Heat convection equation with nonhomogeneous boundary condition

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Abstract.

We consider the stationary heat convection equations and the time periodic heat convection equations (Boussinesq approximation) with non-homogeneous boundary condition, and obtain the existence result similar to the Navier-Stokes equations' case.

The boundary value for the fluid velocity should satisfy so-called general outflow condition (GOC). For the 2 or 3 dimensional bounded domain, the existence of the solution can be shown if the boundary condition satisfies the stringent outflow condition (SOC). Similarly to the Navier-Stokes equations, we obtain the existence result for the 2 dimensional symmetric domain and symmetric data with the boundary value satisfying only (GOC).

Key words.

Stationary Boussinesq flow with non-homogeneous boundary condition, time periodic Boussinesq flow with non-homogeneous boundary condition, 2 or 3 dimensional bounded domain with multiply connected boundary, general outflow condition, 2 dimensional symmetric flow

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n = 2, 3)$. The boundary $\partial\Omega$ consists of $N + 1$ smooth connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$ where $N \geq 1$, Ω being inside of Γ_0.

Firstly, we consider the stationary motion of heat convection of incompressible viscous fluid governed by the Boussinesq approximation.

$$(1.1) \quad \begin{cases} \nu\Delta v - (v\cdot\nabla)v - \nabla p + \eta g\theta + f_1 &= 0 \quad \text{in } \Omega \\ \operatorname{div} v &= 0 \quad \text{in } \Omega \\ \kappa\Delta\theta - (v\cdot\nabla)\theta + f_2 &= 0 \quad \text{in } \Omega \end{cases}$$

From now on, we call this equation "Boussinesq equation" in short. The boundary condition is:

(1.2) $v(x) = \beta_0(x)$ and $\theta(x) = \gamma_0(x)$ on $\partial\Omega$.

Here $v = v(x)$ is the fluid velocity, $p = p(x)$ is the pressure, $\theta = \theta(x)$ is the temperature. $f_i = f_i(x)\,(i = 1, 2)$ are external forces, g is the gravitational constant vector, and ν(kinematic viscosity), η(coefficient of volume expansion), κ (thermal conductivity) are positive constants. $\beta_0 = \beta_0(x)$ and $\gamma_0 = \gamma_0(x)$ are given functions defined on $\partial\Omega$.

We consider also time dependent problem. Let $T > 0$.

$$(1.3)\quad \begin{cases} \dfrac{\partial v}{\partial t} = \nu\Delta v - (v\cdot\nabla)v - \nabla p + \eta g\theta + f_1 & \text{in } \Omega\times(0,T) \\ \operatorname{div} v = 0 & \text{in } \Omega\times(0,T) \\ \dfrac{\partial \theta}{\partial t} = \kappa\Delta\theta - (v\cdot\nabla)\theta + f_2 & \text{in } \Omega\times(0,T) \end{cases}$$

The boundary condition is:

(1.4) $v(x,t) = \beta_0(x,t)$ and $\theta(x,t) = \gamma_0(x,t)$ on $\partial\Omega\times(0,T)$.

Periodicity condition is:

(1.5) $v(x,0) = v(x,T)$ and $\theta(x,0) = \theta(x,T)$ in Ω.

Here $v = v(x,t)$ is the fluid velocity, $p = p(x,t)$ is the pressure, $\theta = \theta(x,t)$ is the temperature. $f_i = f_i(x,t)\,(i = 1, 2)$ are external forces. g, ν, η, κ are as above. $\beta_0 = \beta_0(x,t)$ and $\gamma_0 = \gamma_0(x,t)$ are given functions defined on $\partial\Omega\times[0,T]$.

According to the Gauss Theorem, the boundary value β_0 should satisfy

$$(GOC)\quad \int_{\partial\Omega} \beta_0\cdot n\,d\sigma = \sum_{k=0}^{N}\int_{\Gamma_k}\beta_0\cdot n\,d\sigma = 0$$

If $N \geq 1$, the condition

$$(SOC)\quad \int_{\Gamma_k}\beta_0\cdot n\,d\sigma = 0 \quad (\forall k = 0, 1, \cdots, N)$$

is stronger than (GOC).

It is well known that if β_0 enjoys the condition (SOC), then the existence of stationary solution to the Navier-Stokes equations can be shown. See

Hopf[2], [3], Ladyzhenskaya[10], Fujita[4], Galdi[6]. If the boundary value satsifies only (GOC), the existence results for stationary Navier-Stokes problem are partly known (Amick[1], Fujita[5], Morimoto[15]). In order to solve the nonstationary problem, we need no condition about the boundary value except (GOC) because we can use the Gronwall inequality. But, for the time periodic problem, we can not apply the Gronwall inequality and it is known only partial answer (Morimoto[16]).

In this note, we report that the similar results are obtained for the Boussinesq equations, without smallness condition for the data.

There are several results concerning the Boussinesq equation. As for the stationary problem, in our previous works [12] and [13], we treated the problem where the boundary of the domain is simply connected, the boundary condition for the velocity is Dirichlet zero, and that for the temperature is mixed one. As for the periodic problem, for cylindrical domain ($2 \leq n \leq 4$), $f_1 = 0, f_2 = 0$ and the Dirichlet zero boundary condition for v but Dirichlet-Neumann condition for θ, Morimoto[14] showed the existence of weak periodic solutions under smallness condition for η. For the noncylindrical domain, Ōeda[17] showed the existence of strong periodic solutions for $n = 3, f_1 = 0, f_2 = 0$ under some smallness condition for the data, and Inoue-Ôtani[8] also obtained strong periodic solutions for $n = 2, 3$ under some smallness conditon for the data.

2 Notation and results

We assume the following for the domain.

(A0) Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ be a bounded domain. The boundary $\partial\Omega$ is smooth and consists of $N + 1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$. Ω is inside of Γ_0.

In order to state our results exactly, we need the following function spaces.

$C_0^\infty(\Omega)$ and $L^2(\Omega)$ are as usual. $H^1(\Omega)$ is a usual Sobolev space.

The inner product and the norm of $L^2(\Omega)^n$ are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$.

$C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega)^n; \mathrm{div} u = 0 \text{ in } \Omega\}$

$H = H(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^n$ and

$H_\sigma^1(\Omega) = \{u \in H^1(\Omega)^n; \mathrm{div} u = 0 \text{ in } \Omega\}$

$V = V(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $H^1(\Omega)^n$.

V' is the dual space of V.

Since Ω is bounded, we use the Dirichlet norm $\|\nabla u\|$ for $u \in V$, which is equivalent to the $(H^1)^n$ norm.

Theorem 2.1 *Suppose* $(A0)$ *and*
$(A1)$ $\beta_0(x) \in H^{1/2}(\partial\Omega)$ *satisfies*

$$(SOC) \int_{\Gamma_k} \beta_0 \cdot n d\sigma = 0 \quad (0 \leq k \leq N)$$

where n *is the unit outward normal vector to* $\partial\Omega$.
$(A2)$ $\gamma_0 \in H^{1/2}(\partial\Omega)$
$(A3)$ $f_1 \in V'$, $\quad f_2 \in H^{-1}(\Omega)$.
Then, there exist v, θ *such that*

$$v - b_0 \in V(\Omega), \quad \theta - \vartheta_0 \in H_0^1(\Omega)$$

for some extension b_0, ϑ_0 *of* β_0, γ_0 *and satisfy the equations*

$$(2.1)\ \nu(\nabla v, \nabla\varphi) + ((v \cdot \nabla)v, \varphi) - \eta(g\theta, \varphi) - {}_{V'}< f_1, \varphi >_V = 0 \quad (\forall \varphi \in V(\Omega))$$

$$(2.2)\ \kappa(\nabla\theta, \nabla\psi) + ((v \cdot \nabla)\theta, \psi) - {}_{H^{-1}}< f_2, \psi >_{H_0^1} = 0 \quad (\forall \psi \in H_0^1(\Omega)).$$

Next we consider the (GOC) case. We obtain an existence result for two dimensional case assuming the symmetry.

$(A0)^s$ Let $\Omega \subset \mathbb{R}^2$ be a bounded domain symmetric with respect to the x_2-axis. The boundary $\partial\Omega$ consists of $N+1$ connected components $\Gamma_k (0 \leq k \leq N)$ and every Γ_k intersects the x_2-axis.

A two dimensional vector function $u = (u_1, u_2)$ is called symmetric with respect to the x_2-axis or symmetric in short if and only if $u_1(x_1, x_2)$ is odd in x_1 and $u_2(x_1, x_2)$ is even in x_1, that is,

$$u_1(-x_1, x_2) = -u_1(x_1, x_2),\ u_2(-x_1, x_2) = u_2(x_1, x_2).$$

Symmetric function spaces:

$$H^s = H^s(\Omega) = \{u \in H(\Omega); u \text{ is symmetric with respect to the } x_2\text{-axis } \}$$

$$V^s = V^s(\Omega) = \{u \in V(\Omega)\ ;\ u \text{ is symmetric with respect to the } x_2\text{-axis } \}$$

$$L^{2,e} = L^{2,e}(\Omega) = \{\theta \in L^2(\Omega); \theta(x_1, x_2) \text{ is an even scalar function of } x_1\}$$

$$H^{1,e} = H^{1,e}(\Omega) = H^1(\Omega) \cap L^{2,e}(\Omega)$$

$$H_0^{1,e} = H_0^{1,e}(\Omega) = H_0^1(\Omega) \cap L^{2,e}(\Omega)$$

Now we state our results with symmetry.

Theorem 2.2 *Suppose* $(A0)^s$ *and*
$(A1)^s$ $\beta_0 \in H^{1/2}(\partial\Omega)$ *is symmetric and satisfies*

$$(GOC)\quad \int_{\partial\Omega} \beta_0 \cdot n d\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_0 \cdot n d\sigma = 0$$

where n *is the unit outward normal vector to* $\partial\Omega$*.*
$(A2)^s$ $\gamma_0 \in H^{1/2}(\partial\Omega)$ *is an even function of* x_1*.*
$(A3)^s$ $f_1 \in (V^s)', \quad f_2 \in (H_0^{1,e}(\Omega))'$
$(A4)^s$ g *is a symmetric constant vector.*
Then, there exist v, θ *such that*

$$v - b_0 \in V^s(\Omega), \quad \theta - \vartheta_0 \in H_0^{1,e}(\Omega)$$

for some extension b_0, ϑ_0 *of* β_0, γ_0 *and satisfy the equations*

$$(2.3)\ \nu(\nabla v, \nabla\varphi) + ((v\cdot\nabla)v, \varphi) - \eta(g\theta, \varphi) - {}_{(V^s)'} < f_1, \varphi >_{V^s} = 0 \quad (\forall \varphi \in V^s(\Omega))$$

$$(2.4)\quad \kappa(\nabla\theta, \nabla\psi) + ((v\cdot\nabla)\theta, \psi) - {}_{(H_0^{1,e})'} < f_2, \psi >_{H_0^{1,e}} = 0 \quad (\forall \psi \in H_0^{1,e}(\Omega)).$$

Now we show the results for periodic problems. We use the following notation.

$$L^p(0,T;X) = \{u : [0,T] \to X; \int_0^T \|u(t)\|_X^p dt < \infty\} (1 \le p < \infty)$$

$$L^\infty(0,T;X) = \{u : [0,T] \to X; \mathrm{ess}\sup_{t\in[0,T]} \|u(t)\|_X < \infty\}$$

$$C^1([0,T];X) = \{u : [0,T] \to X; \text{continuous}\}$$

$$C^1_\pi([0,T];X) = \{u \in C^1([0,T];X); u(\cdot,0) = u(\cdot,T)\}$$

where X is a Banach space.

Theorem 2.3 *Suppose* $(A0)$ *and*
$(A1)_\pi$ $\beta_0(x,t) \in C^1_\pi([0,T]; H^{1/2}(\partial\Omega))$ *satisfies*

$$(SOC)\quad \int_{\Gamma_k} \beta_0 \cdot n d\sigma = 0 \quad (0 \le k \le N)$$

where n *is the unit outward normal vector to* $\partial\Omega$
$(A2)_\pi$ $\gamma_0 \in C^1_\pi([0,T]; H^{1/2}(\partial\Omega))$
$(A3)_\pi$ $f_1 \in L^2(0,T;V')$, $\quad f_2 \in L^2(0,T;H^{-1}(\Omega))$.
Then, there exist periodic functions v *and* θ *of period* T *such that*

$$v - b_0 \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

$$\theta - \vartheta_0 \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

for some extensions b_0, ϑ_0 *of the boundary values* β_0, γ_0*, and satisfying the equations*

$$(2.5)\quad \frac{d}{dt}(v,\varphi) = -\nu(\nabla v, \nabla\varphi) - ((v\cdot\nabla)v,\varphi) + \eta(g\theta,\varphi) + {}_{V'}< f_1, \varphi >_V \qquad (\forall\varphi \in V),$$

$$(2.6)\quad \frac{d}{dt}(\theta,\psi) = -\kappa(\nabla\theta,\nabla\psi) - ((v\cdot\nabla)\theta,\psi) + {}_{H^{-1}}< f_2, \psi >_{H^1_0} \quad (\forall\psi \in H^1_0(\Omega)).$$

As for the uniqueness, we obtain the following result.

Theorem 2.4 *If the periodic solution is small, then it is unique.*

If the boundary value β_0 satisfies only (GOC), we assume the symmetry and obtain the existence of periodic solutions.

Theorem 2.5 *Suppose* $(A0)^s$ *and*
$(A1)^s_\pi$ $\beta_0 \in C^1_\pi([0,T]; H^{1/2}(\partial\Omega))$ *is symmetric and satisfies*

$$(GOC)\quad \int_{\partial\Omega} \beta_0 \cdot n d\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_0 \cdot n d\sigma = 0$$

where n *is the unit outward normal vector to* $\partial\Omega$
$(A2)^s_\pi$ $\gamma_0 \in C^1_\pi([0,T]; H^{1/2}(\partial\Omega))$ *is an even function of* x_1
$(A3)^s_\pi$ $f_1 \in L^2(0,T;(V^s)')$, $\quad f_2 \in L^2(0,T;(H^{1,e}_0(\Omega))')$
$(A4)^s_\pi$ g *is a symmetric constant vector.*
Then, there exist periodic functions v *and* θ *of period* T *such that*

$$v - b_0 \in L^2(0,T;V^s) \cap L^\infty(0,T;H^s)$$

$$\theta - \vartheta_0 \in L^2(0,T;H^{1,e}_0(\Omega)) \cap L^\infty(0,T;L^{2,e}(\Omega))$$

for some extensions b_0, ϑ_0 *of the boundary values* β_0, γ_0, *and satisfying the equations*

$$(2.7)\quad \frac{d}{dt}(v,\varphi) = -\nu(\nabla v, \nabla\varphi) - ((v\cdot\nabla)v,\varphi) + \eta(g\theta,\varphi) + {}_{(V^s)'}< f_1, \varphi >_{V^s} \qquad (\forall\varphi \in V^s),$$

$$(2.8)\quad \frac{d}{dt}(\theta,\psi) = -\kappa(\nabla\theta,\nabla\psi) - ((v\cdot\nabla)\theta,\psi) + {}_{(H_0^{1,e})'}< f_2, \psi >_{H_0^{1,e}} \qquad (\forall\psi \in H_0^{1,e}(\Omega)).$$

3 Preliminary

We need extension of the boundary values β_0. It is classical if β_0 does not depend on t and satisfies (SOC). See, e.g., Fujita[5]. For (GOC) case, see Fujita[4]. The following lemmas are time depending case, and due to Kobayashi[9]

Lemma 3.1 *Suppose* β_0 *satisfies* $(A1)_\pi$. *Then for every* $\varepsilon > 0$, *there exits a function* $b_0 \in C^1_\pi([0,T]; H^1_\sigma(\Omega))$ *such that*

$$b_0(x,t) = \beta_0(x,t) \quad (\forall x \in \partial\Omega,\ \forall t \in [0,T])$$

$$(L)\quad |((u\cdot\nabla)b_0, u)| \le \varepsilon\|\nabla u\|^2 \quad (\forall u \in V, \forall t \in [0,T]).$$

Lemma 3.2 *Assume* $(A0)^s$ *holds and* β_0 *satisfies* $(A1)^s_\pi$. *Then, for every* $\varepsilon > 0$, *there exists a symmetric function* $b_0 \in C^1_\pi([0,T]; H^1_\sigma(\Omega))$ *such that*

$$b_0(x,t) = \beta_0(x,t) \quad (\forall x \in \partial\Omega,\ \forall t \in [0,T])$$

$$(LF)\quad |((u\cdot\nabla)b_0, u)| \le \varepsilon\|\nabla u\|^2 \quad (\forall u \in V^s, \forall t \in [0,T]).$$

As for γ_0, we have the following results easily.

Lemma 3.3 *Suppose* γ_0 *satisfies* $(A2)_\pi$. *Then, for every* $\varepsilon > 0$, *there exists* $\vartheta_0 \in C^1_\pi([0,T]; H^1(\Omega))$ *satisfying*

$$\vartheta_0(x,t) = \gamma_0(x,t) \quad (\forall x \in \partial\Omega,\ \forall t \in [0,T])$$

$$|((u\cdot\nabla)\vartheta_0, \vartheta)| \le \varepsilon\|\nabla u\|\|\nabla\vartheta\| \quad (\forall u \in V,\ \forall\vartheta \in H^1_0,\ \forall t \in [0,T]).$$

Lemma 3.4 *Suppose* $(A0)^s$ *holds and* γ_0 *satisfies* $(A2)^s_\pi$. *Then, for every* $\varepsilon > 0$, *there exists* $\vartheta_0 \in C^1_\pi([0,T]; H^{1,e}(\Omega))$ *satisfying*

$$\vartheta_0(x,t) = \gamma_0(x,t) \quad (\forall x \in \partial\Omega,\ \forall t \in [0,T])$$

$$|((u\cdot\nabla)\vartheta_0, \vartheta)| \le \varepsilon \|\nabla u\|\ \|\nabla\vartheta\|\ (\forall u \in V^s, \forall\vartheta \in H_0^{1,e}, \forall t \in [0,T]).$$

Lemma 3.5 *(For the proof, see, e.g.,Temam[19]) Let*

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad \vartheta \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).$$

Then

$$(u\cdot\nabla)u \in L^p(0,T;V'), \quad (u\cdot\nabla)\vartheta \in L^p(0,T;H^{-1}(\Omega))$$

where $p = 2$ *for* $n = 2$ *and* $p = 4/3$ *for* $n = 3$. *More precisely,*

$$\|(u\cdot\nabla)u\|_{L^2(0,T;V')} \le c\|u\|_{L^\infty(0,T;H)}\|u\|_{L^2(0,T;V)},$$

$$\|(u\cdot\nabla)\vartheta\|_{L^2(0,T;H^{-1})} \le \{\|u\|_{L^\infty(0,T;H)}\|u\|_{L^2(0,T;V)}\|\vartheta\|_{L^\infty(0,T;L^2)}\|\vartheta\|_{L^2(0,T;H^1)}\}^{1/2}$$

for $n = 2$, *and*

$$\|(u\cdot\nabla)u\|_{L^{4/3}(0,T;V')} \le c\|u\|^{1/2}_{L^\infty(0,T;H)}\|u\|^{3/2}_{L^2(0,T;V)}$$

$$\|(u\cdot\nabla)\vartheta\|_{L^{4/3}(0,T;H^{-1})} \le \|u\|^{1/4}_{L^\infty(0,T;H)}\|u\|^{3/4}_{L^2(0,T;V)}\|\vartheta\|^{1/4}_{L^\infty(0,T;L^2)}\|\vartheta\|^{3/4}_{L^2(0,T;H^1)}$$

for $n = 3$.

Lemma 3.6 *(Theorem 5.1 [11] p.58) Let* B_0, B, B_1 *be three Banach spaces,* B_0, B_1 *reflexive,* $B_0 \subset B \subset B_1$ *with injection bounded and canonical injection* $B_0 \subset B$ *compact. Put*

$$W = \{v \in L^{p_0}(0,T;B_0); v' \in L^{p_1}(0,T;B_1)\}\ (1 < p_0, p_1 < \infty).$$

Then the injection $W \subset L^{p_0}(0,T;B)$ *is compact.*

4 Proof of Theorem2.1

Let $\varepsilon_1, \varepsilon_2$ be arbitrary positive number and choose b_0(resp. ϑ_0) as in Lemma 3.1(resp. Lemma 3.3) for $\varepsilon = \varepsilon_1$(resp. $\varepsilon = \varepsilon_2$) respectively. Put $v = u + b_0$, $\theta = \vartheta + \vartheta_0$. Then the equations (2.1) and (2.2) are rewritten as follows.

$$(4.1)\ \nu(\nabla u, \nabla\varphi) + ((u\cdot\nabla)u, \varphi) + ((u\cdot\nabla)b_0, \varphi) + ((b_0\cdot\nabla)u, \varphi) - \eta(g\vartheta, \varphi)$$

$$= L_1(\varphi) \quad (\forall\varphi \in V(\Omega))$$

$$(4.2)\ \kappa(\nabla\vartheta, \nabla\psi) + ((u\cdot\nabla)\vartheta, \psi) + ((u\cdot\nabla)\vartheta_0, \psi) + ((b_0\cdot\nabla)\vartheta, \psi)$$

$$= L_2(\psi) \quad (\forall\psi \in H_0^1(\Omega)).$$

where $L_1(\varphi)$ (resp. $L_2(\psi)$) is defined for $\varphi \in V$(resp. $\psi \in H_0^1$) as follows.

$$(4.3)\quad L_1(\varphi) = -\nu(\nabla b_0, \nabla\varphi) - ((b_0\cdot\nabla)b_0, \varphi) + \eta(g\vartheta_0, \varphi) + {}_{V'}< f_1, \varphi >_V$$

$$(4.4)\quad L_2(\psi) = -\kappa(\nabla\vartheta_0, \nabla\psi) - ((b_0\cdot\nabla)\vartheta_0, \psi) + {}_{H^{-1}}< f_2, \psi >_{H_0^1}.$$

Let $\varphi = u$ in (4.1), $\psi = \vartheta$ in (4.2).

$$(4.5)\ \nu\|\nabla u\|^2 + ((u\cdot\nabla)b_0, u) - \eta(g\vartheta, u) = L_1(u)$$

$$(4.6)\ \kappa\|\nabla\vartheta\|^2 + ((u\cdot\nabla)\vartheta_0, \vartheta) = L_2(\vartheta)$$

Since L_1 (resp. L_2) is a continuous linear functional on V(resp. H_0^1), we obtain, using Lemma 3.1, Lemma 3.3,

$$(4.7)\quad (\nu - \varepsilon_1)\|\nabla u\|^2 \le \eta|g|\|u\|\|\vartheta\| + \|L_1\|\|\nabla u\|$$

$$(4.8)\quad \kappa\|\nabla\vartheta\|^2 \le \varepsilon_2\|\nabla u\|\|\nabla\vartheta\| + \|L_2\|\|\nabla\vartheta\|$$

If we choose $\varepsilon_1, \varepsilon_2$ sufficiently small, we can find some positive constants M_1, M_2 such that

$$\|\nabla u\| \le M_1, \quad \|\nabla\vartheta\| \le M_2$$

hold and we get the existence of $u \in V(\Omega)$ and $\vartheta \in H_0^1(\Omega)$ satisfying (4.1), (4.2) and Theorem 2.1 is porved.

5 Proof of Theorem2.2

Lemma 3.2 is crucial for the proof of Theorem2.2. Let $\varepsilon_1, \varepsilon_2$ be arbitrary positive numbers and choose b_0(resp. ϑ_0) as in Lemma 3.2 (resp. Lemma 3.4) for $\varepsilon = \varepsilon_1$(resp. $\varepsilon = \varepsilon_2$) respectively. Then, we have only to repeat the process in the preceding section, in the symmetric spaces $V^s(\Omega)$ and $H_0^{1,e}(\Omega)$ using Lemma 3.2, Lemma 3.4 and omit the details.

6 Proof of Theorem2.3

Let $\varepsilon_1, \varepsilon_2$ be arbitrary positive numbers and choose b_0(resp. ϑ_0) as in Lemma 3.1 (resp. Lemma 3.3) for $\varepsilon = \varepsilon_1$(resp. $\varepsilon = \varepsilon_2$) respectively.

Suppose v and θ satisfy (2.5), (2.6), and $v - b_0 \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\theta - \vartheta_0 \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Put $v = u + b_0$ and $\theta = \vartheta + \vartheta_0$. Then, u and ϑ satisfy

$$(6.1)\quad \frac{d}{dt}(u,\varphi) = -\nu(\nabla u, \nabla\varphi) - ((u\cdot\nabla)u,\varphi) - ((u\cdot\nabla)b_0,\varphi)$$
$$-((b_0\cdot\nabla)u,\varphi) + \eta(g\vartheta,\varphi) + \mathcal{L}_1(\varphi) \quad (\forall\varphi \in V(\Omega))$$

$$(6.2)\quad \frac{d}{dt}(\vartheta,\psi) = -\kappa(\nabla\vartheta,\nabla\psi) - ((u\cdot\nabla)\vartheta,\psi) - ((u\cdot\nabla)\vartheta_0,\psi)$$
$$-((b_0\cdot\nabla)\vartheta,\psi) + \mathcal{L}_2(\psi) \quad (\forall\psi \in H_0^1(\Omega))$$

where

$$\mathcal{L}_1(\varphi) = -\nu(\nabla b_0,\nabla\varphi) - ((b_0\cdot\nabla)b_0,\varphi) + \eta(g\vartheta_0,\varphi) - (\partial_t b_0,\varphi) + {}_{V'}< f_1,\varphi >_V$$

$$\mathcal{L}_2(\psi) = -\kappa(\nabla\vartheta_0,\nabla\psi) - ((b_0\cdot\nabla)\vartheta_0,\psi) - (\partial_t\vartheta_0,\psi) + {}_{H^{-1}}< f_2,\psi >_{H_0^1}$$

We look for periodic solutions u and ϑ to (6.1) and (6.2).

Let $\{\varphi_m\}_{m=1}^\infty \subset V(\Omega)$ be eigenfunctions of the Stokes operator, that is,

$$(\nabla\varphi_m,\nabla w) = \lambda_m(\varphi_m, w) \quad (\forall w \in V(\Omega)).$$

After orthonormalizing them and using the same symbol, we can consider they are a complete ortho-normal basis of $H(\Omega)$.

Let $\{\psi_m\}_{m=1}^\infty \subset H_0^1(\Omega)$ be eigenfunctions of the Laplace operator,

$$(\nabla\psi_m,\nabla\psi) = \mu_m(\psi_m,\psi) \quad (\forall\psi \in H_0^1(\Omega)).$$

They can be considered as a complete ortho-normal basis of $L^2(\Omega)$. We use Galerkin method.

Let us look for functions

$$u^{(m)}(x,t)=\sum_{k=1}^{m} f_k^{(m)}(t)\varphi_k(x), \quad \vartheta^{(m)}(x,t)=\sum_{k=1}^{m} g_k^{(m)}(t)\psi_k(x)$$

satisfying the following system of ordinary differential equations:

$$(6.3)\ \ \frac{d}{dt}(u^{(m)},\varphi_j)=-\nu(\nabla u^{(m)},\nabla\varphi_j)-((u^{(m)}\cdot\nabla)u^{(m)},\varphi_j)-((u^{(m)}\cdot\nabla)b_0,\varphi_j)$$
$$-((b_0\cdot\nabla)u^{(m)},\varphi_j)+\eta(g\vartheta^{(m)},\varphi_j)+\mathcal{L}_1(\varphi_j) \quad (1\le j\le m)$$

$$(6.4)\ \ \frac{d}{dt}(\vartheta^{(m)},\psi_j)=-\kappa(\nabla\vartheta^{(m)},\nabla\psi_j)-((u^{(m)}\cdot\nabla)\vartheta^{(m)},\psi_j)-((u^{(m)}\cdot\nabla)\vartheta_0,\psi_j)$$
$$-((b_0\cdot\nabla)\vartheta^{(m)},\psi_j)+\mathcal{L}_2(\psi_j)\ (1\le j\le m)$$

and the initial condition

$$(6.5)\ \ u^{(m)}(0)=u_{m0}\in[\varphi_1,\varphi_2,\cdots,\varphi_m], \quad \vartheta^{(m)}(0)=\vartheta_{m0}\in[\psi_1,\psi_2,\cdots,\psi_m].$$

It is easy to show the local existence in time of solution to (6.3), (6.4) with the initial condition (6.5). Next we show an a priori estimate of the solutions.

$$(6.6)\quad \frac{1}{2}\frac{d}{dt}\|u^{(m)}\|^2=-\nu\|\nabla u^{(m)}(t)\|^2-((u^{(m)}\cdot\nabla)b_0,u^{(m)})$$
$$+\eta(g\vartheta^{(m)},u^{(m)})+\mathcal{L}_1(u^{(m)})$$

$$(6.7)\quad \frac{1}{2}\frac{d}{dt}\|\vartheta^{(m)}\|^2=-\kappa\|\nabla\vartheta^{(m)}(t)\|^2-((u^{(m)}\cdot\nabla)\vartheta_0,\vartheta^{(m)})+\mathcal{L}_2(\vartheta^{(m)})$$

According to our choice, b_0 and ϑ_0 satisfy the following inequalities

$$(6.8)\quad |((u\cdot\nabla)b_0,u)|\le\varepsilon_1\|\nabla u\|^2 \ \ (\forall u\in V(\Omega),\ \forall t\in[0,T])$$

$$(6.9)\quad |((u\cdot\nabla)\vartheta_0,\vartheta)|\le\varepsilon_2\|\nabla u\|\,\|\nabla\vartheta\| \ \ (\forall u\in V(\Omega),\ \forall\vartheta\in H_0^1(\Omega),\ \forall t\in[0,T]).$$

Use Poincaré's inequality:

$$\|u\|\le c_0\|\nabla u\| \quad (u\in H_0^1(\Omega))$$

and Hölder's inequality to estimate (6.6), and we obtain

$$(6.10)\ \frac{d}{dt}\|u^{(m)}\|^2 + 2(\nu - C\varepsilon_1)\|\nabla u^{(m)}\|^2 \le \frac{\eta^2|g|^2}{2\varepsilon_1}\|\vartheta^{(m)}\|^2 + C_1(t).$$

where $|g|$ is the length of the vector g, C is a constant depending only on Ω and

$$C_1(t) = \frac{1}{2\varepsilon_1}\left\{\nu^2\|\nabla b_0\|^2 + \|b_0\|^4_{L^4(\Omega)} + \eta^2|g|^2\|\vartheta_0\|^2 + \|\partial_t b_0\|^2 + \|f_1\|^2_{V'}\right\}$$

We estimate (6.7) similarly and obtain,

$$(6.11)\ \frac{d}{dt}\|\vartheta^{(m)}\|^2 + 2(\kappa - C\varepsilon_2)\|\nabla\vartheta^{(m)}\|^2 \le \varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t)$$

where C is a constant depending only on Ω and

$$C_2(t) = \frac{1}{2\varepsilon_2}\left\{\|b_0\|^2_{L^4}\|\vartheta_0\|^2_{L^4} + \kappa^2\|\nabla\vartheta_0\|^2 + \|\partial_t\vartheta_0\|^2 + \|f_2\|^2_{H^{-1}}\right\}.$$

Choose $\varepsilon_1 > 0, \varepsilon_2 > 0$ so small that $\nu - C\varepsilon_1 > 0$ and $\kappa - C\varepsilon_2 > 0$ hold true. We fix ε_1. Put

$$\alpha = 2(\nu - C\varepsilon_1), \quad \beta = \frac{\eta^2|g|^2}{2\varepsilon_1}, \quad \gamma = 2(\kappa - C\varepsilon_2)$$

Then α, β, γ are positive constants and the inequalities

$$(6.12)\ \frac{d}{dt}\|u^{(m)}\|^2 + \alpha\|\nabla u^{(m)}\|^2 \le \beta\|\vartheta^{(m)}\|^2 + C_1(t),$$

$$(6.13)\ \frac{d}{dt}\|\vartheta^{(m)}\|^2 + \gamma\|\nabla\vartheta^{(m)}\|^2 \le \varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t)$$

hold true. According to our assumptions, C_1 and C_2 belong to $L^1(0,T)$ and are independent of m. Integrating (6.12), we have

$$(6.14)\ \|u^{(m)}(t)\|^2 + \alpha\int_0^t \|\nabla u^{(m)}(s)\|^2 ds$$

$$\le \|u_{m0}\|^2 + \beta\int_0^t \|\vartheta^{(m)}(s)\|^2 ds + \int_0^t C_1(s) ds$$

Therefore

$$(6.15)\ \int_0^t \|\nabla u^{(m)}(s)\|^2 ds \le \frac{1}{\alpha}\left\{\|u_{m0}\|^2 + \beta\int_0^t \|\vartheta^{(m)}(s)\|^2 ds + \int_0^t C_1(s)ds\right\}$$

$$\le \frac{1}{\alpha}\left\{\|u_{m0}\|^2 + \beta c_0^2\int_0^t \|\nabla\vartheta^{(m)}(s)\|^2 ds + \int_0^t C_1(s)ds\right\}$$

After integrating (6.13), we use (6.15) and obtain

$$(6.16)\ \|\vartheta^{(m)}(t)\|^2 + (\gamma - \frac{\beta c_0^2}{\alpha}\varepsilon_2)\int_0^t \|\nabla\vartheta^{(m)}(s)\|^2 ds \le M_1 \quad (\forall t \in [0,T])$$

where

$$M_1 = \|\vartheta_{m0}\|^2 + \frac{\varepsilon_2}{\alpha}\left\{\|u_{m0}\|^2 + \int_0^T C_1(s)ds\right\} + \int_0^T C_2(s)ds.$$

We can choose, if necessary, ε_2 so small that $\gamma - \frac{\beta c_0^2}{\alpha}\varepsilon_2 > 0$ holds true. Then, integrating (6.16), we obtain

$$\int_0^t \|\vartheta^{(m)}(s)\|^2 ds \le M_1 T$$

Applying this estimate for the right hand side of (6.14), we find

$$(6.17)\ \|u^{(m)}(t)\|^2 + \alpha\int_0^t \|\nabla u^{(m)}(s)\|^2 ds \le \|u_{m0}\|^2 + \beta T M_1 + \int_0^T C_1(s)ds$$

$$(\forall t \in [0,T]).$$

Estimates (6.16) and (6.17) yield the global existence in time of solutions of (6.3), (6.4), (6.5).

Using (6.16) for the right hand side of (6.12), we obtain

$$\frac{d}{dt}\|u^{(m)}\|^2 + \alpha\|\nabla u^{(m)}\|^2 \le \beta M_1 + C_1(t).$$

Put $\alpha' = \alpha c_0^{-2}$ where c_0 is the constant appearing in Poincaré's inequality. Then the above inequality is transformed to

$$\frac{d}{dt}\|u^{(m)}\|^2 + \alpha'\|u^{(m)}\|^2 \le \beta\|\vartheta_{m0}\|^2 + \frac{\beta\varepsilon_2}{\alpha}\|u_{m0}\|^2 + M_2 + C_1(t)$$

where

$$M_2 = \frac{\beta\varepsilon_2}{\alpha}\int_0^T C_1(t)dt + \beta\int_0^T C_2(t)dt.$$

Therefore

$$\frac{d}{dt}\left\{e^{\alpha' t}\|u^{(m)}\|^2\right\} \le e^{\alpha' t}\left\{\beta\|\vartheta_{m0}\|^2 + \frac{\beta\varepsilon_2}{\alpha}\|u_{m0}\|^2 + M_2 + C_1(t)\right\}.$$

Integrating the both side,

$$e^{\alpha' t}\|u^{(m)}(t)\|^2$$

$$\le \|u_{m0}\|^2 + \frac{e^{\alpha' t}-1}{\alpha'}\left\{\beta\|\vartheta_{m0}\|^2 + \frac{\beta\varepsilon_2}{\alpha}\|u_{m0}\|^2 + M_2\right\} + \int_0^t e^{\alpha' s}C_1(s)ds.$$

Therefore

(6.18) $\|u^{(m)}(T)\|^2 \le e^{-\alpha' T}\|u_{m0}\|^2 + \dfrac{1-e^{-\alpha' T}}{\alpha'}\beta\left\{\|\vartheta_{m0}\|^2 + \dfrac{\varepsilon_2}{\alpha}\|u_{m0}\|^2\right\} + C_3$

where

$$C_3 = \frac{1-e^{-\alpha' T}}{\alpha'}M_2 + \int_0^T C_1(t)dt.$$

Put $\gamma' = \gamma c_0^{-2}$. From (6.13), we have

$$\frac{d}{dt}\|\vartheta^{(m)}\|^2 + \gamma'\|\vartheta^{(m)}\|^2 \le \varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t).$$

Therefore

$$\frac{d}{dt}\left\{e^{\gamma' t}\|\vartheta^{(m)}\|^2\right\} \le e^{\gamma' t}(\varepsilon_2\|\nabla u^{(m)}\|^2 + C_2(t)).$$

Integrating this inequality from 0 to T, and using (6.17), we obtain

(6.19) $\|\vartheta^{(m)}(T)\|^2$

$$\le e^{-\gamma' T}\|\vartheta_{m0}\|^2 + \frac{\varepsilon_2}{\alpha}\left\{\frac{\alpha+\beta T\varepsilon_2}{\alpha}\|u_{m0}\|^2 + \beta T\|\vartheta_{m0}\|^2 + D_2\right\} + \int_0^T C_2(t)dt$$

where

$$D_2 = (1 + \frac{\beta T \varepsilon_2}{\alpha}) \int_0^T C_1(t)dt + \beta T \int_0^T C_2(t)dt.$$

Put

$$C_4 = \frac{D_2 \varepsilon_2}{\alpha} + \int_0^T C_2(t)dt.$$

Consider the following system of linear equations for X_1, X_2.

$$(6.20)\quad \left(1 - \frac{\beta \varepsilon_2}{\alpha \alpha'}\right) X_1 - \frac{\beta}{\alpha'} X_2 = \frac{C_3}{1 - e^{-\alpha' T}}$$

$$(6.21)\quad -\frac{(\alpha + \beta T \varepsilon_2)\varepsilon_2}{\alpha^2} X_1 \ + \ \left(1 - e^{-\gamma' T} - \frac{\beta T \varepsilon_2}{\alpha}\right) X_2 = C_4$$

Let us choose once again ε_2 sufficiently small, if necessary, and the equations (6.20) (6.21) have a pair of unique positive solutions $\{X_1, X_2\}$. Note that X_1, X_2 do not depend on m. Put $R_1 = \sqrt{X_1}, R_2 = \sqrt{X_2}$

Suppose $\|u_{m0}\| \le R_1$ and $\|\vartheta_{m0}\| \le R_2$. Then, from the estimates (6.18) and (6.19), we can derive easily $\|u^{(m)}(T)\| \le R_1$ and $\|\vartheta^{(m)}(T)\| \le R_2$. Let $u^{(m)}(t)$ and $\vartheta^{(m)}(t)$ be such solution to (6.3), (6.4) with (6.5). Let us define an operator in $\mathbb{R}^{2m}$ as

$$\mathcal{T} : (u_{m0}, \vartheta_{m0}) \to (u^{(m)}(T), \vartheta^{(m)}(T))$$

Put

$$K = \{(\xi_1, \xi_2, \cdots, \xi_m, \eta_1, \eta_2, \cdots, \eta_m) \in \mathbb{R}^{2m}; \sum_{k=1}^m \xi_k^2 \le R_1^2, \sum_{k=1}^m \eta_k^2 \le R_2^2\}$$

Then K is a compact convex subset of $\mathbb{R}^{2m}$ and $\mathcal{T}$ is a continuous operator from K to K. Brouwer's fixed point theorem assures that there exists a fixed point of the operator $\mathcal{T}$ in K. We denote the fixed point by (u_{m0}, ϑ_{m0})

Let $\{u^{(m)}(t), \vartheta^{(m)}(t)\}$ be the solution to (6.3), (6.4),(6.5), with the initial value $\{u_{m0}, \vartheta_{m0}\}$. Then $\{u^{(m)}(t), \vartheta^{(m)}(t)\}$ is a periodic solution of (6.3) and (6.4). Since the initial values $\{u_{m0}\}_m \subset H$ and $\{\vartheta_{m0}\}_m \subset L^2$ are bounded, it is shown easily from (6.16) and (6.17) that

$$(6.22)\quad \{u^{(m)}(t)\}_m : \text{ bounded sequence in } L^2(0,T;V) \cap L^\infty(0,T;H)$$

(6.23) $\{\vartheta^{(m)}(t)\}_m$: bounded sequence in $L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$.

Moreover, it follows from our choice of the basis and Lemma 3.5 that

(6.24) $\left\{\frac{d}{dt}u^{(m)}\right\}_m$: bounded sequence in $L^p(0,T;V')$

and

(6.25) $\left\{\frac{d}{dt}\vartheta^{(m)}\right\}_m$: bounded sequence in $L^p(0,T;H^{-1}(\Omega))$

where $p = 2$ if $n = 2$, and $p = 4/3$ if $n = 3$.

Now we use Lemma 3.6 for $B_0 = V, B = H, B_1 = V'$ and $p_0 = p_1 = 2(n = 2)$ or $p_0 = p_1 = 4/3(n = 3)$. Taking an appropriate converging subsequence from $\{u^{(m)}, \vartheta^{(m)}\}$, we obtain periodic functions u and ϑ such that

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \vartheta \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$$

satisfying (6.1), (6.2) and Theorem 2.3 is proved.

7 Proof of Theorem2.4

We give here an outline of the proof of Theorem 2.4.

Let $v_i, \vartheta_i (i = 1,2)$ be two solutions. Put $v = v_1 - v_2$ and $\vartheta = \vartheta_1 - \vartheta_2$. After similar calculation to the existence proof, we find the following inequalities hold.

(7.1) $$\frac{d}{dt}\|v\|^2 + \alpha\|\nabla v\|^2 \leq \beta\|\vartheta\|^2$$

(7.2) $$\frac{d}{dt}\|\vartheta\|^2 + \gamma\|\nabla\vartheta\|^2 \leq \delta\|\nabla v\|^2$$

where

$$\alpha = \nu - 2c \max_{0\leq t\leq T} \|v_1(t)\|_{L^4}$$

$$\beta = \frac{\eta^2|g|^2}{C_\nu}$$

$$\gamma = 2(\kappa - c \max_{0\leq t\leq T} \|\vartheta_1(t)\|_{L^4})$$

$$\delta = \frac{c}{4} \max_{0 \le t \le T} \|\vartheta_1(t)\|_{L^4}.$$

Here the constant c depends only on Ω and C_ν depends only on ν.

Put $\alpha' = \alpha c_0^{-2}$, $\gamma' = \gamma c_0^{-2}$ where c_0 is the Poincaré constant. After tedious calculation, we obtain

$$\text{(7.3)} \quad \|v(t)\|^2 \le \{e^{-\alpha' t} + \frac{\beta\delta}{\alpha\alpha'}(1 - e^{-\alpha' t})\}\|v(0)\|^2 + \frac{\beta}{\alpha'}(1 - e^{-\alpha' t})\|\vartheta(0)\|^2,$$

$$\text{(7.4)} \quad \|\vartheta(t)\|^2 \le \frac{\delta}{\alpha}(\frac{\beta\delta T}{\alpha} + 1)\|v(0)\|^2 + (e^{-\gamma' t} + \frac{\beta\delta T}{\alpha})\|\vartheta(0)\|^2.$$

Put $t = T$ in (7.3) and (7.4), and use the relations $v(0) = v(T)$ and $\vartheta(0) = \vartheta(T)$, then we have

$$\text{(7.5)} \quad (1 - \frac{\beta\delta}{\alpha\alpha'})\|v(0)\|^2 - \frac{\beta}{\alpha'}\|\vartheta(0)\|^2 \le 0,$$

$$\text{(7.6)} \quad -\frac{\delta}{\alpha}(\frac{\beta\delta T}{\alpha} + 1)\|v(0)\|^2 + \{1 - e^{-\gamma' T} - \frac{\beta\delta T}{\alpha}\}\|\vartheta(0)\|^2 \le 0.$$

Therefore, if $\max_{0 \le t \le T} \|v_1(t)\|_{L^4}$ and $\max_{0 \le t \le T} \|\vartheta_1(t)\|_{L^4}$ are sufficiently small, we have $\|v(0)\| = \|\vartheta(0)\| = 0$. According to the estimates (7.3) (7.4), we obtain $\|v(t)\| = \|\vartheta(t)\| = 0$.

Remark 7.1 *Our equations contain external forces f_1, f_2 depending on time variable. Therefore, small periodic solutions, if they exist, are not stationary solutions.*

8 Proof of Theorem 2.5

We give here a sketch of the proof of Theorem 2.5. As in the proof of Theorem 2.2, Lemma 3.2 is crucial to prove this theorem.

Let $\{\varphi_m^s\}_{m=1}^\infty \subset V^s(\Omega)$ be eigenfunctions of the Stokes operator in $V^s(\Omega)$, that is,

$$(\nabla\varphi_m^s, \nabla w) = \lambda_m^s(\varphi_m^s, w) \quad (\forall w \in V^s(\Omega)).$$

After orthonormalizing them and using the same symbol, we can consider they are a complete ortho-normal basis of $H^s(\Omega)$ (C.f. [16]). Let $\{\psi_m^s\}_{m=1}^\infty \subset H_0^{1,e}(\Omega)$ be eigenfunctions of the Laplace operator in $H_0^{1,e}(\Omega)$, that is,

$$(\nabla\psi_m^s, \nabla\psi) = \mu_m^s(\psi_m^s, \psi) \quad (\forall\psi \in H_0^{1,e}(\Omega)).$$

They can be considered as a complete ortho-normal basis of $L^{2,e}(\Omega)$. We use Galerkin method.

Let ε_1 and ε_2 be arbitrary positive numbers. According to $(A0)^s$ and $(A1)_\pi^s$, we can find b_0 satisfying the following inequality (Lemma 3.2)

$$(8.1) \quad |((u\cdot\nabla)b_0, u)| \le \varepsilon_1\|\nabla u\|^2 \ (\forall u \in V^s(\Omega),\ \forall t \in [0,T])$$

and, $(A2)_\pi^s$ allows us to choose ϑ_0 satisfying the inequality (Lemma 3.4)

$$(8.2)\, |((u\cdot\nabla)\vartheta_0, \vartheta)| \le \varepsilon_2\|\nabla u\|\,\|\nabla\vartheta\| \ (\forall u \in V^s(\Omega),\ \forall\vartheta \in H_0^{1,e}(\Omega),\ \forall t \in [0,T]).$$

Using (8.1) and (8.2), we obtain a periodic solution $\{u^{(m)}, \vartheta^{(m)}\}$ to (6.3) and (6.4) with $\varphi_j = \varphi_j^s$, $\psi_j = \psi_j^s$ $(1 \le \forall j \le m)$. Furthermore,

$$\{u^{(m)}\}_m : \text{ bounded sequence in } L^2(0,T;V^s) \cap L^\infty(0,T;H^s)$$

$$\{\vartheta^{(m)}\}_m : \text{ bounded sequence in } \in L^2(0,T;H_0^{1,e}) \cap L^\infty(0,T;L^{2,e})$$

$$\left\{\frac{d}{dt}u^{(m)}\right\}_m : \text{ bounded sequence in } L^2(0,T;(V^s)')$$

$$\left\{\frac{d}{dt}\vartheta^{(m)}\right\}_m : \text{ bounded sequence in } L^2(0,T;(H_0^{1,e})'(\Omega))$$

Now we use Lemma 3.6 for $B_0 = V^s, B = H^s, B_1 = (V^s)'$ and $p_0 = p_1 = 2$. Choosing a subsequence from $\{u^{(m)}, \vartheta^{(m)}\}$ appropriately, the limit functions $\{u, \vartheta\}$ are periodic and satisfy (6.1), (6.2) for all $\varphi \in V^s$ and $\psi \in H_0^{1,e}$, and Theorem 2.5 is proved.

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