtain
\[
\| \vartheta(m)(t) \|^2 + \left( \gamma - \frac{\beta c_0^2}{\alpha} \varepsilon_2 \right) \int_0^t \| \nabla \vartheta(m)(s) \|^2 ds \leq M_1 \quad (\forall t \in [0, T])
\]
where
\[
M_1 = \| \vartheta_{m0} \|^2 + \frac{\varepsilon_2}{\alpha} \left\{ \| u_{m0} \|^2 + \int_0^T C_1(s) ds \right\} + \int_0^T C_2(s) ds.
\]
We can choose, if necessary, \( \varepsilon_2 \) so small that \( \gamma - \frac{\beta c_0^2}{\alpha} \varepsilon_2 > 0 \) holds true. Then, integrating (6.16), we obtain
\[
\int_0^t \| \vartheta'(m)(s) \|^2 ds \leq M_1 T
\]
Applying this estimate for the right hand side of (6.14), we find
\[
\int_0^t \| u^{(m)}(s) \|^2 ds \leq \| u_{m0} \|^2 + \beta T M_1 + \int_0^T C_1(s) ds \quad (\forall t \in [0, T]).
\]
Estimates (6.16) and (6.17) yield the global existence in time of solutions of (6.3), (6.4), (6.5).

Using (6.16) for the right hand side of (6.12), we obtain
\[
\frac{d}{dt} \| u^{(m)} \|^2 + \alpha \| \nabla u^{(m)} \|^2 \leq \beta M_1 + C_1(t).
\]
Put \( \alpha' = \alpha c_0^{-2} \) where \( c_0 \) is the constant appearing in Poincaré's inequality. Then the above inequality is transformed to
\[
\frac{d}{dt} \| u^{(m)} \|^2 + \alpha' \| u^{(m)} \|^2 \leq \beta \| \vartheta_{m0} \|^2 + \frac{\beta \varepsilon_2}{\alpha} \| u_{m0} \|^2 + M_2 + C_1(t)
\]
where
\[ M_2 = \frac{\beta \varepsilon_2}{\alpha} \int_0^T C_1(t) dt + \beta \int_0^T C_2(t) dt. \]

Therefore
\[ \frac{d}{dt} \left\{ e^{\alpha t} \| u^{(m)} \|^2 \right\} \leq e^{\alpha t} \left\{ \beta \| \vartheta_{m0} \|^2 + \frac{\beta \varepsilon_2}{\alpha} \| u_{m0} \|^2 + M_2 + C_1(t) \right\}. \]

Integrating the both side,
\[
e^{\alpha t} \| u^{(m)}(t) \|^2 \leq \| u_{m0} \|^2 + e^{\alpha t} \left\{ \frac{\alpha}{\alpha'} \left\{ \beta \| \vartheta_{m0} \|^2 + \frac{\beta \varepsilon_2}{\alpha} \| u_{m0} \|^2 \right\} + \int_0^t e^{\alpha' s} C_1(s) ds \right\}.
\]

Therefore
\[
(6.18) \quad \| u^{(m)}(T) \|^2 \leq e^{-\alpha T} \| u_{m0} \|^2 + \frac{1 - e^{-\alpha T}}{\alpha'} \beta \left\{ \| \vartheta_{m0} \|^2 + \frac{\varepsilon_2}{\alpha} \| u_{m0} \|^2 \right\} + C_3
\]

where
\[ C_3 = \frac{1 - e^{-\alpha T}}{\alpha'} M_2 + \int_0^T C_1(t) dt. \]

Put \( \gamma' = \frac{\gamma}{\gamma_0^2} \). From (6.13), we have
\[
\frac{d}{dt} \| \vartheta^{(m)} \|^2 + \gamma' \| \vartheta^{(m)} \|^2 \leq \varepsilon_2 \| \nabla u^{(m)} \|^2 + C_2(t).
\]

Therefore
\[
\frac{d}{dt} \left\{ e^{\gamma' t} \| \vartheta^{(m)} \|^2 \right\} \leq e^{\gamma' t} \left\{ \varepsilon_2 \| \nabla u^{(m)} \|^2 + C_2(t) \right\}.
\]

Integrating this inequality from 0 to \( T \), and using (6.17), we obtain
\[
(6.19) \quad \| \vartheta^{(m)}(T) \|^2 \leq e^{-\gamma T} \| \vartheta_{m0} \|^2 + \frac{\varepsilon_2}{\alpha} \left\{ \frac{\alpha + \beta T \varepsilon_2}{\alpha} \| u_{m0} \|^2 + \beta T \| \vartheta_{m0} \|^2 \right\} + D_2 + \int_0^T C_2(t) dt
\]

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where
\[
D_2 = (1 + \frac{\beta T \varepsilon_2}{\alpha}) \int_0^T C_1(t) dt + \beta T \int_0^T C_2(t) dt.
\]

Put
\[
C_4 = \frac{D_2 \varepsilon_2}{\alpha} + \int_0^T C_2(t) dt.
\]

Consider the following system of linear equations for \(X_1, X_2\).

\[
(6.20) \quad \left(1 - \frac{\beta \varepsilon_2}{\alpha \alpha'}\right) X_1 - \frac{\beta}{\alpha'} X_2 = \frac{C_3}{1 - e^{-\alpha T}}
\]

\[
(6.21) \quad -\left(\frac{\alpha + \beta T \varepsilon_2}{\alpha^2}\right) \varepsilon_2 X_1 + \left(1 - e^{-\gamma T} - \frac{\beta T \varepsilon_2}{\alpha}\right) X_2 = C_4
\]

Let us choose once again \(\varepsilon_2\) sufficiently small, if necessary, and the equations (6.20) (6.21) have a pair of unique positive solutions \(\{X_1, X_2\}\). Note that \(X_1, X_2\) do not depend on \(m\). Put \(R_1 = \sqrt{X_1}, R_2 = \sqrt{X_2}\).

Put \(\|u_m\| \leq R_1, R_2 \leq R_2\). Then, from the estimates (6.18) and (6.19), we can derive easily \(\|u^{(m)}(T)\| \leq R_1\) and \(\|\vartheta^{(m)}(T)\| \leq R_2\). Let \(u^{(m)}(t)\) and \(\vartheta^{(m)}(t)\) be such solution to (6.3), (6.4) with (6.5). Let us define an operator in \(\mathbb{R}^{2m}\) as

\[
T : (u_{m0}, \vartheta_{m0}) \rightarrow (u^{(m)}(T), \vartheta^{(m)}(T))
\]

Put

\[
K = \left\{ (\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_m) \in \mathbb{R}^{2m} ; \sum_{k=1}^m \xi_k^2 \leq R_1^2, \sum_{k=1}^m \eta_k^2 \leq R_2^2 \right\}
\]

Then \(K\) is a compact convex subset of \(\mathbb{R}^{2m}\) and \(T\) is a continuous operator from \(K\) to \(K\). Brouwer’s fixed point theorem assures that there exists a fixed point of the operator \(T\) in \(K\). We denote the fixed point by \((u_{m0}, \vartheta_{m0})\).

Let \(\{u^{(m)}(t), \vartheta^{(m)}(t)\}\) be the solution to (6.3), (6.4), (6.5), with the initial value \(\{u_{m0}, \vartheta_{m0}\}\). Then \(\{u^{(m)}(t), \vartheta^{(m)}(t)\}\) is a periodic solution of (6.3) and (6.4). Since the initial values \(\{u_{m0}\}_m \subset H\) and \(\{\vartheta_{m0}\}_m \subset L^2\) are bounded, it is shown easily from (6.16) and (6.17) that

\[
(6.22) \quad \{u^{(m)}(t)\}_m : \text{ bounded sequence in } L^2(0,T;V) \cap L^\infty(0,T;H)
\]
Moreover, it follows from our choice of the basis and Lemma 3.5 that
\[
\left\{ \frac{d}{dt} u^{(m)} \right\}_m : \text{bounded sequence in } L^p(0,T;V') \tag{6.24}
\]
and
\[
\left\{ \frac{d}{dt} \vartheta^{(m)} \right\}_m : \text{bounded sequence in } L^p(0,T;H^{-1}(\Omega)) \tag{6.25}
\]
where \( p = 2 \) if \( n = 2 \), and \( p = 4/3 \) if \( n = 3 \).

Now we use Lemma 3.6 for \( B_0 = V, B = H, B_1 = V' \) and \( p_0 = p_1 = 2(n = 2) \) or \( p_0 = p_1 = 4/3(n = 3) \). Taking an appropriate converging subsequence from \( \{u^{(m)}, \vartheta^{(m)}\} \), we obtain periodic functions \( u \) and \( \vartheta \) such that
\[
\begin{align*}
\quad u & \in L^2(0,T;V) \cap L^\infty(0,T;H), \\
\vartheta & \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))
\end{align*}
\]
satisfying (6.1), (6.2) and Theorem 2.3 is proved.

7 Proof of Theorem 2.4

We give here an outline of the proof of Theorem 2.4.

Let \( v_i, \vartheta_i \) (\( i = 1,2 \)) be two solutions. Put \( v = v_1 - v_2 \) and \( \vartheta = \vartheta_1 - \vartheta_2 \). After similar calculation to the existence proof, we find the following inequalities hold.
\[
\begin{align*}
\frac{d}{dt} \|v\|^2 + \alpha \|\nabla v\|^2 & \leq \beta \|\vartheta\|^2 \tag{7.1} \\
\frac{d}{dt} \|\vartheta\|^2 + \gamma \|\nabla \vartheta\|^2 & \leq \delta \|\nabla v\|^2 \tag{7.2}
\end{align*}
\]
where
\[
\begin{align*}
\alpha &= \nu - 2c \max_{0 \leq t \leq T} \|v_1(t)\|_{L^4} \\
\beta &= \frac{\eta^2 |g|^2}{C_{\nu}} \\
\gamma &= 2(\kappa - c \max_{0 \leq t \leq T} \|\vartheta_1(t)\|_{L^4})
\end{align*}
\]
\[ \delta = \frac{c}{4} \max_{0 \leq t \leq T} \| \vartheta_1(t) \|_{L^4}. \]

Here the constant \( c \) depends only on \( \Omega \) and \( C_{\nu} \) depends only on \( \nu \).

Put \( \alpha' = \alpha c_0^{-2}, \gamma' = \gamma c_0^{-2} \) where \( c_0 \) is the Poincaré constant. After tedious calculation, we obtain

\[
\| v(t) \| \leq \left\{ e^{-\alpha't} + \frac{\beta \delta}{\alpha} (1 - e^{-\alpha't}) \right\} \| v(0) \|^2 + \frac{\beta}{\alpha'} (1 - e^{-\alpha't}) \| \vartheta(0) \|^2,
\]

(7.3)

\[
\| \vartheta(t) \| \leq \left\{ \frac{\delta}{\alpha} \left( \frac{\beta \delta T}{\alpha} + 1 \right) \| v(0) \|^2 + \left( e^{-\gamma'T} + \frac{\beta \delta T}{\alpha} \right) \| \vartheta(0) \|^2 \right\}.
\]

(7.4)

Put \( t = T \) in (7.3) and (7.4), and use the relations \( v(0) = v(T) \) and \( \vartheta(0) = \vartheta(T) \), then we have

\[
(7.5) \quad \left( 1 - \frac{\beta \delta}{\alpha \alpha'} \right) \| v(0) \|^2 - \frac{\beta}{\alpha'} \| \vartheta(0) \|^2 \leq 0,
\]

\[
(7.6) \quad -\frac{\delta}{\alpha} \left( \frac{\beta \delta T}{\alpha} + 1 \right) \| v(0) \|^2 + \left\{ 1 - e^{-\gamma'T} - \frac{\beta \delta T}{\alpha} \right\} \| \vartheta(0) \|^2 \leq 0.
\]

Therefore, if \( \max_{0 \leq t \leq T} \| v_1(t) \|_{L^4} \) and \( \max_{0 \leq t \leq T} \| \vartheta_1(t) \|_{L^4} \) are sufficiently small, we have \( \| v(0) \| = \| \vartheta(0) \| = 0 \). According to the estimates (7.3) (7.4), we obtain \( \| v(t) \| = \| \vartheta(t) \| = 0 \).

**Remark 7.1** Our equations contain external forces \( f_1, f_2 \) depending on time variable. Therefore, small periodic solutions, if they exist, are not stationary solutions.

**8 Proof of Theorem 2.5**

We give here a sketch of the proof of Theorem 2.5. As in the proof of Theorem 2.2, Lemma 3.2 is crucial to prove this theorem.

Let \( \{ \varphi_m^s \}_{m=1}^\infty \subset V^s(\Omega) \) be eigenfunctions of the Stokes operator in \( V^s(\Omega) \), that is,

\[
\left( \nabla \varphi_m^s, \nabla w \right) = \lambda_m^s (\varphi_m^s, w) \quad (\forall w \in V^s(\Omega)).
\]
After orthonormalizing them and using the same symbol, we can consider they are a complete ortho-normal basis of $H^s(\Omega)$ (C.f. [16]). Let $\{\psi^s_m\}_{m=1}^\infty \subset H^1_0(\Omega)$ be eigenfunctions of the Laplace operator in $H^1_0(\Omega)$, that is,

$$\langle \nabla \psi^s_m, \nabla \psi \rangle = \mu^s_m(\psi^s_m, \psi) \quad (\forall \psi \in H^1_0(\Omega)).$$

They can be considered as a complete orthonormal basis of $L^2(\Omega)$. We use Galerkin method.

Let $\varepsilon_1$ and $\varepsilon_2$ be arbitrary positive numbers. According to (A0)$_s$ and (A1)$_s$, we can find $b_0$ satisfying the following inequality (Lemma 3.2)

$$(8.1) \quad \|(u \cdot \nabla)b_0, u\| \leq \varepsilon_1\|\nabla u\|_2^2 \quad (\forall u \in V^s(\Omega), \forall t \in [0, T])$$

and, (A2)$_s$ allows us to choose $\vartheta_0$ satisfying the inequality (Lemma 3.4)

$$(8.2) \quad \|(u \cdot \nabla)\vartheta_0, \vartheta\| \leq \varepsilon_2\|\nabla u\|\|\nabla \vartheta\| \quad (\forall u \in V^s(\Omega), \forall \vartheta \in H_0^1(\Omega), \forall t \in [0, T]).$$

Using (8.1) and (8.2), we obtain a periodic solution $\{u^{(m)}, \vartheta^{(m)}\}$ to (6.3) and (6.4) with $\varphi_j = \varphi^s_j$, $\psi_j = \psi^s_j$ (1 \leq j \leq m). Furthermore,

$${\{u^{(m)}\}}_m : \text{bounded sequence in } L^2(0, T; V^s) \cap L^\infty(0, T; H^s)$$

$${\{\vartheta^{(m)}\}}_m : \text{bounded sequence in } L^2(0, T; H^1_0) \cap L^\infty(0, T; L^2)$$

$$\left\{\frac{d}{dt}u^{(m)}\right\}_m : \text{bounded sequence in } L^2(0, T; (V^s)')$$

$$\left\{\frac{d}{dt}\vartheta^{(m)}\right\}_m : \text{bounded sequence in } L^2(0, T; (H^1_0)'(\Omega))$$

Now we use Lemma 3.6 for $B_0 = V^s, B = H^s, B_1 = (V^s)'$ and $p_0 = p_1 = 2$. Choosing a subsequence from $\{u^{(m)}, \vartheta^{(m)}\}$ appropriately, the limit functions $\{u, \vartheta\}$ are periodic and satisfy (6.1), (6.2) for all $\varphi \in V^s$ and $\psi \in H^1_0$, and Theorem 2.5 is proved.
References


