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$RR = \#BS$ VIA LOCALIZATION OF INDEX

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Abstract. This is a lecture note on the joint work [9, 10] with Fujita and Furuta about an index theory on open Riemannian manifolds and its application to the geometric quantization of Lagrangian fibrations. This note is based on the lecture given in KAIST Toric Topology Workshop 2010.

Contents

1. Introduction

This note is based on the lecture given in KAIST Toric Topology Workshop 2010.

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For a prequantized closed symplectic manifold the *Riemann-Roch number* is defined to be the index of a Spin^c Dirac operator with coefficients in the prequantization line bundle.

Suppose the prequantized symplectic manifold is equipped with a structure of a Lagrangian fibration. A fiber of the Lagrangian fibration is said to be *Bohr-Sommerfeld* if the restriction of the prequantization line bundle to the fiber is trivially flat. Bohr-Sommerfeld fibers appear discretely. Then Andersen showed in [1] that the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers.

Similar phenomena have been observed for several examples of Lagrangian fiber bundles with singular fibers, such as,

- moment maps of toric varieties [6],
- Gelfand-Cetlin's completely integrable system for complex flag manifolds [16],
- Goldman's completely integrable system on the moduli space of flat SU(2)-bundle on a Riemann surface [13, 20],

and for the following generalizations

- pre-symplectic toric manifolds [21],
- Spin^c manifolds [14],
- torus manifolds [24, 19],

and so on.

In this note we try to make clear the mechanism of these phenomena by using a localization of index. The plan of this note is as follows. In Section 2 we recall some fundamental fact on Dirac-type operators. In Section 3 we introduce the notion of a Lagrangian fibration and prove its classification theorem. In Section4 we explain some background of the above phenomena from physics. In Section 5 we explain a localization technique of the index of a Dirac-type operator. The results in this section are based on a joint work [9, 10, 11] with Mikio Furuta and Hajime Fujita. The idea used here is the Witten deformation that is used to prove the Morse inequalities in [32]. As an application, we show that if a prequantized symplectic manifold admits the structure of a Lagrangian fibration with singular fibers, then, the Riemann-Roch number is localized on nonsingular Bohr-Sommerfeld fibers and singular fibers.

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2. Dirac-type operator

In this section we introduce the notion of Dirac-type operator. We first recall the general facts about differential operators in Subsection 2.1. Then, we explain Dirac-type operators in Subsection 2.2. For differential operators see [30, 31] and for Dirac operators and Dirac-type operators see [23, 27, 12].

2.1. **Differential operator.**

Let M be a manifold, W_0 and W_1 complex vector bundles on M.

Definition 2.1. A *linear differential operator of order k* is a linear map $D\colon \Gamma(W_0) \to \Gamma(W_1)$ which satisfies the following conditions:

(i) For an arbitrary section $s \in \Gamma(W_0)$, the support of Ds is contained in that of s.

(ii) For each point $x \in M$, a sufficiently small open neighborhood U of x with local coordinate $x = (x_1, \ldots, x_n)$, a local trivialization $W_i|_U \cong U \times \mathbb{C}^{r_i}$ of W_i $(i = 0, 1)$, and a multi index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, \ldots, k\}^n$, there exists a map $a_\alpha \colon U \to \mathrm{Hom}(\mathbb{C}^{r_0}, \mathbb{C}^{r_1})$ such that for any section $s \in W_0$ with support in U D is written as

$$Ds(x) = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha s(x), \tag{2.1}$$

where $|\alpha| = \sum_i \alpha_i$, $\partial^\alpha = \prod_i \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$.

For a linear differential operator D of order k, by using the local description (2.1), we define the map $\sigma(D) \colon T^*M \to \mathrm{Hom}(W_0, W_1)$ by

$$\sigma(D)\left(\sum_i u_i dx_i\right) = \sum_{|\alpha|=k} a_\alpha(x) \prod_i u_i^{\alpha_i}$$

for each $x \in U \subset M$ and $\sum_i u_i dx_i \in T_x^*M$.

Exercise 2.2. Let $D \colon \Gamma(W_0) \to \Gamma(W_1)$ be a linear differential operator of order k. For $x \in M$, $u \in T_x^*M$, and $e \in W_0$, let $f \in C^\infty(M)$ be a function satisfying $f(x) = 0$ and $df(x) = u$, and $s \in \Gamma(W_0)$ a section with $s(x) = e$. Then, show that $\sigma(D)$ can be written as

$$\sigma(D)(u)(e) = D\left(\frac{1}{k!} f^k s\right)(x)$$

In particular, $\sigma(D)$ does not depend on the choice of local coordinates and local trivializations.

Definition 2.3. For a linear differential operator D of order k, we call the map $\sigma(D) \colon T^*M \to \mathrm{Hom}(W_0, W_1)$ the *principal symbol* of D.

Example 2.4. Let $W = \wedge^\bullet(T^*M \otimes_\mathbb{R} \mathbb{C})$. The exterior derivative $d \colon \Gamma(W) \to \Gamma(W)$ is a first order linear differential operator. The principal symbol of d is given by

$$\sigma(d)(u) = u\wedge$$

for $x \in M$, $u \in T_x^*M$. In fact, for $e \in W_x$, by taking $f \in C^\infty(M)$ and $s \in \Gamma(W)$ with $f(x) = 0$, $df(x) = u$ and $s(x) = e$, respectively,

$$\begin{aligned} \sigma(d)(u)(e) &= d(fs)(x) \\ &= (df \wedge s + f ds)(x) \\ &= u \wedge e. \end{aligned}$$

Example 2.5. Let (M, J) be an almost complex manifold. We extend J to $TM \otimes_\mathbb{R} \mathbb{C}$ complex linearly, and denote its $\sqrt{-1}$ and $-\sqrt{-1}$-eigenspace by $T^{1,0}M$ and $T^{0,1}M$, respectively. We put

$$\wedge^{p,q}T^*M = \wedge^p(T^{1,0}M)^* \otimes_\mathbb{C} \wedge^q(T^{0,1}M)^*.$$

Then, $\wedge^k(T^*M \otimes_\mathbb{R} \mathbb{C})$ is decomposed as

$$\wedge^k(T^*M \otimes_\mathbb{R} \mathbb{C}) = \bigoplus_{p+q=k} \wedge^{p,q}T^*M.$$

For $k=p+q$, we denote the natural projection from $\wedge^k(T^*M\otimes_{\mathbb{R}}\mathbb{C})$ to $\wedge^{p,q}T^*M$ by

$$\pi_{p,q}\colon \wedge^k(T^*M\otimes_{\mathbb{R}}\mathbb{C})\to\wedge^{p,q}T^*M,$$

and define the operator $\bar\partial\colon\Gamma(\wedge^{p,q}T^*M)\to\Gamma(\wedge^{p,q+1}T^*M)$ by

$$\bar\partial=\pi_{p,q+1}\circ d\colon\Gamma(\wedge^{p,q}T^*M)\xrightarrow{d}\Gamma(\wedge^{p+q+1}T^*M)\xrightarrow{\pi_{p,q}}\Gamma(\wedge^{p,q+1}T^*M).$$

$\bar\partial$ is a first order linear differential operator and its principal symbol is given for $x\in M$ and $u\in T_x^*M$ by

$$\sigma(\bar\partial)(u)=\pi_{0,1}(u)\wedge .$$

In fact, for $e\in\wedge^{p,q}T_x^*M$, by taking $f\in C^\infty(M)$ and $s\in\Gamma(\wedge^{p,q}T^*M)$ with $f(x)=0$, $df(x)=u$, and $s(x)=e$, respectively,

$$\begin{aligned}\sigma(\bar\partial)(u)(e)&=\bar\partial(fs)(x)\\&=\pi_{p,q+1}(df\wedge s+fds)(x)\\&=(\pi_{0,1}(df)\wedge s+f\pi_{p,q+1}(ds))(x)\\&=\pi_{0,1}(u)\wedge e.\end{aligned}$$

Definition 2.6. A linear differential operator D is said to be *elliptic* if for $x\in M$ and any non zero $u\in T_x^*M\backslash\{0\}$, $\sigma(D)(u)\in\mathrm{Hom}((W_0)_x,(W_1)_x)$ is an isomorphism.

Next, let (M,g) be an oriented Riemannian manifold. We denote by vol the volume element with respect to g. Let $\langle\ ,\ \rangle_{W_i}$ be a Hermitian metric on W_i. For a linear differential operator $D\colon\Gamma(W_0)\to\Gamma(W_1)$, we define the linear differential operator $D^*\colon\Gamma(W_1)\to\Gamma(W_0)$ to be the one that satisfies the following equation

$$\int_M\langle Ds_0,s_1\rangle_{W_1}vol=\int_M\langle s_0,D^*s_1\rangle_{W_0}vol\quad(\forall s_i\in\Gamma_c(W_i)).$$

For D D^* exists uniquely.

Exercise 2.7. Suppose that locally D has the form (2.1) and the volumes form vol is written as $vol=G(x)^{\frac12}dx_1\wedge\cdots\wedge dx_n$. Then, show that D^* is written as

$$D^*s(x)=G^{-\frac12}\sum_{|\alpha|\le k}(-1)^{|\alpha|}\partial^\alpha\left(G^{\frac12}a_\alpha^*s\right)(x),$$

where a_α^* is is the adjoint of a_α.

Definition 2.8. $D^*\colon\Gamma(W_1)\to\Gamma(W_0)$ is called the *formal adjoint* of D. Moreover, in the case where $W_0=W_1$ D is said to be *formally self-adjoint* if D satisfies $D=D^*$.

Example 2.9. Let (M,g) be an oriented n-dimensional Riemannian manifold and W the complexified exterior algebra bundle $\wedge^\bullet T^*M\otimes_{\mathbb{R}}\mathbb{C}$ on M. First let us define the Hodge $*$-operator. For a p-form $\alpha\in\Gamma(\wedge^pT^*M\otimes_{\mathbb{R}}\mathbb{C})$ on M, suppose that α locally has the form

$$\alpha=\sum_{i_1<\ldots<i_p}\alpha_{i_1,\ldots i_p}dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

Then, we define $*\alpha\in\Gamma(\wedge^{n-p}T^*M\otimes_{\mathbb{R}}\mathbb{C})$ by

$$*\alpha=\sum_{i_1<\ldots<i_p}\alpha_{i_1,\ldots i_p}dx_{i_p}\llcorner\cdots dx_{i_1}\llcorner vol,$$

where we identify TM with T^*M by g and $\llcorner$ is the interior product. It does not depend on the choice of local coordinates and the linear map $*\colon \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C}) \to \Gamma(\wedge^{n-p} T^*M \otimes_{\mathbb{R}} \mathbb{C})$ is uniquely determined. It is called the *Hodge $*$-operator*. The Hodge $*$-operator has the following properties:

(i) $*1 = vol$.

(ii) For a p-form $\alpha \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$, the complex conjugation $\overline{\alpha} \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$ is defined by

$$\overline{\alpha}(u_1, \dots, u_p) = \overline{\alpha(u_1, \dots, u_p)}$$

for $u_1, \dots, u_p \in \Gamma(TM)$. Then,

$$\overline{*\alpha} = *\overline{\alpha}.$$

(iii) For a p-form $\alpha \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$,

$$* * \alpha = (-1)^{p(n-p)}\alpha = (-1)^{np+p}\alpha.$$

(iv) For a p-form $\alpha, \beta \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$,

$$\alpha \wedge *\beta = \beta \wedge *\alpha.$$

(Exercise: Check that the Hodge $*$-operator satisfies the above properties.)

We define $\bar{*}$ by

$$\bar{*}\alpha = *\overline{\alpha}.$$

By the property (ii) and (iii) $\bar{*}$ satisfies the equality

$$\bar{*}\bar{*}\alpha = (-1)^{p(n-p)}\alpha.$$

Using this, we define the Hermitian metric $\langle \ , \ \rangle$ on W in the following way. For $\alpha \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$ and $\beta \in \Gamma(\wedge^q T^*M \otimes_{\mathbb{R}} \mathbb{C})$, if $p = q$, then we define $\langle \alpha, \beta \rangle$ so that $\langle \alpha, \beta \rangle$ satisfies

$$\alpha \wedge \bar{*}\beta = \langle \alpha, \beta \rangle \, vol,$$

and otherwise we define $\langle \alpha, \beta \rangle = 0$. Locally $\langle \ , \ \rangle$ can be described as follows. Let $e_1, \dots e_n$ be a local orthonormal frame of T^*M. For p-forms $\alpha, \beta \in \Gamma(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$ with local forms

$$\alpha = \sum_{i_1 < \dots < i_p} \alpha_{i_1, \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}, \ \beta = \sum_{j_1 < \dots < j_p} \beta_{j_1, \dots j_p} e_{j_1} \wedge \dots \wedge e_{j_p},$$

$\langle \alpha, \beta \rangle$ can be written as

$$\langle \alpha, \beta \rangle = \sum_{i_1 < \dots < i_p} \sum_{j_1 < \dots < j_p} \alpha_{i_1, \dots i_p} \overline{\beta}_{j_1, \dots j_p} \delta^{i_1, \dots, i_p}_{j_1, \dots, j_p}.$$

In particular, $\langle \ , \ \rangle$ is characterized by the condition that $\{e_{i_1} \wedge \cdots e_{i_p}\}_{i_1 < \dots i_p}$ is a local orthonormal frame of $\wedge^p T^*M$.

The formal adjoint d^* of the exterior derivative d with respect to $\langle \ , \ \rangle$ is written as

$$d^* = (-1)^{np+n+1} \bar{*} d \bar{*} = (-1)^{np+n+1} * d * .$$

In fact, for $\alpha \in \Gamma_c(\wedge^{p-1}T^*M \otimes_{\mathbb{R}} \mathbb{C})$ and $\beta \in \Gamma_c(\wedge^p T^*M \otimes_{\mathbb{R}} \mathbb{C})$

$$\begin{aligned}
\int_M \langle d\alpha, \beta \rangle \, vol &= \int_M d\alpha \wedge \bar{*}\beta \\
&= \int_M d(\alpha \wedge \bar{*}\beta) + (-1)^p \alpha \wedge d\bar{*}\beta \\
&= (-1)^{p+(n-p+1)(p-1)} \int_M \alpha \wedge \bar{*}\bar{*}d\bar{*}\beta \\
&= \int_M \left\langle \alpha, (-1)^{np+n+1} \bar{*}d\bar{*}\beta \right\rangle vol. \\
&= \int_M \left\langle \alpha, (-1)^{np+n+1} * d * \beta \right\rangle vol.
\end{aligned}$$

Exercise 2.10. Show that the principal symbol $\sigma(d^*)$ of d^* is given by

$$\sigma(d^*)(u)(e) = -u \llcorner e$$

for $u \in TM$ and $e \in W$.

Example 2.11. Let M be a $2n$-dimensional manifold, J an almost complex structure on M, and g a Riemannian metric on M which saitisfies $g(Ju, Jv) = g(u, v)$. Such a triple (M, J, g) is called an *almost Hermitian manifold*. The almost complex structure J induces an orientation on M. Then, by Example 2.9 the Hermitian metric $\langle \ , \ \rangle$ and the Hodge $*$-operator are defined on $\wedge^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$. Moreover, by construction, the decomposition of $\wedge^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$

$$\wedge^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{p,q} \wedge^{p,q} T^*M$$

is a direct sum decomposition with respect to the Hermitian metric $\langle \ , \ \rangle$, and the Hodge $*$-operator sends a (p, q)-form to a $(n - q, n - p)$-form. Hence, $\bar{*}$ sends a (p, q)-form to a $(n - p, n - q)$-form. Then, the formal adjoint of the operator $\bar{\partial}$ defined in Example 2.5 with respect to the Hermite metric $\langle \ , \ \rangle$ is given by

$$\bar{\partial}^* = -\bar{*}\bar{\partial}\bar{*}.$$

In fact, for $\alpha \in \Gamma_c(\wedge^{p,q-1}T^*M)$ and $\beta \in \Gamma_c(\wedge^{p,q}T^*M)$, since $\alpha \wedge \bar{*}\beta$ is a $(n, n-1)$-form the following equation holds

$$d(\alpha \wedge \bar{*}\beta) = \bar{\partial}(\alpha \wedge \bar{*}\beta).$$

By using this equality,

$$\begin{aligned}
\int_M \left\langle \bar{\partial}\alpha, \beta \right\rangle vol &= \int_M \bar{\alpha} \wedge \bar{*}\beta \\
&= \int_M d(\alpha \wedge \bar{*}\beta) + (-1)^{p+q} \alpha \wedge \bar{\partial}\bar{*}\beta \\
&= (-1)^{p+q+2n(2n-(p+q)+1)+2n-(p+q)+1} \int_M \alpha \wedge \bar{*}\bar{*}\bar{\partial}\bar{*}\beta \\
&= \int_M \left\langle \alpha, \bar{*}\bar{\partial}\bar{*}\beta \right\rangle vol.
\end{aligned}$$

Exercise 2.12. Show that the principal symbol $\sigma(\bar{\partial}^*)$ of $\bar{\partial}^*$ is given by

$$\sigma(\bar{\partial}^*)(u) = -\pi_{0,1}(u) \llcorner,$$

where, $\llcorner$ is the interior product that is defined by using the Hermitian metric $\langle\ ,\ \rangle$ by

$$\pi_{0,1}(u)\llcorner(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := \sum_{i=1}^{k}(-1)^{i-1}\langle v_i, u\rangle\, v_1 \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

2.2. **Clifford module bundle and Dirac-type operator.** Let (M, g) be a Riemannian manifold. In the rest of this note we identify TM with T^*M by the Riemannian metric g.

Definition 2.13. A $\mathbb{Z}_2$*-graded Clifford module bundle* is the following data (W, c).

- (i) W is the direct sum $W = W^0 \oplus W^1$ of two Hermitian vector bundles W^0 and W^1.
- (ii) c is a $\mathbb{R}$-linear map $c\colon TM \to \operatorname{End} W$ that satisfies the following conditions:
  - (a) For any $u \in TM$, $c(u)$ sends each element of W^i to that of W^{i+1}.
  - (b) For any $u \in TM$, $w_1, w_2 \in W$,
  $$\langle c(u)w_1, w_2\rangle_W = -\langle w_1, c(u)w_2\rangle_W\,,$$
  where $\langle\ ,\ \rangle_W$ is the Hermitian metric on W.
  - (c) For any $u, v \in TM$,
  $$c(u) \circ c(v) + c(v) \circ c(u) = -2g(u, v)\,\mathrm{id}_W\,.$$

Example 2.14. Let (M, g) be a Riemannian manifold. We put

$$W = \wedge^{\bullet}T^*M \otimes_{\mathbb{R}} \mathbb{C},\ W^0 = \wedge^{\mathrm{even}}T^*M \otimes_{\mathbb{R}} \mathbb{C},\ W^0 = \wedge^{\mathrm{odd}}T^*M \otimes_{\mathbb{R}} \mathbb{C}.$$

We define $c\colon TM \to \operatorname{End} W$ by

$$c(u)\alpha = u \wedge \alpha - u\llcorner\alpha$$

for $u \in TM$ and $\alpha \in W$. Then, the data (W, c) is a $\mathbb{Z}_2$-graded Clifford module bundle.

Example 2.15. Let (M, J, g) be an almost Hermitian manifold. We put

$$W = \wedge^{0,\bullet}T^*M,\ W^0 = \wedge^{0,\mathrm{even}}T^*M,\ W^1 = \wedge^{0,\mathrm{odd}}T^*M.$$

We define $c\colon TM \to \operatorname{End} W$ by

$$c(u)\alpha = \sqrt{2}\,(\pi_{0,1}(u) \wedge \alpha - \pi_{0,1}(u)\llcorner\alpha)$$

for $u \in TM$ and $\alpha \in W$. Then, the data (W, c) is a $\mathbb{Z}_2$-graded Clifford module bundle.

Example 2.16. Let (M, J, g) be a $2n$-dimensional almost Hermitian manifold. g and J induces the Hermitian metric h on (TM, J) which is defined by

$$h(u, v) = g(u, v) + \sqrt{-1}g(u, Jv) \tag{2.2}$$

for $u, v \in TM$. We put

$$W = \wedge^{\bullet}(TM, J),\ W^0 = \wedge^{\mathrm{even}}(TM, J),\ W^1 = \wedge^{\mathrm{odd}}(TM, J).$$

Let $e_1, \dots, e_n$ be a local orthonormal frame of (TM, J, h). By putting $e_{n+i} = Je_i$, $e_1, \dots, e_{2n}$ is a local orthonormal frame of (TM, g). Then, we define the Hermitian metric on W so that $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{i_1<\dots i_k}$ is a local orthonormal frame of W. This Hermitian metric does not depend on the choice of $e_1, \dots, e_n$ and determines uniquely. We define $c\colon TM \to \operatorname{End} W$ by

$$c(u)\alpha = u \wedge \alpha - u\llcorner_h\alpha$$

for $u \in TM$ and $\alpha \in W$, where $\llcorner_h$ is the interior product with respect to h which is defined by

$$u\llcorner_h(v_1 \wedge v_2 \wedge \cdots \wedge v_k) := \sum_{i=1}^{k}(-1)^{i-1}h(v_i, u)v_1 \wedge \hat{v}_i \wedge \cdots \wedge v_k.$$

Then, (W, c) is a $\mathbb{Z}_2$-graded Clifford module bundle.

Remark 2.17. The Clifford module bundle defined in Example 2.15 is isomorphic to that in Example 2.16 as Clifford module bundles. In fact, as complex vector bundles, (TM, J) is isomorphic to $T^{1,0}M$ by the isomorphism

$$TM \ni u \mapsto \frac{u - \sqrt{-1}Ju}{\sqrt{2}} \in T^{1,0}M.$$

The $\mathbb{C}$-bilinear extension of the Riemannian metric g to $TM \otimes_{\mathbb{R}} \mathbb{C}$ gives an isomorphism

$$T^{1,0}M \ni u \mapsto g(\cdot, u) \in (T^{0,1}M)^*.$$

Let $\varphi\colon TM \to (T^{0,1}M)^*$ be the composition of these isomorphisms. Then, φ preserves the Hermitian metric h on (TM, J) and the Hermitian metric $\langle\ ,\ \rangle$ on $(T^{0,1}M)^*$ defined in Example 2.11. Moreover, φ induces the isomorphism between $\wedge^{\bullet}(TM, J)$ and $\wedge^{0,\bullet}T^*M$ as $\mathbb{Z}_2$-graded Clifford module bundles.

Let (W, c) be a $\mathbb{Z}_2$-graded Clifford module bundle on (M, g).

Definition 2.18. A linear differential operator $D\colon \Gamma(W) \to \Gamma(W)$ of order at most one is said to be *of Dirac-type* if D satisfies the following conditions:

(i) D shifts the degree of $\Gamma(W)$.
(ii) D is formally self-adjoint.
(iii) $\sigma(D) = c$.

Example 2.19. Let (M, g) be an n-dimensional oriented Riemannian manifold and (W, c) the $\mathbb{Z}_2$-graded Clifford module bundle defined in Example 2.14. We define the linear differential operator $D\colon \Gamma(W) \to \Gamma(W)$ on W by

$$D = d + d^*.$$

We call D the *de Rham operator*. By Example 2.4 and Exercise 2.10 the de Rham operator is a Dirac-type operator.

Example 2.20. Let M be a complex manifold and (W, c) the $\mathbb{Z}_2$-graded Clifford module bundle defined in Example 2.15. We define the linear differential operator $D\colon \Gamma(W) \to \Gamma(W)$ on W by

$$D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*).$$

We call D the *Dolbeault operator*. By Example 2.5 and Exercise 2.12 the Dolbeault operator is a Dirac-type operator.

One of the most typical example of a Dirac-type operator is the *Dirac operator* which is constructed as follows. Let (M, g) be a Riemannian manifold, (W, c) a $\mathbb{Z}_2$-graded Clifford module bundle on (M, g).

Definition 2.21. If W is equipped with a connection

$$\nabla^W : \Gamma(W) \to \Gamma(T^*M \otimes W)$$

which satisfies the following properties, then we call the triple (W, c, ∇^W) a **Dirac bundle**:

(i) ∇ is Hermitian, i.e., let $\langle\ ,\ \rangle$ be the Hermitian metric on W. For $X \in \Gamma(TM)$, $s_1, s_2 \in \Gamma(W)$ ∇ satisfies

$$X \langle s_1, s_2 \rangle = \left\langle \nabla^W_X s_1, s_2 \right\rangle + \left\langle s_1, \nabla^W_X s_2 \right\rangle .$$

(ii) ∇ is compatible with the Clifford multiplication c, i.e.,

$$\nabla^W_X (c(Y)\alpha) = c\left(\nabla^{LC}_X Y\right)\alpha + c(Y)\nabla^W_X \alpha$$

for $X, Y \in \Gamma(TM)$ and $\alpha \in \Gamma(W)$, where ∇^{LC} in the first term of the right hand side is the Levi-Civita connection with respect to g.

In general, for a Dirac bundle (W, c, ∇) on (M, g) we define the linear differential operator D on W by

$$D = c \circ \nabla. \tag{2.3}$$

Definition 2.22. We call D the *Dirac operator.*

Example 2.23. Let (M, g) be a Riemannian manifold and (W, c) the $\mathbb{Z}_2$-graded Clifford module bundle that is defined in Example 2.14. The Levi-Civita connection ∇^{LC} on TM with respect to g induces a connection ∇ on W that satisfies the above properties. Then, it is well-known that the Dirac operator associated to these data agrees with the de Rham operator.

Example 2.24. Let (M, J, g) be an almost Hermitian manifold and (W, c) the $\mathbb{Z}_2$-graded Clifford module bundle that is defined in Example 2.15. The Levi-Civita connection ∇^{LC} on TM with respect to g induces a connection ∇ on W that satisfies the above properties. Then, the Dirac operator associated to these data is called the *Spin^c Dirac operator* associated to the almost Hermitian structure.

Remark 2.25. In Example 2.24 suppose that (M, J, g) is a Kähler manifold. Then, it is well-known that the Spin^c Dirac operator defined by (2.3) is written as

$$D = \sqrt{2}\left(\bar\partial + \bar\partial^*\right).$$

In this sense, a Spin^c Dirac operator is a generalization of a Dolbeault operator.

Remark 2.26. Let (W, c, ∇^W) be a Dirac bundle on M and $(E, \langle\ ,\ \rangle_E, \nabla^E)$ a Hermitian vector bundle on M. On the Hermitian vector bundle $W \otimes_{\mathbb{C}} E$ we define the connection and the structure of a $\mathbb{Z}_2$-graded Clifford module bundle by

$$\nabla^{W\otimes E} = \nabla^E \otimes \mathrm{id}_E + \mathrm{id}_W \otimes \nabla^E$$

and

$$c(u)(w \otimes e) = (c(u)w) \otimes e \ (u \in TM,\ w \otimes e \in W \otimes_{\mathbb{C}} E).$$

Then, $W \otimes_{\mathbb{C}} E$ with these data becomes a Dirac bundle again. The Dirac operator associated to this Dirac bundle is called the *Spin^c Dirac operator with coefficients in E.*

Let $D\colon \Gamma(W) \to \Gamma(W)$ be a Dirac-type operator. By the condition (iii) in Definition 2.18 D is elliptic. Here we put

$$D^0 = D|_{W^0} \colon \Gamma(W^0) \to \Gamma(W^1),\ D^1 = D|_{W^1} \colon \Gamma(W^1) \to \Gamma(W^0).$$

Fact 2.27. *If M is closed, then both of $\ker D^0$ and $\ker D^1$ are finite dimensional vector spaces.*

Definition 2.28. In case where M is closed, we define the *index* of the Dirac-type operator D by

$$\operatorname{ind} D = \ker D^0 - \ker D^1.$$

Proposition 2.29. *$\operatorname{ind} D$ is invariant under continuous deformation of the data. In particular, $\operatorname{ind} D$ depends only on (W, c) and does not depend on the choice of D itself.*

Theorem 2.30. *Let (M, g) be an n-dimensional closed oriented Riemannian manifold, and $D\colon \Gamma(\wedge^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}) \to \Gamma(\wedge^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C})$ the de Rham operator. Then, we have the natural isomorphism*

$$\ker \left(D|_{\wedge^k T^*M \otimes_{\mathbb{R}} \mathbb{C}}\right) \cong H^k(M; \mathbb{C}).$$

In particular, the index of the de Rham operator is nothing but the Euler characteristic of M.

Theorem 2.31. *Let M be a closed complex manifold and $D\colon \Gamma(\wedge^{0,\bullet} T^*M) \to \Gamma(\wedge^{0,\bullet} T^*M)$ the Dolbeault operator. Then, we have the natural isomorphism*

$$\ker (D|_{\wedge^{0,q} T^*M}) = H^q(M; \mathcal{O}_M),$$

where $H^q(M; \mathcal{O}_M)$ is the Dolbeault cohomology. In particular, the index of the Dolbeault operator is nothing but the Euler characteristic of the Dolbeault cohomology.

Theorem 2.32 (Hirzebruch-Riemann-Roch)**.** *Let (M, J, g) be a closed almost Hermitian manifold, E a Hermitian vector bundle on M, and $D\colon \Gamma(\wedge^{0,\bullet} T^*M \otimes E) \to \Gamma(\wedge^{0,\bullet} T^*M \otimes E)$ a Spin^c Dirac operator with coefficients in E. Then, $\operatorname{ind} D$ is given by*

$$\operatorname{ind} D = \int_M ch(E) Td(TM, J),$$

where $Td(TM, J)$ is the Todd class for the complex vector bundle (TM, J).

3. Lagrangian fibration

3.1. **Symplectic manifold.**

In this subsection we introduce the notion of a symplectic manifold. The books [26, 5] are good introductory references of this direction.

Definition 3.1. Let M be a manifold. A *symplectic structure* on M is a two-form $\omega \in \Omega^2(M)$ that satisfies the following conditions:

(i) ω is closed, namely, $d\omega = 0$.
(ii) ω is nondegenerate, namely, for each $x \in M$ and $u \neq 0 \in T_xM$, there exists a tangent vector $v \in T_xM$ such that $\omega_x(u, v) \neq 0$.

The pair (M, ω) is called the *symplectic manifold*.

Proposition 3.2. *Let (M, ω) be a finite dimensional symplectic manifold. Then, the dimension of M is even.*

Exercise 3.3. Prove Proposition 3.2. (Show that a finite dimensional vector space equipped with a nondegenerate bilinear form is even dimensional.)

Exercise 3.4. Let (M, ω) be a $2n$-dimensional symplectic manifold. Show that $\dfrac{\omega^n}{n!}$ is a volume form on M. In particular, any symplectic manifold is orientable. In the rest of this note, we consider the orientation determined by this volume form on a symplectic manifold.

Example 3.5. We define the two-form ω_0 on $\mathbb{R}^{2n}$ by

$$\omega_0(u,v) = \sum_{i=1}^{n}(u_i v_{n+i} - v_i u_{n+i})$$

for $u = (u_1, \ldots, u_{2n})$, $v = (v_1, \ldots, v_{2n}) \in \mathbb{R}^{2n}$. Then, $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic manifold. By using a standard coordinate $(x_1, \ldots, x_n, y_1 \ldots, y_n)$ on $\mathbb{R}^{2n}$ ω_0 is written as

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

It is well-known that every $2n$-dimensional symplectic manifold is locally identified with $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 3.6 (Darboux)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold. For each $x \in M$, there exists a coordinate neighborhood (U, φ) around x such that $\varphi^*\omega_0 = \omega|_U$.*

Example 3.7. Let $T^2 = (\mathbb{R}/\mathbb{Z})^2$ and (x, y) a coordinate of T^2. The pair (T^2, ω) of T^2 and the two-form $\omega = dx \wedge dy$ on T^2 is a symplectic manifold.

Example 3.8. Let N be a manifold. On the cotangent bundle T^*N of N there is a canonical one-form λ which is defined by

$$\lambda_\xi(v) = \xi\left(\pi_*(v)\right) \ \ \xi \in T^*N, \ v \in T_{u^*}(T^*N),$$

where $\pi \colon T^*N \to N$ is the bundle projection. λ is called the *Liouville form*. Then, the exterior derivative $d\lambda$ defines a symplectic structure on T^*N. In fact, a local coordinate $(q_1, \ldots, q_n)$ of N induces a local coordinate of $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ of T^*N which is defined by

$$p_i(\xi) = \xi\left(\frac{\partial}{\partial q_i}\right).$$

By using this coordinate, λ is written as

$$\lambda = \sum_{i=1}^{n} p_i dq_i.$$

Hence, we have

$$d\lambda = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Exercise 3.9. In Example 3.8, let α be a one-form on N, and we use the notation $\iota_\alpha \colon N \to T^*N$ to denote the one-form α when we think of it as a section of T^*N. Show the following equality holds

$$\iota_\alpha^*\lambda = \alpha.$$

Example 3.10. An oriented surface with volume form on it is an example of a symplectic manifold.

Example 3.11. Let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. We define the adjoint action of G on $\mathfrak{g}$ by

$$\mathrm{Ad}_g(\xi) = \frac{d}{dt}\Big|_{t=0} g e^{t\xi} g^{-1}.$$

Ad induces the G-action on the dual space $\mathfrak{g}^*$ by

$$\left(\mathrm{Ad}_g^*(\xi)\right)(\eta) = \xi\left(\mathrm{Ad}_{g^{-1}}(\eta)\right) \quad \xi \in \mathfrak{g}^*,\ \eta \in \mathfrak{g}.$$

This action is called the *coadjoint action* of G on $\mathfrak{g}^*$. An orbit of the coadjoint action is equipped with a symplectic structure which is defined as follows. For any $\xi \in \mathcal{O}$, we define the map $\varphi_\xi \colon G \to \mathcal{O}$ by

$$\varphi_\xi(g) = \mathrm{Ad}_g(\xi).$$

Since the map $(\varphi_\xi)_* \colon \mathfrak{g} \to T_\xi\mathcal{O}$ induced by φ_ξ is surjective , for any $u \in T_\xi\mathcal{O}$, there exists an element $\eta_u \in \mathfrak{g}$ such that

$$(\varphi_\xi)_*(\eta_u) = u.$$

η_u is determined uniquely up to adding an element of the Lie algebra of the stabilizer of ξ. Then, for u and $v \in T_\xi\mathcal{O}$, $\xi([\eta_u, \eta_v])$ does not depend on the choice of η_u and η_v, and depends only on ξ, u, and v. Thus, we define

$$\omega_\xi(u, v) = \xi([\eta_u, \eta_v]). \tag{3.1}$$

ω is a symplectic structure on $\mathcal{O}$.

Exercise 3.12. (1) For $\xi \in \mathcal{O}$, u, $v \in T_\xi\mathcal{O}$, show that $\xi([\eta_u, \eta_v])$ does not depend on the choice of η_u and η_v, and depends only on ξ, u, and v.
(2) Show that (3.1)is a symplectic structure on $\mathcal{O}$.

Example 3.13. Let $G = SU(n)$ $(2 \leq n)$ and $\mathfrak{g}$ the Lie algebra $\mathfrak{su}(n)$ of G. Let Σ be a closed connected Riemann surface with genus $g \geq 2$ and P the principal G-bundle on Σ. Since G is simply connected P is necessarily trivial. We fix a trivialization $P \cong \Sigma \times G$.

The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ determines the trivial Lie algebra bundle $P \times_{\mathrm{Ad}} \mathfrak{g} \cong \Sigma \times \mathfrak{g}$ on Σ. We identify a differential form which takes values in $P \times_{\mathrm{Ad}} \mathfrak{g}$ with a $\mathfrak{g}$-valued differential form, and denote the space of $\mathfrak{g}$-valued p-forms by $\Omega^p(\Sigma; \mathfrak{g})$.

We denote by $\mathcal{A}$ the space of connections on P. $\mathcal{A}$ is an affine space modeled on the space $\Omega^1(\Sigma; \mathfrak{g})$. We identify $\mathcal{A}$ with $\Omega^1(\Sigma; \mathfrak{g})$ by fixing the trivial connection on P as a base point. We denote by $\mathcal{G}$ the gauge group of P, namely, the group of bundle automorphisms which covers the identity $\mathrm{id} \colon \Sigma \to \Sigma$. Since P is trivial, $\mathcal{G}$ is identified with the space $C^\infty(\Sigma, G)$ of maps from Σ to G. The gauge group $\mathcal{G}$ acts on $\mathcal{A}$ from the right by pull-back. Under the identifications $\mathcal{A} \cong \Omega^1(\Sigma; \mathfrak{g})$ and $\mathcal{G} \cong C^\infty(\Sigma, G)$, the action is written by

$$A \cdot g = g^{-1}Ag + g^{-1}dg$$

for $A \in \mathcal{A}$, and $g \in \mathcal{G}$.

Let $\mathcal{M}_g$ be the moduli space of flat G-bundles on Σ which is defined by

$$\mathcal{M}_g = \{A \in \mathcal{A} \mid F_A = 0\}/\mathcal{G}_\Sigma,$$

where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature form of A.

On $\mathcal{A}$, there exists a symplectic form ω which is defined by

$$\omega_A(a, a') = -\int_\Sigma \langle a, a' \rangle$$

for $A \in \mathcal{A}$, and $a, a' \in T_A\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$, where $\langle a, a' \rangle$ is the two-form on Σ that is defined by

$$\langle a, a' \rangle_x (u_1, u_2) = \langle a_x(u_1), a'_x(u_2) \rangle - \langle a_x(u_2), a'_x(u_1) \rangle$$

for $x \in \Sigma$ and $u_1, u_2 \in T_x\Sigma$. Then, it is easy to check that the $\mathcal{G}$-action on $\mathcal{A}$ preserves ω. Moreover, ω descends to a symplectic strurcutre on (the smooth part of) $\mathcal{M}_g$. For more details, see [3].

Exercise 3.14. Show that the $\mathcal{G}$-action on $\mathcal{A}$ preserves ω. Moreover, ω descends to a symplectic strurcutre on (the smooth part of) $\mathcal{M}_g$.

Let (M, ω) be a symplectic manifold.

Definition 3.15. An almost complex structure $J \in \mathrm{End}(TM)$ on (M, ω) is said to be *compatible with* ω if J satisfies the following conditions:

(i) $\omega(Ju, Jv) = \omega(u, v)$,
(ii) We put $g(u, v) = \omega(u, Jv)$. Then, g defines a Riemannian metric on M.

Example 3.16. Let J be the standard complex structure on $\mathbb{R}^{2n}$ defined by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},\ J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}.$$

Then, J is compatible with the symplectic structure ω_0 on $\mathbb{R}^{2n}$ defined in Example 3.5.

Then, the following proposition is well-known.

Proposition 3.17. *Any symplectic manifold admits a compatible almost complex structure. Moreover, the space of compatible almost complex structures on a symplectic manifold is contractible.*

One of the most important examples of symplectic manifolds is the following Kähler manifold.

Definition 3.18. A symplectic manifold which has an integrable compatible almost complex structure is called the *Kähler manifold*.

It is known that symplectic manifolds in Example 3.10, Example 3.11, and Example 3.13 are Kähler. Another important example of a Kähler manifold is the following symplectic toric manifold. To introduce a symplectic toric manifold we prepare some terminology for a group action.

Let G be a compact, connected Lie group and $\mathfrak{g}$ its Lie algebra. Let (M, ω) be a symplectic manifold equipped with a left action of G preserving ω.

Definition 3.19. A *moment map* of the G-action on (M, ω) is a map $\mu \colon M \to \mathfrak{g}^*$ which satisfies the following two conditions

(i) $X_\xi \lrcorner \omega = -d\langle \mu, \xi \rangle$
(ii) $\mu(g \cdot x) = \mathrm{Ad}^*(g^{-1}) \circ \mu(x)$,

for $\xi \in \mathfrak{g}$, $x \in M$, and $g \in G$, where X_ξ is the infinitesimal action which is defined by

$$X_\xi(x) := \frac{d}{dt} \exp(t\xi)x\Big|_{t=0},$$

$\langle\ ,\ \rangle$ in the condition (i) is the natural pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$, and Ad^* in the condition (ii) is the coadjoint action of G on $\mathfrak{g}^*$.

Proposition 3.20 ([15])**.** *Suppose that* (M, ω) *is equipped with an effective Hamiltonian action of a torus* T*. Then,* $\dim M \geq 2 \dim T$*.*

The maximal case in Proposition 3.20 is the symplectic toric manifold.

Definition 3.21. A $2n$-dimensional symplectic manifold equipped with an effective Hamiltonian action of an n-dimensional torus is called the *symplectic toric manifold.*

It is known in [7] that a symplectic toric manifold is a Kähler manifold. For Hamiltonian group actions see [4, 17] and especially for symplectic toric manifolds, see [15].

For an example of a non Kähler symplectic manifold, see Example 3.26.

3.2. **Nonsingular case.** In this subsection we introduce the notion of a Lagrangian fibration and explain their classification. For Lagrangian fibrations see [8, 34].

Let (M, ω) be a symplectic manifold.

Definition 3.22. A map $\pi\colon (M, \omega) \to B$ from (M, ω) to a manifold B is called a *Lagrangian fibration* if π satisfies the following conditions:

(i) π is a fiber bundle.
(ii) For each $b \in B$, the fiber $\pi^{-1}(b)$ is a Lagrangian submanifold of (M, ω), namely, it satisfies
  (a) $\omega|_{\pi^{-1}(b)} \equiv 0$,
  (b) $\dim \pi^{-1}(b) = \frac{1}{2} \dim M$.

Example 3.23. Let $\pi\colon (\mathbb{R}^{2n}, \omega_0) \to \mathbb{R}^n$ be the map that is defined by

$$\pi(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_n).$$

Then, π is a Lagrangian fibration.

Example 3.24. Let (T^2, ω) be the torus with the standard symplectic structure defined in Example 3.7. Then, the projection $\pi\colon (T^2, \omega) \to S^1$ to the first factor is a Lagrangian fibration.

Example 3.25. The cotangent bundle $(T^*T^n, d\lambda)$ of an n-dimensional torus $T^n = (\mathbb{R}/\mathbb{Z})^n$ is identified with $(\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i)$, where p_i and q_i are coordinates of the $\mathbb{R}^n$-factor and the T^n-factor,respectively. With this identification, the projection $\pi\colon (\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i) \to \mathbb{R}^n$ to $\mathbb{R}^n$ is a Lagrangian fibration.

Example 3.26 (Kodaira-Thurston's example [18, 29])**.** We consider $\mathbb{R}^2 \times T^2$ with symplectic structure $dx_1 \wedge dx_2 + dy_1 \wedge dy_2$, where (x_1, x_2) and (y_1, y_2) are coordinates of the $\mathbb{R}^2$-factor and the T^2-factor, respectively. Define the action of $\mathbb{Z}^2$ on $\mathbb{R}^2 \times T^2$ by

$$n(x, y) = (x + n, A_n y) \ (n \in \mathbb{Z}^2,\ (x, y) \in \mathbb{R}^2 \times T^2),$$

where

$$A_n = \begin{pmatrix} 1 & n_2 \\ 0 & 1 \end{pmatrix}.$$

This $\mathbb{Z}^2$-action preserves the symplectic structure $dx_1 \wedge dx_2 + dy_1 \wedge dy_2$ on $\mathbb{R}^2 \times T^2$ and the quotient space $(\mathbb{R}^2 \times T^2, dx_1 \wedge dx_2 + dy_1 \wedge dy_2)/\mathbb{Z}^2$ becomes a symplectic manifold. Note that this symplectic manifold does not admit any Kähler structure [29]. The map form $(\mathbb{R}^2 \times T^2, dx_1 \wedge dx_2 + dy_1 \wedge dy_2)/\mathbb{Z}^2$ to T^2 defined by

$$(\mathbb{R}^2 \times T^2, dx_1 \wedge dx_2 + dy_1 \wedge dy_2)/\mathbb{Z}^2 \to T^2\colon (x, y) \mapsto (x_2, y_2)$$

is a Lagrangian fibration.

The next lemma gives a local model for a Lagrangian fibration.

Lemma 3.27 (Arnold-Liouville's theorem [2])**.** *Let $\pi\colon (M,\omega) \to B$ be a Lagrangian fibration with compact connected fibers. Then, π is locally identified with the Lagrangian fibration $(\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i) \to \mathbb{R}^n$ in Example 3.25. Namely, for each $b \in B$ and coordinate neighborhood (U, φ) around b there exists a symplectomorphism $\psi\colon (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}) \to (\varphi(U) \times T^n, \sum_i dp_i \wedge dq_i)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\pi^{-1}(U), \omega|_{\pi^{-1}(U)}) & \xrightarrow{\psi} & (\varphi(U) \times T^n, \sum_i dp_i \wedge dq_i) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ U & \xrightarrow{\varphi} & \varphi(U). \end{array}$$

In the rest of this note we assume that every Lagrangian fibration has compact connected fibers.

Now we investigate automorphisms of the local model.

Definition 3.28. Let $\pi\colon (M,\omega) \to B$ be a Lagrangian fibration and s a section of π. The section s is said to be *Lagrangian* if s satisfies $s^*\omega = 0$.

Exercise 3.29. For a section $\mathbb{R}^n \to \mathbb{R}^n \times T^n \colon p \mapsto (p, s(p))$ of $\mathbb{R}^n \times T^n \to \mathbb{R}^n$, we define the diffeomorphism $\psi\colon \mathbb{R}^n \times T^n \to \mathbb{R}^n \times T^n$ by

$$\psi(p,q) = (p, q + s(p)),$$

where we use the product structure $+$ of $T^n = (\mathbb{R}/\mathbb{Z})^n$ as an abelian group. Then, show that ψ is a symplectomorphism if and only if s is Lagrangian.

Lemma 3.30. *Let $\pi\colon (\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i) \to \mathbb{R}^n$ is the Lagrangian fibration in Example 3.25, $\psi\colon (\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i) \to (\mathbb{R}^n \times T^n, \sum_i dp_i \wedge dq_i)$ is a fiber-preserving symplectomorphism. Then, there exists a matrix $X \in \mathrm{GL}_n(\mathbb{Z})$, a constant $c \in \mathbb{R}^n$, and a Lagrangian section $\mathbb{R}^n \to \mathbb{R}^n \times T^n \colon p \mapsto (p, s(p))$ of π such that ψ is written as*

$$\psi(p,q) = \left({}^tX^{-1}p + c, Xq + s(p)\right).$$

In the rest of this note we assume that every Lagrangian fibration has compact connected fibers. By Lemma 3.27 and Lemma 3.30 we can obtain the following proposition.

Proposition 3.31. *Let $\pi\colon (M,\omega) \to B$ be a Lagrangian fibration. Then, there exists a coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of B and for each $\alpha \in A$ there exists a symplectomorphism $\psi_\alpha\colon \left(\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}\right) \to (\varphi_\alpha(U_\alpha) \times T^n, \sum_i dp_i \wedge dq_i)$ such that the following diagram commutes*

$$\begin{array}{ccc} (\pi^{-1}(U_\alpha), \omega|_{\pi^{-1}(U_\alpha)}) & \xrightarrow{\psi_\alpha} & (\varphi_\alpha(U_\alpha) \times T^n, \sum_i dp_i \wedge dq_i) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ U_\alpha & \xrightarrow{\varphi_\alpha} & \varphi_\alpha(U_\alpha). \end{array}$$

Moreover, on each nonempty overlap $U_\alpha \cap U_\beta$ there exist locally constant maps $X_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{Z})$, $c_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{R}^n$, and a Lagrangian section $u_{\alpha\beta}\colon \varphi_\alpha(U_\alpha \cap U_\beta) \to (\varphi_\alpha(U_\alpha \cap U_\beta) \times T^n, \sum_i dp_i \wedge dq_i)$ such that the overlap map is written as

$$\psi_\alpha \circ \psi_\beta^{-1}(p,q) = \left({}^tX_{\alpha\beta}^{-1}p + c_{\alpha\beta}, X_{\alpha\beta}q + u_{\alpha\beta}\left(\varphi_\alpha \circ \varphi_\beta^{-1}(p)\right)\right).$$

Definition 3.32. We call the coordinate neighborhood system $\{(U_\alpha, \varphi_\alpha)\}$ of B in Proposition 3.31 the **integral affine structure**.

The existence of an integral affine structure is a sufficiently and necessary condition for the existence of a Lagrangian fibration.

Proposition 3.33. *Let B be a manifold. B is a base space of a Lagrangian fibration if and only if B admits an integral affine structure.*

It follows form the above argument that if B is a base space of a Lagrangian fibration, then B admits an integral affine structure. Conversely, suppose that B admits an integral affine structure $\{(U_\alpha, \varphi_\alpha)\}$. Then, we can construct a Lagrangian fibration on B in the following way. For each $\alpha \in A$ let $\psi_\alpha^{T^*B} \colon T^*B|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ be the local trivialization of the cotangent bundle T^*B induced from $\{(U_\alpha, \varphi_\alpha)\}$. On each nonempty overlap $U_\alpha \cap U_\beta$, by using the map $X_{\alpha\beta}$ in Proposition 3.31, the overlap map is writen as

$$\psi_\alpha^{T^*B} \circ (\psi_\beta^{T^*B})^{-1}(b, u) = (b, X_{\alpha\beta} u). \tag{3.2}$$

In particular, the structure group of T^*B reduces to $\mathrm{GL}_n(\mathbb{Z})$. By (3.2), the trivial $\mathbb{Z}^n$-bundles $U_\alpha \times \mathbb{Z}^n (\subset U_\alpha \times \mathbb{R}^n)$ are patched together to form a $\mathbb{Z}^n$-bundle on B which is denoted by $\pi_\mathbb{Z} \colon T^*B_\mathbb{Z} \to B$. We also denote the quotient bundle $\pi_T \colon T^*B/T^*B_\mathbb{Z} \to B$ by T^*B_T. The standard symplectic structure on T^*B defined in Example 3.8 induces a symplectic structure on T^*B_T. We denote it by $\omega_{T^*B_T}$. Then, $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ becomes a Lagrangian fibration.

Definition 3.34. We call $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ the **canonical model**.

Remark 3.35. (1) By construction, $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ is a fiber bundle with fiber an Abelian group T^n. In particular T^*B_T acts fiberwise on itself.
(2) The zero section of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ is Lagrangian.
(3) $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ fiberwise acts on $\pi \colon (M, \omega) \to B$. In fact, for $b \in B$, $u \in \pi_T^{-1}(b)$, and $x \in \pi^{-1}(b)$ we define the action by

$$u \cdot x = \psi_\alpha^{-1}\left(\psi_\alpha^{T^*B_T}(u) + \psi_\alpha(x)\right), \tag{3.3}$$

where we regard both of $\psi_\alpha^{T^*B_T}(u) \in \{b\} \times T^n$ and $\psi_\alpha(x) \in \{b\} \times T^n$ as elements of T^n and the notation "+" in the right hand side represents the product structure of the Abelian group T^n. The action (3.3) does not depend on the choice of a local trivialization, hence it is well-defined. This action is free and transitive.

Let $\pi \colon (M, \omega) \to B$ be a Lagrangian fibration, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ the integral affine structure on B associated to π, and $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ the canonical model. By Lemma 3.27 $\pi \colon (M, \omega) \to B$ is locally symplectomorphic to $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$. In fact, let $\psi_\alpha^{T^*B_T} \colon T^*B_T|_{U_\alpha} \to \varphi_\alpha(U_\alpha) \times T^n$ be the local trivialization of T^*B_T naturally induced from the local trivialization $\psi_\alpha^{T^*B}$ of T^*B. Then the composition of $\psi_\alpha^{T^*B_T}$ and the local trivialization ψ_α of $\pi \colon (M, \omega) \to B$ as in Proposition 3.31

$$h_\alpha = \psi_\alpha^{-1} \circ \psi_\alpha^{T^*B_T} \colon (T^*B_T, \omega_{T^*B_T})|_{U_\alpha} \to (M, \omega)|_{U_\alpha}$$

is equivariant with respect to the fiberwise T^*B_T-actions. Moreover, it is a fiber-preserving symplectomorphism such that the diagram

$$\begin{array}{ccc} (T^*B_T, \omega_{T^*B_T})|_{U_\alpha} & \xrightarrow{h_\alpha} & (M,\omega)|_{U_\alpha} \\ & \searrow^{\pi_T} \quad \swarrow_{\pi} & \\ & U_\alpha & \end{array}$$

is commutative.

Lemma 3.36. *On each $U_\alpha \cap U_\beta$, there exists a Lagrangian section $u_{\alpha\beta} \colon U_\alpha \cap U_\beta \to (T^*B_T, \omega_{T^*B_T})|_{U_\alpha \cap U_\beta}$ of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ such that $h_\alpha \circ h_\beta^{-1}$ is written as*

$$h_\alpha \circ h_\beta^{-1}(x) = u_{\alpha\beta}(\pi(x)) \cdot x.$$

Proof. Since $h_\alpha \circ h_\beta^{-1}(x)$ and x belong to the same fiber of π and the fiberwise action (3.3) of $T^*B_T \to B$ on $(M,\omega) \to B$ is free and transitive, there uniquely exists $u_{\alpha\beta} \in \pi_T^{-1}(\mu(x))$ such that $h_\alpha \circ h_\beta^{-1}(x) = u_{\alpha\beta} \cdot x$. $u_{\alpha\beta}$ depends only on $\mu(x)$ since h_α is equivariant with respect to the fiberwise T^*B_T-actions. Moreover, Lemma 3.29 and the fact that h_α is symplectomorphism imply that $u_{\alpha\beta}$ is a Lagrangian section. □

Let $\mathscr{S}$ be the sheaf of germs of Lagrangian section of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$. $|SS$ is the sheaf of Abelian groups since the fiber of $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ has the structure of an Abelian group as explained in Remark 3.35. By definition $\{u_{\alpha\beta}\}$ forms a Čech one-cocycle on B with coefficients in $\mathscr{S}$. The cohomology class determined by $\{u_{\alpha\beta}\}$ does not depend on the choice of a specific integral affine structure and depends only on $\pi \colon (M,\omega) \to B$. We denote the cohomology class by $u \in H^1(B; \mathscr{S})$.

Definition 3.37. u is called the *Lagrangian class* of $\pi \colon (M,\omega) \to B$.

By the construction of u we can show the lemma.

Lemma 3.38. *There exists a symplectomorphism $\psi \colon (M,\omega) \to (T^*B_T, \omega_{T^*B_T})$ such that the diagram*

$$\begin{array}{ccc} (M,\omega) & \xrightarrow{\psi} & (T^*B_T, \omega_{T^*B_T}) \\ & \searrow^{\pi} \quad \swarrow_{\pi_T} & \\ & B & \end{array}$$

commutes if and only if u vanishes.

Proposition 3.39. *If there exists a Lagrangian section of $\pi \colon (M,\omega) \to B$ if and only if u vanishes.*

Proof. If $u = 0$, by Lemma 3.38 $\pi \colon (M,\omega) \to B$ and $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ are symplectomorphic. Moreover, by construction $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ is equipped with a Lagrangian section. Therefore, $\pi \colon (M,\omega) \to B$is also equipped with a Lagrangian section.

Conversely, suppose that $\pi_T \colon (T^*B_T, \omega_{T^*B_T}) \to B$ has a Lagrangian section s. By using the T^*B_T-action (3.3) on M we define the map from T^*B_T to M by

$$u \mapsto u \cdot s(\pi_T(u)).$$

Then, it is a symplectomorphism between $\pi_T\colon (T^*B_T, \omega_{T^*B_T}) \to B$ and $\pi\colon (M,\omega) \to B$ covering the identity id_B. □

Lagrangian fibrations are classified with the integral affine structures and the Lagrangian classes.

Theorem 3.40 ([8, 34])**.** *Two Lagrangian fibrations $\pi_1\colon (M_1,\omega_1) \to B_1$ and $\pi_2\colon (M_2,\omega_2) \to B_2$ are fiber-preserving symplectomorphic if and only if there exists a diffeomorphism $\varphi\colon B_1 \to B_2$ such that φ preserves the integral affine structures and $\varphi^* u_2 = u_1$. Moreover, if a manifold B, an integral affine structure on B, and the cohomology class $u \in H^1(B;\mathscr{S})$ are given, then, there exists a Lagrangian fibration $\pi\colon (M,\omega) \to B$ that realizes them.*

3.3. **Singular case.** In the previous subsection we dealt with Lagrangian fibration without singular fibers. But Lagrangian fibrations which admit singular fibers appear in the various scenes in Mathematics. In this subsection we introduce some examples of them.

3.3.1. *Locally toric Lagrangian fibration.* First we introduce locally toric Lagrangian fibrations. Let $\omega_{\mathbb{C}^n}$ be the symplectic structure of $\mathbb{C}^n$ which is defined by

$$\omega_{\mathbb{C}^n} = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i.$$

We define the map $\mu_{\mathbb{C}^n}\colon (\mathbb{C}^n,\omega) \to \mathbb{R}^n_{\geq 0}$ by

$$\mu_{\mathbb{C}^n}(z) = (|z_1|^2, \ldots, |z_n|^2), \tag{3.4}$$

where

$$\mathbb{R}^n_{\geq 0} = \{p = (p_1,\ldots,p_n) \in \mathbb{R}^n \mid p_i \geq 0 \ i = 1,\ldots,n\}.$$

Note that $\mu_{\mathbb{C}^n}$ is a moment map of the standard $T^n = (\mathbb{R}/\mathbb{Z})^n$-action on $\mathbb{C}^n$

$$t \cdot z = (e^{2\pi\sqrt{-1}t_1} z_1, \ldots, e^{2\pi\sqrt{-1}t_n} z_n).$$

Let (M,ω) be a $2n$-dimensional symplectic manifold and B an n-dimensional manifold with corners.

Definition 3.41. A map $\mu\colon (M,\omega) \to B$ is called the *locally toric Lagrangian fibration* if there exist a coordinate neighborhood system $\{(U_\alpha,\varphi_\alpha)\}$ of B as a manifold with corners and a symplectomorphism $\psi_\alpha\colon (\mu^{-1}(U_\alpha), \omega|_{\mu^{-1}(U_\alpha)}) \to \left(\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha(U_\alpha)), \omega_{\mathbb{C}^n}|_{\mu_{\mathbb{C}^n}^{-1}(\varphi_\alpha(U_\alpha))}\right)$ such that $\mu_{\mathbb{C}^n} \circ \psi_\alpha = \varphi_\alpha \circ \mu$.

Example 3.42. In case where $\partial B \neq \emptyset$ any Lagrange fibration $\pi\colon (M,\omega) \to B$ is an example of locally toric Lagrangian fibrations.

Example 3.43. A moment map of a symplectic toric manifold is an example of locally toric Lagrangian fibrations.

Example 3.44. Let $c \in \mathbb{N}$ be a positive integer. We consider the diagonal Hamiltonian S^1-action on $(\mathbb{C}^2, \omega_{\mathbb{C}^2})$ with moment map

$$\Phi(z) := \|z\|^2 - c.$$

It is well-known that the symplectic quotient $\left(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}\right)/S^1$ is $\mathbb{C}P^1$ with c times Fubini-Study form ω_{FS}. In the rest of this example we identify $(\mathbb{C}P^1, c\omega_{FS})$ with $\left(\Phi^{-1}(0), \omega_{\mathbb{C}^2}|_{\Phi^{-1}(0)}\right)/S^1$.

Let $\tilde{\mu}\colon (\tilde{M}, \tilde{\omega}) \to \tilde{B}$ be the singular Lagrangian fibration which is defined by

$$\begin{aligned}(\tilde{M}, \tilde{\omega}) &:= (\mathbb{R} \times S^1 \times \mathbb{C}P^1, dr \wedge d\theta \oplus c\omega_{FS}),\\ \tilde{B} &:= \mathbb{R} \times [0, c],\\ \tilde{\mu}(r, u, [z_0 : z_1]) &:= (r, |z_1|^2),\end{aligned}$$

where we use the coordinate $(r, e^{2\pi\sqrt{-1}\theta}) \in \mathbb{R} \times S^1$. For a negative integer $a \in \mathbb{Z}$ $(a < 0)$ and a positive integer $b \in \mathbb{N}$, we define the $\mathbb{Z}$-actions on $\tilde{M}$ and $\tilde{B}$ by

$$n(r, u, [z_0 : z_1]) := \left(r + n(-a|z_1|^2 + b), u, [z_0 : u^{na} z_1]\right), \tag{3.5}$$

$$n(r_1, r_2) := (r_1 + n(-ar_2 + b), r_2). \tag{3.6}$$

It is easy to see that (3.5) and (3.6) are free $\mathbb{Z}$-actions and (3.5) preserves $\tilde{\omega}$. Then we put

$$\begin{aligned}(M, \omega) &:= (\tilde{M}, \tilde{\omega})/\mathbb{Z},\\ B &:= \tilde{B}/\mathbb{Z}.\end{aligned}$$

It is also easy to see that $\tilde{\mu}$ is equivariant with respect to (3.5) and (3.6). Hence $\tilde{\mu}$ induces the map from M to B which we denote by $\mu\colon (M, \omega) \to B$. By construction, B is a cylinder and μ is a locally toric Lagrangian fibration which has singular fibers on ∂B.

We can generalize Theorem 3.40 to locally toric Lagrangian fibrations. See [34].

3.3.2. *Gelfand-Cetlin's completely integrable system.* Let $G = U(n)$ and $\mathfrak{g}$ the Lie algebra of G. We make the standard canonical identification of $\mathfrak{g}^*$ with the space $\sqrt{-1}\mathfrak{u}(n)$ of $n \times n$ Hermitian matrices. Under this identification there is a one-to-one correspondence between coadjoint orbits in $\mathfrak{g}^*$ and a point in the set

$$\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}. \tag{3.7}$$

The explicit correspondence is given as follows. For $\lambda = (\lambda_1, \dots \lambda_n)$ in the set (3.7) the corresponding coadjoint orbit consists of all Hermitian matrices whose eigenvalues are $\lambda_1, \dots \lambda_n$, i.e.,

$$\mathcal{O}_\lambda = \{A \in \sqrt{-1}\mathfrak{u}(n) \mid \operatorname{Spec} A = \lambda_1, \dots \lambda_n\}. \tag{3.8}$$

Suppose that λ satisfies $\lambda_1 > \cdots > \lambda_n$. Then, $\mathcal{O}_\lambda$ is $n(n-1)$-dimensional symplectic manifold (recall Example 3.11). We define the map $\pi\colon \mathcal{O}_\lambda \to \mathbb{R}^{\frac{n(n-1)}{2}}$ by

$$\begin{aligned}\pi(A) &= \left(\pi_1(A), \dots, \pi_{\frac{n(n-1)}{2}}(A)\right)\\ &= \left(\mu_1^{(1)}, \dots, \mu_{n-1}^{(1)}, \mu_1^{(2)}, \dots, \mu_{n-2}^{(2)}, \dots, \mu_1^{(n-1)}\right)\end{aligned}$$

for $A = (a_{ij}) \in \mathcal{O}_\lambda$, where $\mu_1^{(i)}, \dots, \mu_{n-i}^{(i)}$ $(1 \leq i \leq n-1)$ are the eigenvalues of the matrix

$$A_i = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-i} \\ \vdots & & \vdots \\ a_{n-i,1} & \cdots & a_{n-i,n-i} \end{pmatrix}$$

ordered so that $\mu_1^{(i)} \geq \cdots \geq \mu_{n-i}^{(i)}$. Then, Guillemin-Sternberg showed that π has the following properties.

Proposition 3.45 ([16]). *(1) π satisfies the following conditions*

$$\lambda_r \geq \mu_r^{(1)} \geq \lambda_{r+1},\ 1 \leq r \leq n-1, \tag{3.9}$$

$$\mu_r^{(i-1)} \geq \mu_r^{(i)} \geq \mu_{r+1}^{(i-1)},\ 1 \leq r \leq n-i.. \tag{3.10}$$

(2) π is smooth and the derivative of π is surjective at points where inequalities (3.9) *and* (3.10) *are strict. Moreover, At such points, $\{\pi_i, \pi_j\} = 0$ for $1 \leq i, j \leq \frac{n(n-1)}{2}$, where $\{\pi_i, \pi_j\}$ is the Poisson bracket of π_i and π_j which is defined by* (4.1)*.*

π is called *Gelfand-Cetlin's completely integrable system.* In particular, π defines a Lagrangian fibration with singular fibers at points where inequalities (3.9) and (3.10) are strict.

3.3.3. *Goldman's completely integrable system.* Let $G = SU(n)$. Let Σ be a closed connected Riemann surface with genus $g \geq 2$ and P be a principal G-bundle. Let $\mathcal{M}_g$ be the moduli space of flat G-bundles on Σ which is defined in Example 3.13.

We fix a trinion decomposition on Σ. This trinion decomposition defines $3g-3$ simple closed curves $C_1, \ldots, C_{3g-3}$. Then, we define the map $\pi\colon \mathcal{M}_g \to \mathbb{R}^{3g-3}$ by associating to an arbitrary element $A \in \mathcal{M}_g$ the holonomy of A with respect to C_1, $\ldots$, C_{3g-3}. In case of $n = 2$, Goldman showed that π has the properties similar to that of the Gelfand-Cetlin system in [13, 20]. π is called *Goldman's completely integrable system.* In particular, Goldman's completely integrable system defines a Lagrangian fibration with singular fibers (a least on the smooth part of $\mathcal{M}_g$). π plays an important role in the geometric quantization of $\mathcal{M}_g$, see [20].

4. Geometric quantization

In this section we explain some physical background. For more details, see [22, 33].

4.1. **Classical mechanics on a symplectic manifold.** Let (M, ω) be a symplectic manifold. Since ω is nondegenerate, it induces a bundle isomorphism from TM to T^*M by $TM \ni u \mapsto \iota_u\omega = \omega(u,\ ) \in T^*M$. In particular, for a function $f \in C^\infty(M)$ on M the vector field X_f is uniquely determined by the formula

$$df = -\iota_{X_f}\omega.$$

Definition 4.1. The vector field X_f is called the *Hamiltonian vector field* associated to f.

Example 4.2. For the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, the Hamiltonian vector field associated to $f \in C^\infty(\mathbb{R}^{2n})$ is given by

$$X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} \right).$$

For functions $f, g \in C^\infty(M)$, we define the function $\{f, g\} \in C^\infty(M)$ by

$$\{f, g\} = \omega(X_f, X_g). \tag{4.1}$$

Definition 4.3. $\{f, g\}$ is called the *Poisson bracket* of f and g.

Example 4.4. For $(\mathbb{R}^{2n}, \omega_0)$, a Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i} \right).$$

Proposition 4.5. *Let $f, g, h \in C^\infty(M)$ and $\lambda \in \mathbb{R}$. A Poisson bracket has the following properties*

(i) $\{\lambda f, g\} = \{f, \lambda g\} = \lambda\{f, g\}$,
(ii) $\{f + g, h\} = \{f, h\} + \{g, h\}$,
(iii) $\{f, g\} = -\{g, f\}$,
(iv) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ *(Jacobi identity),*
(v) $[X_f, X_g] = X_{\{f,g\}}$.

Exercise 4.6. Prove Proposition 4.5.

Remark 4.7. Proposition 4.5 (i)∼(iv) implies that a Poisson bracket defines the structure of a Lie algebra on $C^\infty(M)$. (v) implies that the map defined by $C^\infty(M) \ni f \mapsto X_f \in \chi(M)$ is a Lie algebra homomorphism.

In classical mechanics, for a function $H \in C^\infty(M)$, the time development of the classical mechanical system on (M, ω) with Hamiltonian H is described as the integral curve of the Hamiltonian vector field

$$\frac{dx}{dt}(t) = X_H(x(t)). \tag{4.2}$$

Hence, when a function $f \in C^\infty(M)$ is given, the time development of the value of f in this system is described, in terms of the Poisson bracket, by

$$\frac{d}{dt} f(x(t)) = \{H, f\}(x(t)). \tag{4.3}$$

In physics f is often called the *observable*. The equation (4.3) represents the time development of the observable f in the classical mechanical system with Hamiltonian H.

In (4.3) we put $f = H$. Then, by Proposition 4.5 (iii), we obtain

$$\frac{d}{dt} H(x(t)) = 0.$$

This implies that the Hamiltonian H is invariant along the time development of this classical system. In particular, as in Exercise 4.8, in the case where H itself is the total energy of the system this is nothing but the conservation law.

Exercise 4.8 (Harmonic oscillator)**.** We consider the cotangent bundle $(T^*\mathbb{R}, d\lambda)$ of $\mathbb{R}$ in Example 3.8. For a positive constant m and k let H be the function on $(T^*\mathbb{R}, d\lambda)$ which is defined by

$$H(p, q) = \frac{p^2}{2m} + \frac{kq^2}{2}. \tag{4.4}$$

Then, show that the equation (4.2) for (p, q) is written by

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{p}{m} \\ \dfrac{dp}{dt} = -kq. \end{cases} \tag{4.5}$$

In particular, by (4.5) we obtain the equation of motion for the harmonic oscillator

$$m\frac{d^2q}{dt^2} = -kq.$$

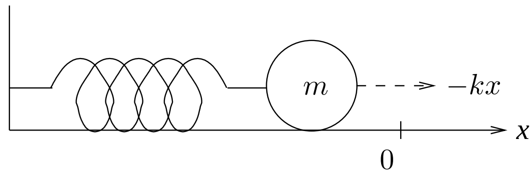

FIGURE 1. one-dimensional harmonic oscillator

Exercise 4.9. Let H be a function on the cotangent bundle $(T^*\mathbb{R}^n, d\lambda)$ of $\mathbb{R}^n$ in Example 3.8. Then, show that the equation (4.2) is written by

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}. \end{cases} \tag{4.6}$$

(4.6) is called the **Hamilton's canonical equation**. In particular, by putting $f = q_i$, or $f = p_i$, Hamilton's canonical equation (4.6) is written by

$$\frac{df}{dt} = \{H, f\}. \tag{4.7}$$

Exercise 4.10. Let (N, g) be a Riemannian manifold. g induces the isomorphism between TN and T^*N. With the identification, define the function $k \in C^\infty(T^*N)$ by

$$k(u) = g_{\pi(u)}(u, u),$$

where $\pi\colon T^*N \to N$ denotes the natural projection. Then, show that the image of the integral curve of the Hamiltonian vector field associated to k by π is a geodesic of (N, g).

4.2. **Mathematical formulation of quantization by Dirac.** As explained in the previous section, in the classical mechanical system on a symplectic manifold (M, ω) with Hamiltonian $H \in C^\infty(M)$ the time development of an observable $f \in C^\infty(M)$ is described as (4.3). On the other hand, the quantum physics is described in terms of a Hilbert space. In particular, a Hamiltonian and an observable are formulated as operators on some Hilbert space $\mathcal{H}$, and the time development of an observable $\hat{f}$ in the quantum mechanical system with Hamiltonian $\hat{H}$ is described by the following *Heisenberg's equation*

$$\frac{d}{dt}\hat{f} = [\hat{H}, \hat{f}]_h = \frac{2\pi\sqrt{-1}}{h}\left(\hat{H}\hat{f} - \hat{f}\hat{H}\right),$$

where h is the constant called the Planck constant.

According to Dirac, the terminology "quantization" is, roughly speaking, a procedure that associates the classical pictures with the quantum pictures. Namely,

Definition 4.11. A *quantization of* (M, ω) is a linear map $f \mapsto \mathcal{Q}(f)$ from the Poisson algebra $(C^\infty(M, \mathbb{C}), \{\ ,\ \})$, or the subalgebra of it, to some complex Hilbert space $\mathcal{Q}(M, \omega)$ that satisfies the following properties:

(i) $\mathcal{Q}(1) = \mathrm{id}_{\mathcal{Q}(M,\omega)}$.
(ii) $\mathcal{Q}(\{f, g\}) = [\mathcal{Q}(f), \mathcal{Q}(g)]_h$.
(iii) $\mathcal{Q}(f^*) = \mathcal{Q}(f)^*$, where f^* in the left hand side means the complex conjugation of f and $\mathcal{Q}(f)^*$ in the right hand side is the adjoint of $\mathcal{Q}(f)$.

(iv) For a complete set of functions $\{f_1, \dots, f_m\}$ on M, the set of the corresponding operators $\{\mathcal{Q}(f_1), \dots, \mathcal{Q}(f_m)\}$ is also complete[1].

Example 4.12. Let us consider the harmonic oscillator in Exercise 4.8. For a function $f \in C^\infty(T^*\mathbb{R})$, we define the operator $\mathcal{Q}(f)$ on the space $L^2(T^*\mathbb{R}, \mathbb{C})$ of complex valued L^2-functions on $T^*\mathbb{R}$ by

$$\mathcal{Q}(f)g = fg + \frac{h}{2\pi\sqrt{-1}} X_f g - \lambda(X_f)g. \tag{4.8}$$

(4.8) satisfies Definition 4.11 (i), (ii), and (iii). But, it does not satisfy (iv). In fact, for $p, q \in C^\infty(T^*\mathbb{R})$, which are complete set, the corresponding operators are

$$\mathcal{Q}(p) = \frac{h}{2\pi\sqrt{-1}} \frac{\partial}{\partial q}, \ \mathcal{Q}(q) = q - \frac{h}{2\pi\sqrt{-1}} \frac{\partial}{\partial p}. \tag{4.9}$$

It is easy to check that the operators $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial q} - 2\pi\sqrt{-1}p$ commute with $\mathcal{Q}(q)$ and $\mathcal{Q}(p)$. Therefore they do not form a complete set. But the restriction of $\{\mathcal{Q}(p), \mathcal{Q}(q)\}$ to the space $L^2(T^*\mathbb{R}_q)$ consisting of L^2-functions on $T^*\mathbb{R}$ depending only on the variable q

$$\begin{cases} p \mapsto \mathcal{Q}(p)|_{L^2(T^*\mathbb{R}_q)} = \frac{h}{2\pi\sqrt{-1}} \frac{\partial}{\partial q} \\ q \mapsto \mathcal{Q}(q)|_{L^2(T^*\mathbb{R}_q)} = q \end{cases} \tag{4.10}$$

is complete. The correspondence (4.10) is called the *Schrödinger representation.*

Remark 4.13. (4.9) satisfy the so called *canonical commutator relation*

$$[\mathcal{Q}(p), \mathcal{Q}(q)]_h = 1, \ \ [\mathcal{Q}(p), \mathcal{Q}(p)] = [\mathcal{Q}(q), \mathcal{Q}(q)] = 0. \tag{4.11}$$

In general, a quantization satisfying (4.11) is called the *canonical quantization.*

4.3. **From local to global.** In the rest of this section we set $h = 1$ for simplicity. One of the point that the quantization (4.8) in Example 4.12 satisfies (ii) is that the symplectic structure $d\lambda$ on $T^*\mathbb{R}$ is exact. In the case where the symplectic structure is not exact (for example M is compact), (4.8) does not make sense. In order to generalize (4.8) to the case where the symplectic structure is not exact, we give the following observation.

Observation 4.14. In Example 4.12 let L be the trivial complex vector bundle $\mathbb{R}^2 \times \mathbb{C}$, $\langle \ \rangle$ the Hermitian metric on L induced from the standard one on $\mathbb{C}$, ∇ the Hermitian connection on $(L, \langle \ \rangle)$ defined by $\nabla = d - 2\pi\sqrt{-1}\lambda$. By using these, (4.8) can be rewritten, as an operator on $\Gamma(L)$, as

$$\mathcal{Q}(f)s = \frac{1}{2\pi\sqrt{-1}} \nabla_{X_f} s + fs. \tag{4.12}$$

Based on Observation 4.14, we give the following definition.

Definition 4.15. Let (M, ω) be a symplectic manifold. A *prequantization line bundle* is a triple $(L, \langle \ , \ \rangle, \nabla)$ consisting of a complex line bundle $L \to M$, a Hermitian metric $\langle \ , \ \rangle$ of L and a connection ∇ on M that satisfy the following conditions

(i) ∇ is Hermitian, i.e., $\nabla \langle s_1, s_2 \rangle = \langle \nabla s_1, s_2 \rangle + \langle s_1, \nabla s_2 \rangle$ $(s_1, s_2 \in \Gamma(L))$,

[1] The set of functions $\{f_1, \dots, f_m\}$ is complete if a function g satisfying $\{f_i, g\} = 0$ for all $i = 1, \dots, m$ is a constant function, and the set of operators $\{\mathcal{Q}(f_1), \dots, \mathcal{Q}(f_m)\}$ is complete if an operator $\mathcal{Q}(g)$ satisfying $[\mathcal{Q}(f_i), \mathcal{Q}(g)] = 0$ for all $i = 1, \dots, m$ is the identity operator up to constant multiplication.

(ii) the curvature F_∇ of ∇ is equal to $\frac{2\pi}{\sqrt{-1}}\omega$.

Proposition 4.16. *(M,ω) possesses a prequantization line bundle if and only if the cohomology class $[\omega]$ determined by ω is contained in the image of the natural map $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$.*

Let $(L, \langle\ ,\ \rangle, \nabla) \to (M,\omega)$ be a $2n$-dimensional symplectic manifold with prequantization line bundle. On the space $\Gamma_c(L)$ of compact supported sections of L, the Hermitian inner product is defined by

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle \frac{\omega^n}{n!}. \tag{4.13}$$

Then, for $f \in C^\infty(M)$, (4.12) defines an operator on the space $L^2(L)$ of L^2-bounded sections with respect to the Hermitian inner product (4.13). In particular, it defines the linear map $\mathcal{Q}\colon C^\infty(M,\mathbb{C}) \to \mathcal{O}p(L^2(L))$, where $\mathcal{O}p(L^2(L))$ denotes the space of operators on $L^2(L)$.

Proposition 4.17. *The linear map $\mathcal{Q}\colon C^\infty(M,\mathbb{C}) \to \mathcal{O}p(L^2(L))$ defined by* (4.12) *satisfies the conditions* (i) *and* (ii) *in Definition 4.11.*

Proof. This can be checked by the direct computation. The points here are

- the condition (ii) in Definition 4.15,
- $F_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$,
- the condition (v) in Proposition 4.5,
- $\omega(X_f, X_g) = X_f g = -X_g f = \{f,g\}$.

□

Remark 4.18. On a sufficiently small open neighborhood U of a point $x \in M$, we have an identification $(L,\nabla)|_U \cong (U \times \mathbb{C}, d - 2\pi\sqrt{-1}\alpha)$ for some one-form α with $d\alpha = \omega|_U$. Under this identification,

$$\begin{aligned}(4.12) &= \frac{1}{2\pi\sqrt{-1}}\left(X_f s - 2\pi\sqrt{-1}\alpha(X_f)\right)s + fs \\ &= \frac{1}{2\pi\sqrt{-1}}X_f s - \alpha(X_f)s + fs.\end{aligned}$$

In particular, this is a generalization of (4.8).

4.4. **Polarization.** For a symplectic manifold (M,ω) with prequantization line bundle $(L, \langle\ ,\ \rangle, \nabla) \to (M,\omega)$, the linear map $\mathcal{Q}\colon C^\infty(M,\mathbb{C}) \to \mathcal{O}p(L^2(L))$ given by (4.12) satisfies Definition 4.11 (i), (ii), and (iii). But, as explained in Example 4.12, $L^2(L)$ is too big for (4.12) satisfies the condition (iv) in Definition 4.11.

For this issue, let us introduce the following notion.

Definition 4.19. We call an integrable Lagrangian subbundle F of $TM \otimes_{\mathbb{R}} \mathbb{C}$ a *polarization* of (M,ω).

Suppose a polarization F of (M,ω) is given. Let $\mathscr{S}$ be the sheaf of germs of sections s of L which satisfy

$$\nabla_V s = 0 \tag{4.14}$$

for $V \in \Gamma(F)$. Then, the degree zero cohomology $H^0(M;\mathscr{S})$ is the space of sections of L which are covariant constant along F. In the theory of the geometric quantization, it is fundamental to consider $H^0(M;\mathscr{S})$ as $\mathcal{Q}(M,\omega)$.

Example 4.20. In Example 4.12 let $(L, \langle\ \rangle, \nabla)$ be the data in Observation 4.14. We define the polarization F of $(T^*\mathbb{R}, d\lambda)$ by

$$F = \frac{\partial}{\partial p} \otimes_{\mathbb{R}} \mathbb{C} \subset T(T^*\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

F is an example of a real polarization that will be explained later. With respect to F, (4.14) is written by

$$0 = \nabla_{\frac{\partial}{\partial p}} s = \frac{\partial s}{\partial p}.$$

In particular, this implies that $H^0(T^*\mathbb{R}; \mathscr{S})$ consists of sections depending only on q.

Remark 4.21. In general, the operators $\mathcal{Q}(f)$ defined by (4.12) do not preserve $H^0(M; \mathscr{S})$. But, we do not describe this difficulty anymore, and in the rest of this note, we shall pay attention to $\mathcal{Q}(M, \omega)$, especially, the dimension of $\mathcal{Q}(M, \omega)$.

In the rest of this section, we consider two special polarizations, named the Kähler polarization and the real polarization. Assume that M is closed unless otherwise stated.

4.4.1. *Kähler polarization.* Let (M, ω) be a Kähler manifold. Then, the $(0,1)$-part $T^{0,1}M$ with respect to a compatible complex structure J gives a polarization of (M, ω).

Definition 4.22. This polarization $T^{0,1}M$ is called the *Kähler polarization*.

Suppose that (M, ω) has a prequantization line bundle $(L, \langle\ ,\ \rangle, \nabla)$ so that L is holomorphic. Then, $H^*(M; \mathscr{S})$ is nothing but the Dolbeault cohomology $H^*(M; \mathcal{O}_L)$. In this case, by adding a correction term to $H^0(M; \mathscr{S})$, we put

$$\mathcal{Q}(M, \omega) := \sum_i (-1)^i H^i(M; \mathscr{S}).$$

By Theorem 2.31, using the Dolbeault operator with coefficients in L

$$D = \sqrt{2}\left(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L\right) : \wedge^\bullet T^{0,1}M^* \otimes L \to \wedge^\bullet T^{0,1}M^* \otimes L,$$

we obtain

$$\mathcal{Q}(M, \omega) = \ker D^0 - \ker D^1. \tag{4.15}$$

If (M, ω) is not a Kähler manifold, then, $T^{0,1}M$ is not integrable, hence it is not a polarization. But, even in this case, as we mentioned in Remark 2.25, we can consider a spin^c Dirac operator with coefficients in L instead of a Dolbeault operator. Thus, for a general symplectic manifold we define $\mathcal{Q}(M, \omega)$ by (4.15) for the spin^c Dirac operator with coefficients in L. In particular, the dimension of $\mathcal{Q}(M, \omega)$ is given as the index of the spin^c Dirac operator. By Theorem 2.32, we have

$$\dim \mathcal{Q}(M, \omega) = \int_M e^\omega Td(TM, J). \tag{4.16}$$

By Proposition 3.17 (4.16)does not depend on the choice of a compatible almost complex structure and depends only on ω.

Definition 4.23. we call (4.16) the *Riemann-Roch number*.

4.4.2. *Real polarization.* A Kähler polarization F satisfies $F \cap \overline{F} = 0$, whereas

Definition 4.24. A polarization F that satisfies $\overline{F} = F$ is called the *real polarization.*

For example, suppose that (M, ω) is equipped with the structure of a Lagrangian fibration $\pi\colon (M, \omega) \to B$. Then, the complexification $T[\pi] \otimes_{\mathbb{R}} \mathbb{C}$ of the tangent bundle $T[\pi]$ along fibers of π determines a real polarization of (M, ω). In general, a real polarization F is a complexification of some integrable Lagrangian subbundle $F_{\mathbb{R}}$ of TM, and $F_{\mathbb{R}}$ gives a Lagrangian foliation on (M, ω).

Suppose that the symplectic manifold (M, ω) with prequantization line bundle $(L, \langle\ ,\ \rangle, \nabla) \to (M, \omega)$ has a real polarization F. Then, by adding a correction term to $H^0(M; \mathscr{S})$, we define

$$\mathcal{Q}(M, \omega) := \sum_i H^i(M; \mathscr{S}).$$

In the rest of this section we assume that (M, ω) is equipped with the structure of a Lagrangian fibration $\pi\colon (M, \omega) \to B$ and F is given as the complexification $T[\pi] \otimes \mathbb{C}$ of $T[\pi]$. Since π is a Lagrangian fibration, the restriction of (L, ∇) to each fiber of π is a flat line bundle.

Definition 4.25. A fiber $\pi^{-1}(b)$ of π is said to be *Bohr-Sommerfeld* if the restriction $(L, \nabla)|_{\pi^{-1}(b)}$ has a nonzero global parallel section with respect to ∇.

Example 4.26. Let $\pi\colon (T^2, \omega) \to S^1$ be the Lagrangian fibration in Example 3.24. We consider the prequantization line bundle $(L, \langle\ ,\ \rangle, \nabla) \to (T^2, \omega)$ which is defined by

$$(L, \nabla) := (\mathbb{R} \times S^1 \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy)/\mathbb{Z},$$

where the $\mathbb{Z}$-action on $(\mathbb{R} \times S^1 \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy)$ is given by

$$n(x, y, z) = (x + n, y, e^{-2\pi\sqrt{-1}ny}z)$$

for $n \in \mathbb{Z}$ and $(x, y, z) \in \mathbb{R} \times S^1 \times \mathbb{C}$. We detect the Bohr-Sommerfeld fibers. For $x \in S^1$ and $s \in \Gamma(L|_{\pi^{-1}(x)})$, by solving the equation

$$\begin{aligned} 0 &= \nabla_{\partial_y} s \\ &= \partial_y s - 2\pi\sqrt{-1}xs, \end{aligned}$$

we can show that the covariant constant section s is locally of the form

$$s(y) = s(0)e^{2\pi\sqrt{-1}xy}. \tag{4.17}$$

Then, the local section (4.17) is globally defined on $\pi^{-1}(x)$ if and only if $x = 0 \in S^1$. In other words, the fiber $\pi^{-1}(0)$ is the unique Bohr-Sommerfeld fiber.

Bohr-Sommerfeld fibers appear discretely. Then, Śniatycki showed the following theorem.

Theorem 4.27 (Śniatycki [28])**.** *Let $\pi\colon (M, \omega) \to B$ be a Lagrangian fibration. Suppose (M, ω) compact. Then,*

$$H^i(M; \mathscr{S}) = \begin{cases} \bigoplus_{\pi^{-1}(b):BS} \Gamma_{\mathit{flat}}\left((L, \nabla)|_{\pi^{-1}(b)}\right) & i = \dim M/2 \\ 0 & i \neq \dim M/2, \end{cases}$$

where $\Gamma_{\mathit{flat}}\left((L,\nabla)|_{\pi^{-1}(b)}\right)$ *denotes the one-dimensional vector space consisting of parallel sections of* $L|_{\pi^{-1}(b)}$ *with respect to* ∇*, and the sum is taken over all* Bohr-Sommerfeld *fibers.*

By Theorem 4.27,

$$\mathcal{Q}(M,\omega) = H^{\dim M/2}(M;\mathscr{S}). \tag{4.18}$$

In particular, the dimension of $\mathcal{Q}(M,\omega)$ agrees with the number of Bohr-Sommerfeld fibers.

4.5. $RR = \#BS$**?** Let (M,ω) be a symplectic manifold and $(L,\langle\ ,\ \rangle,\nabla) \to (M,\omega)$ a prequantization line bundle. Suppose (M,ω) has the structure of a Lagrangian fibration $\pi\colon (M,\omega) \to B$. In this section we compare the two choices (4.15) and (4.18) of $\mathcal{Q}(M,\omega)$ described in the previous section.

Example 4.28. Let (T^2,ω) be the torus with standard symplectic structure in Example 3.7 and $(L,\langle\ ,\ \rangle,\nabla)$ the prequantization line bundle in Example 4.26.

By (4.16), the index of the spin^c Dirac operator is computed as

$$\begin{aligned}\dim\mathcal{Q}(T^2,\omega) &= \int_{T^2} e^{\omega}Td(TM,J)\\ &= \int_{T^2}\omega\\ &= 1.\end{aligned}$$

On the other hand, by Example 4.26, the number of Bohr-Sommerfeld fibers is one. Therefore, the two choices (4.15) and (4.18) of $\mathcal{Q}(M,\omega)$ agree with each other at least in the level of the dimension.

In general, the following theorem is known.

Theorem 4.29 ([1])**.** *Let* $\pi\colon (M,\omega) \to B$ *be a Lagrangian fibration and* $(L,\langle\ ,\ \rangle,\nabla) \to (M,\omega)$ *a prequantization line bundle on* (M,ω)*. Then, the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers.*

Outline of Proof. By lifting the whole situation to the orientation covering of B if it is necessary, we assume that B is orientable. We fix the volume form of B so that the volume of B is equal to the symplectic volume of (M,ω).

By Proposition 3.31 the Todd class $Td(TM,J)$ is trivial, hence the Riemann-Roch number is equal to the symplectic volume of (M,ω).

For π we denote by $\check{\pi}\colon \check{M} \to B$ the fiber bundle such that for each $b \in B$ the fiber $\check{\pi}^{-1}(b)$ is the moduli space of flat line bundles on the fiber $\pi^{-1}(b)$ of π. By Proposition 3.31 $\check{\pi}\colon \check{M} \to B$ is a torus fiber bundle with structure group $\mathrm{GL}_n(\mathbb{Z})$, where n is the dimension of a fiber of π. Since each fiber of π is Lagrangian, the restriction of $(L,\langle\ ,\ \rangle,\nabla)$ to each fiber of π defines the section of $\check{\pi}$. We denote this section by s_π. By definition the fiber $\pi^{-1}(b)$ is Bohr-Sommerfeld if and only if $s_\pi(b) = 0 \in \check{\pi}^{-1}(b)$. In particular the number of Bohr-Sommerfeld fibers of π is equal to the intersection number of s_π with the zero section of $\check{\pi}$.

We show that the intersection number of s_π with the zero section of $\check{\pi}$ is equal to the symplectic volume. Since B is oriented, the structure group of $\check{\pi}$ is reduced to $\mathrm{SL}_n(\mathbb{Z})$. Since the volume form $dy_1 \wedge \cdots dy_n$ on the n-dimensional torus T^n is invariant under the $\mathrm{SL}_n(\mathbb{Z})$-action on T^n, it defines a closed n-form on $\check{M}$ which we denote by Φ. Then, we can show that the cohomology class represented by Φ is the Poincaré dual of the zero section of $\breve{\pi}$. Hence, the number of Bohr-Sommerfeld fibers is equal to $\int_B s_\pi^*\Phi$. Finally the direct comutation shows that $\int_B s_\pi^*\Phi$ agrees with the volume of$B$. □

Lagrangian fibrations with singular fibers often appear in various scenes in Mathematics. A typical example is a moment map of a symplectic toric manifold. In this case, Danilov showed the following theorem.

Theorem 4.30 ([6])**.** *Let (M,ω) be a compact symplectic toric manifold with moment map $\mu\colon M \to \mathfrak{g}^*$. Suppose that there is a prequantization line bundle $(L,\langle\ ,\ \rangle,\nabla) \to (M,\omega)$ on (M,ω). Then, the dimension of $H^0(M,\mathcal{O}_L)$ is equal to the number of $\mu(M)\cap\mathfrak{g}_\mathbb{Z}^*$, where $\mathfrak{g}_\mathbb{Z}^*$ denotes the weight lattice in $\mathfrak{g}^*$.*

Since the higher degree cohomology groups vanish, the dimension of $H^0(M,\mathcal{O}_L)$ is equal to the Riemann-Roch number.

On the other hand, generic fibers of μ are $\frac{\dim M}{2}$-dimensional tori and singular fibers are tori of dimensions smaller than $\frac{\dim M}{2}$. In particular, since singular fibers are also smooth manifolds, the definition of a Bohr-Sommerfeld condition makes sense. Moreover, it is known that elements in $\mu(M)\cap\mathfrak{g}_\mathbb{Z}^*$ correspond one-to-one to Bohr-Sommerfeld fibers. See also Proposition 6.9. Therefore, Theorem 4.30 implies that the Riemann-Roch number is equal to the number of both of the singular and nonsingular Bohr-Sommerfeld fibers for the symplectic toric manifold with prequantization line bundle.

Similar phenomena have been observed for Gelfand-Cetlin's completely integrable system on a complex flag manifold [16], Goldman's completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemannian surface [20], and several examples of locally toric Lagrangian fibrations [35] by computing the both sides separately and comparing them with each other. Moreover, the similar phenomena have been observed for the case where a symplectic structure is degenerate [14, 21, 25] by counting both of the singular and nonsingular Bohr-Sommerfeld fibers with multiplicities.

These phenomena suggest a localization of the Riemann-Roch number to Bohr-Sommerfeld fibers. So it would be natural to ask the following question.

Question 4.31. Make clear the mechanism which controls these phenomena.

5. Local index

Motivated by Question 4.31, in [9, 10] we gave a formulation on the index for the Dirac-type operator on an open manifold based on the idea of Witten's deformation in [32], and as an application, we showed that for a singular Lagrangian fibration with prequantization line bundle the Riemann-Roch number is described as the number of the nonsingular Bohr-Sommerfeld fibers and the contribution from singular fibers. In this section we explain the first half. The applications will be described in the next section.

5.1. **Main theorem.** Let M be a manifold.

Definition 5.1. A *compatible fibration on M* is the data $\{\pi_\alpha\colon V_\alpha \to U_\alpha \mid \alpha\in A\}$ that satisfies the following conditions:

(i) $\{V_\alpha\}_{\alpha\in A}$ is a finite covering of M.

(ii) U_α is a manifold and $\pi_\alpha\colon V_\alpha \to U_\alpha$ is a fiber bundle with fiber $(\mathbb{R}/\mathbb{Z})^{k_\alpha}$. Here the rank k_α may vary depending on α.

(iii) $\pi_\alpha^{-1}(\pi_\alpha(V_\alpha \cap V_\beta)) = \pi_\beta^{-1}(\pi_\beta(V_\alpha \cap V_\beta)) = V_\alpha \cap V_\beta$.

(iv) On each nonempty overlap $V_\alpha \cap V_\beta \neq \emptyset$, for each $x \in V_\alpha \cap V_\beta$, one of the relation

$$\pi_\alpha^{-1}(\pi_\alpha(x)) \supset \pi_\beta^{-1}(\pi_\beta(x)),\ \pi_\alpha^{-1}(\pi_\alpha(x)) \subset \pi_\beta^{-1}(\pi_\beta(x))$$

holds. Moreover, in the former case, there exists a fiber bundle $\pi_{\alpha\beta}\colon \pi_\beta(V_\alpha \cap V_\beta) \to \pi_\alpha(V_\alpha \cap V_\beta)$ such that $\pi_\alpha = \pi_{\alpha\beta} \circ \pi_\beta$. The similar condition holds for the latter case.

For simplicity we often express a compatible fibration by $\{\pi_\alpha\}$.

Suppose that there is a compatible fibration $\{\pi_\alpha\}$ on M.

Definition 5.2. An open set C of M is said to be *admissible* if for each $\alpha \in A$ C satisfies the following condition

$$\pi_\alpha^{-1}(\pi_\alpha(C \cap V_\alpha)) = C \cap V_\alpha.$$

Lemma 5.3. *For $\{\pi_\alpha\}$ there exists a partition of unity $\{\rho_\alpha^2\}_{\alpha\in A}$ subordinate to $\{V_\alpha\}_{\alpha\in A}$ such that each ρ_α is constant along fibers of all π_β.*

Let g be a Riemannian metric on M, (W, c) a $\mathbb{Z}_2$-graded Clifford module bundle on (M, g).

Definition 5.4. A *compatible system of Dirac-type operators along fibers of π_α's* is the data $\{D_\alpha\}_{\alpha\in A}$ that satisfies the following conditions:

(i) $D_\alpha\colon \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is a formally self-adjoint first order linear differential operator of degree one.

(ii) The principal symbol D_α of $\sigma(D_\alpha)$ is given by

$$\sigma(D_\alpha) = c \circ p_\alpha \circ \iota_\alpha^*\colon T^*V_\alpha \to \mathrm{End}(W|_{V_\alpha}),$$

where $\iota_\alpha\colon T[\pi_\alpha] \to TV_\alpha$ is the natural inclusion of the tangent bundle $T[\pi_\alpha]$ along fibers of π_α to TV_α, $p_\alpha\colon T^*[\pi_\alpha] \to T[\pi_\alpha]$ is the isomorphism induced by the Riemannian metric g. In particular, D_α contains only derivatives along fibers of π_α.

(iii) For each $b \in U_\alpha$ & $u \in T_bU_\alpha$, $\tilde{u} \in \Gamma(TV_\alpha|_{\pi_\alpha^{-1}(b)})$ denotes the horizontal lift of u with respect to g. $\tilde{u}$ acts on $\Gamma(W|_{\pi_\alpha^{-1}(b)})$ as a Clifford multiplication $c(\tilde{u})$. Then, for all $b \in U_\alpha$ and $u \in T_bU_\alpha$, D_α and $c(\tilde{u})$ anti-commute each other, namely,

$$D_\alpha \circ c(\tilde{u}) + c(\tilde{u}) \circ D_\alpha = 0.$$

(iv) On $V_\alpha \cap V_\beta \neq \emptyset$ $D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is a differential operator along smaller fibers between π_α and π_β.

Definition 5.5. A compatible system $\{D_\alpha\}_{\alpha\in A}$ is said to be *acyclic* if it satisfies the following conditions:

(i) For each $\alpha \in A$ and $b \in U_\alpha$ $\ker\left(D_\alpha|_{\pi_\alpha^{-1}(b)}\right) = 0$.

(ii) On each $V_\alpha \cap V_\beta \neq \emptyset$ $D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is non-negative operator, namely,

$$\int_M \langle (D_\alpha \circ D_\beta + D_\beta \circ D_\alpha)s, s\rangle_W \, vol \geq 0 \ \forall s \in \Gamma(W|_{V_\alpha \cap V_\beta}). \tag{5.1}$$

Theorem 5.6 ([9, 10])**.** *Let (M, g) be a Riemannian manifold, (W, c) a $\mathbb{Z}_2$-graded Clifford module bundle on (M, g), V an open set of M with complement $M \setminus V$ compact, $\{\pi_\alpha\}_{\alpha\in A}$ a compatible fibration on V, and $\{D_\alpha\}_{\alpha\in A}$ an acyclic compatible system. Moreover, we assume that there exists a Riemannian metric on each U_α such that π_α is a Riemannian submersion with respect to the restriction of $g|_{V_\alpha}$ and this Riemannian metric. Then, there exists an integer* $\mathrm{ind}(M, V) = \mathrm{ind}(M, W, V, \{\pi_\alpha\}, \{D_\alpha\}) \in \mathbb{Z}$ *depending on these data such that* $\mathrm{ind}(M, V)$ *satisfies the following properties:*

(i) $\mathrm{ind}(M, V)$ *is invariant under continuous deformation of the data.*

(ii) *If M is closed, then,* $\mathrm{ind}(M, V)$ *agrees with the index of a Dirac-type operator on W.*

(iii) *If M' is an admissible open neighborhood of $M \setminus V$, then,*

$$\mathrm{ind}(M, V) = \mathrm{ind}(M', M' \cap V) \text{ (excision).}$$

(iv) *If V' is an admissible open subset of V with the complement $M \setminus V'$ compact, then,*

$$\mathrm{ind}(M, V) = \mathrm{ind}(M, V').$$

(v) *For $M = M_1 \sqcup M_2$*

$$\mathrm{ind}(M, V) = \mathrm{ind}(M_1, M_1 \cap V) + \mathrm{ind}(M_2, M_2 \cap V) \text{ (sum formula).}$$

We call $\mathrm{ind}(M, V)$ *the local index.*

Remark 5.7. (1) If $M = V$, then, $\mathrm{ind}(M, V)$ satisfies

$$\mathrm{ind}(M, V) = 0. \text{ (vanishing)} \tag{5.2}$$

In fact, by putting $M = M \sqcup \emptyset$ in (v), then, we obtain $\mathrm{ind}(\emptyset, \emptyset) = 0$. Since $M = V$, by putting $M' = \emptyset$ in (iii), we have $\mathrm{ind}(M, V) = 0$. In [10], first we show (5.2), then, by using this, we prove Theorem 5.6.

(2) We have a product formula for $\mathrm{ind}(M, V)$. One of the simplest case is as follows. For $i = 0, 1$ let (M_i, g_i), (W_i, c_i), V_i, $\{\pi_{i,\alpha}\}_{\alpha\in A}$, and $\{D_{i,\alpha}\}_{\alpha\in A}$ be the data as in Theorem 5.6. Then, we have

$$\mathrm{ind}\left((M_0, V_0) \times (M_1, V_1)\right) = \mathrm{ind}(M_0, V_0)\,\mathrm{ind}(M_1, V_1).$$

It is necessary for the product formula to generalize the notion of a compatible fibration and a compatible system.

(3) We have an equivariant version of Theorem 5.6 [11].

Corollary 5.8 ([9, 10])**.** *Under the assumption in Theorem 5.6, moreover we assume that M is closed and $M \setminus V$ is covered by mutually disjoint finitely many open sets $O_1, \ldots, O_k$. Then, for a Dirac-type operator D on W we have*

$$\mathrm{ind}\, D = \sum_{i=1}^{k} \mathrm{ind}(O_i, O_i \cap V).$$

Proof.

$$\begin{aligned}
\mathrm{ind}\, D &= \mathrm{ind}(M, V) \ (\because \text{(ii)}) \\
&= \mathrm{ind}(O_1 \sqcup \ldots \sqcup O_k \cup V, V) \\
&= \mathrm{ind}(O_1 \sqcup \ldots \sqcup O_k, O_1 \sqcup \ldots \sqcup O_k \cap V) \ (\because \text{(iii)}) \\
&= \sum_i \mathrm{ind}(O_i, O_i \cap V) \ (\because \text{(v)}).
\end{aligned}$$

□

5.2. **Definition of** $\mathrm{ind}(M, V)$**.** Here we describe the construction of $\mathrm{ind}(M, V)$.

5.2.1. *Witten's deformation.* First let us explain Witten's deformation. The original paper by Witten is [32]. See also [12, 27] for Witten's deformation. The idea of Witten's deformation is one of the point of the construction of $\mathrm{ind}(M, V)$.

Fact 5.9. *Let (M, g) be a complete Riemannian manifold, (W, c) a $\mathbb{Z}_2$-graded Clifford module bundle on (M, g) , and $h \in \mathrm{End}(W)$ an endmorphism which satisfies the following conditions:*

(i) *Hermitian*
(ii) *h shifts the degree of W.*
(iii) $\mathrm{supp}\, h = \{x \in M \mid \ker(h_x : W_x \to W_x) \neq 0\}$ *is compact.*
(iv) $h \circ c + c \circ h = 0$.

Then, for $t \geq 0$ we put

$$D_t = D + th.$$

For a sufficiently large t, $\ker D_t \cap L^2(W)$ *is finite dimensional, and* $\ker D_t^0 \cap L^2(W) - \ker D_t^1 \cap L^2(W)$ *does not depend on the choice of a sufficiently large t and continuous deformations of the data.*

In our case we adopt $\sum_\alpha \rho_\alpha D_\alpha \rho_\alpha$ as h, and for a sufficiently large t we define $\mathrm{ind}(M, V)$ as the "index" of $D_t = D + t \sum_\alpha \rho_\alpha D_\alpha \rho_\alpha$. In the rest of this section we explain the outline of the construction.

5.2.2. *Vanishing lemma.* The key is the following vanishing lemma.

Lemma 5.10. *In the setting of Theorem 5.6, we assume that M is closed and $V = M$. For $t \geq 0$ and a Dirac-type operator D on W we put*

$$D_t = D + t \sum_\alpha \rho_\alpha D_\alpha \rho_\alpha.$$

Then, for a sufficiently large t, $\ker D_t = 0$.

In order to prove this theorem we prepare the following lemma. Put $D'_\alpha = \rho_\alpha D_\alpha \rho_\alpha$.

Lemma 5.11. *For each $\alpha \in A$ $D \circ D'_\alpha + D'_\alpha \circ D$ is a linear differential operator along fibers of $\pi_\alpha : V_\alpha \to U_\alpha$ of degree at most two.*

Proof. This lemma follows from the fact that there is a Riemannian metric on each U_α and π_α is a Riemannian submersion with respect to the restriction of g to V_α and this Riemannian metric and Definition 5.4 (iii). □

Proof of Lemma 5.10. Let $s \in \Gamma(W)$ be a section of W. Since for each $\alpha \in A$, $D \circ D'_\alpha + D'_\alpha \circ D$ is a linear differential operator along fibers of $\pi_\alpha : V_\alpha \to U_\alpha$ of degree at most two, for each $b \in U_\alpha$ and $D \circ D'_\alpha + D'_\alpha \circ D$ there exists a positive constant C such that the following a priori estimate holds

$$\left| \int_{\pi_\alpha^{-1}(b)} \langle (D \circ D'_\alpha + D'_\alpha \circ D)s, s \rangle \, vol \right| \leq C \int_{\pi_\alpha^{-1}(b)} |D'_\alpha s|^2 vol. \tag{5.3}$$

Since M is compact, we can take the constant C uniformly. Hence,

$$\begin{aligned}\int_M |D_t s|^2 vol &= \int_M \left\langle D_t^2 s, s\right\rangle vol \\ &= \int_M \left\langle D^2 s, s\right\rangle vol + t\sum_{\alpha\in A}\int_M \langle (D\circ D'_\alpha + D'_\alpha \circ D)s, s\rangle\, vol \\ &+ t^2\sum_{\alpha\neq\beta}\int_M \left\langle (D'_\alpha\circ D'_\beta + D'_\beta\circ D'_\alpha)s, s\right\rangle vol + t^2\sum_\alpha\int_M |D'_\alpha s|^2 vol. \end{aligned} \tag{5.4}$$

Since for each $\alpha \in A$ D_α contains only the derivatives along fibers of π_α and ρ_α is constant along fibers of all π_β, the estimate (5.1) still holds for D'_αs. By using (5.3) together with this fact we have

$$(5.4) \geq \int_M |Ds|^2 vol + (t^2 - Ct)\sum_{\alpha\in A}\int_M |D'_\alpha s|^2 vol.$$

In particular, if $t > C$ and $D_t s = 0$, then, $D'_\alpha s = 0$ for all $\alpha \in A$. Moreover, Definition 5.5 (i) implies that $s = 0$. □

5.2.3. *Cylindrical end case.*

Lemma 5.12. *In the setting of Theorem 5.6, assume that there exists a codimension-one submanifold N of M such that $V = N \times (0,\infty)$ and all the data are translationally invariant on V. Then, for a sufficiently large t, $\ker D_t \cap L^2(W)$ is finite dimensional. Moreover, $\ker D_t^0 \cap L^2(W) - \ker D_t^1 \cap L^2(W)$ is independent of large t and any other continuous deformation of data.*

Proof. The restriction of D_t to $V = Ntimes(0,\infty)$ is of the form $\alpha(\partial_r + D_{N,t})$, where α is the Clifford multiplication of ∂_r and $D_{N,t}$ is a formally self-adjoint operator on N depending on t. In general, when the kernel of $D_{N,t}$ is trivial, any L^2-solution for the equation $D_t s = 0$ is exponentially decreasing on the end. Then it is well-known that $\ker D_t \cap L^2(W)$ is finite dimensional, and $\ker D_t^0 \cap L^2(W) - \ker D_t^1 \cap L^2(W)$ is deformation invariant as far as $\ker D_{N,t} = 0$. Therefore it is sufficient to show that the kernel of $D_{N,t}$ is trivial.

By assumption, D_t is translationally invariant on $N \times (0,\infty)$, hence we can think of D_t as an operator on $N \times S^1$. Since $N \times S^1$ is closed, by Lemma 5.10 for a sufficiently large t $\ker D_t = 0$. Thus, in particular, the kernel of $D_{N,t}$ is trivial. □

Definition 5.13. Under the assumption in lemma 5.12 we define the *local index* $\mathrm{ind}(M,V)$ by

$$\mathrm{ind}(M,V) = \ker D_t^0 \cap L^2(W) - \ker D_t^1 \cap L^2(W)$$

for a sufficiently large t.

5.2.4. *General end case.* In the case of a general V, we reduce to the cylindrical end case by cutting M along some codimension-one submanifold N and extend the cut locus cylindrically. We check that $\mathrm{ind}(M,V)$ does not depend on the choice of a cut locus. Though we do not explain it here it is possible. See [9, 10].

5.3. **Examples.** We compute $\mathrm{ind}(M,V)$ for two examples.

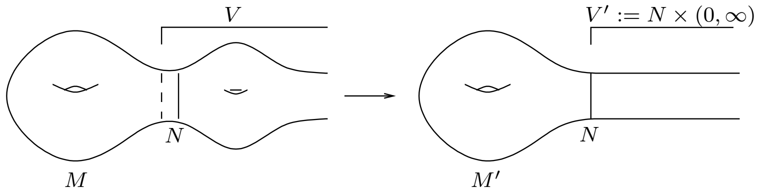

FIGURE 2. Reducing the general end case to the cylindrical end case

5.3.1. *Two-dimensional cylinder.* Let $M := \mathbb{R} \times S^1$ and consider the standard almost Hermitian structure (g, J) on it. Let $L = M \times \mathbb{C}$ be the trivial complex line bundle on M. For $0 < \delta < 1$ let $\rho(x)$ be a smooth increasing function on $\mathbb{R}$ with $\rho(0) = 0$, $\rho(x) \equiv \delta$ for sufficiently large x and $\rho(x) \equiv -\delta$ for sufficiently small x. Consider the connection on L of the form $\nabla = d - 2\pi\sqrt{-1}\rho(x)dy$, where dy is the volume form on S^1.

Let $W := \wedge^\bullet(TM, J) \otimes_{\mathbb{C}} L$ be the $\mathbb{Z}/2$-graded Clifford module bundle over M which is explained in Example 2.16 and Remark 2.26. It is easy to check that the Levi-Civita connection associated to g and the connection on L induces a connection on W so that W is a Dirac bundle. We take a Dirac-type operator D acting on $\Gamma(W)$ to be the spin^c Dirac operator with coefficients in L. In this case, D is written as

$$Ds = \partial_x \otimes (\partial_x s_0 + \sqrt{-1}\partial_y s_0 + 2\pi\rho(x)s_0) - \sqrt{-1}(\partial_x s_1 - \sqrt{-1}\partial_y s_1 - 2\pi\rho(x)s_1)$$

for $s = s_0 + \partial_x \otimes s_1 \in \Gamma(W)$, where $s_0 \in \Gamma(L)$ and $\partial_x \otimes s_1 \in \Gamma((TM, J) \otimes_{\mathbb{C}} L)$ are even and odd parts of s, respectively.

Let $\pi\colon M \to \mathbb{R}$ be the first projection and $T[\pi]$ be the tangent bundle along fibers of π. Then, as complex vector bundles, $T[\pi] \otimes_{\mathbb{R}} \mathbb{C}$ is identified with (TM, J) by

$$T[\pi] \otimes_{\mathbb{R}} \mathbb{C} \to (TM, J),\ \partial_y \otimes_{\mathbb{R}} (a + \sqrt{-1}b) \mapsto\ a\partial_y + bJ\partial_y. \tag{5.5}$$

Under the identification (5.5) let D_{fiber} be the de Rham operators along fibers of π which is defined by

$$D_{\mathrm{fiber}}s = \partial_y \otimes (\partial_y s_0 - 2\pi\sqrt{-1}\rho(x)s_0) - (\partial_y s_1 - 2\pi\sqrt{-1}\rho(x)s_1).$$

Put $V := M \setminus \pi^{-1}(0)$. It is easy to see that on V, π defines a compatible fibration consisting of a single fiber bundle $\pi|_V\colon V \to \pi(V)$ and D_{fiber} restricted on V is an acyclic compatible system. See also Proposition 6.4.

We compute the local index $\mathrm{ind}(M, V)$. The deformed operator $D_t = D + tD_{\mathrm{fiber}}$ is written in the following way

$$\begin{aligned} D_t =& \partial_x \otimes \left(\partial_x s_0 + \sqrt{-1}(1+t)\partial_y s_0 + 2\pi(1+t)\rho(x)s_0\right) \\ &- \sqrt{-1}\left(\partial_x s_1 - \sqrt{-1}(1+t)\partial_y s_1 - 2\pi(1+t)\rho(x)s_1\right). \end{aligned}$$

Let us compute $\ker D_t^0 \cap L^2$. For an L^2-section s_0 of L, we first solve the equation

$$0 = \partial_x s_0 + \sqrt{-1}(1+t)\partial_y s_0 + 2\pi(1+t)\rho(x)s_0. \tag{5.6}$$

By taking the Fourier expansion of s_0 with respect to y, s_0 is written as

$$s_0 = \sum_{n\in\mathbb{Z}} a_n(x)e^{2\pi\sqrt{-1}ny}. \tag{5.7}$$

Then, s_0 satisfies (5.6) if and only if each a_n is of the form

$$a_n(x) = c_n \exp\left(2\pi(1+t)\int_0^x n - \rho(x)dx\right)$$

for some constant c_n. Since $\rho(x) \equiv \pm\delta$ for sufficiently large, or small x and since s is a L^2-section, it is easy to see that $c_n = 0$ except for $n = 0$. This implies that the kernel of the even part of D_t is one-dimensional.

Next to compute $\ker D_t^1 \cap L^2$ we solve the equation

$$0 = \partial_x s_1 - \sqrt{-1}(1+t)\partial_y s_1 - 2\pi(1+t)\rho(x)s_1.$$

By the similar argument we can show that $c_n = 0$ for all $n \in \mathbb{Z}$. This implies that the kernel of the odd part of D_t is zero-dimensional. Thus, $\operatorname{ind}(M,V) = 1$.

5.3.2. *Two-dimensional disc.* Let $M := \mathbb{C}$. For a sufficiently small positive number $0 < \delta < 1$, consider the following almost Hermitian structure (g, J) on M

$$g(a_1\partial_r + b_1\partial_\theta, a_2\partial_r + b_2\partial_\theta) = a_1a_2 + \tau(r)^2 b_1 b_2,$$
$$J\colon \partial_r \mapsto \tau(r)^{-1}\partial_\theta \ , \partial_\theta \mapsto -\tau(r)\partial_r,$$

where we use the polar coordinate $z = re^{2\pi\sqrt{-1}\theta}$ and $\tau(r)$ is an increasing function on $\mathbb{R}_{\geq 0}$ that satisfies $\tau(r) = 2\pi r$ for $0 \leq r < \delta/3$ and $\tau(r) \equiv 1$ for $2\delta/3 \leq r$. Note that the almost Hermitian structure (g, J) agrees with the standard one on $\mathbb{C}$ for $r < \delta/3$ and the standard one on $\mathbb{R} \times S^1$ for $2\delta/3 \leq r$.

Let $L = M \times \mathbb{C}$ be the trivial complex line bundle on M and ∇ the connection on L of the form $\nabla = d - 2\pi\sqrt{-1}h(r)d\theta$, where $h(r)$ is an increasing function on $\mathbb{R}_{\geq 0}$ that satisfies $h(r) = r^2$ on $0 \leq r < \delta/3$ and $h(r) = \delta^2$ on $2\delta/3 \leq r$.

Let $W := \wedge^\bullet(TM, J) \otimes_{\mathbb{C}} L$ be the $\mathbb{Z}/2$-graded Clifford module bundle over M which is explained in Example 2.16 and Remark 2.26. A direct computation shows that the Levi-Civita connection associated to g and the connection on L induces a connection on W so that W is a Dirac bundle. We take a Dirac-type operator D acting on $\Gamma(W)$ to be the spin^c Dirac operator with coefficients in L. By a direct computation, D is written as

$$\begin{aligned} Ds = \partial_\theta \otimes \left\{-\sqrt{-1}\tau(r)^{-1}\partial_r s_0 + \tau(r)^{-2}(\partial_\theta s_0 - 2\pi\sqrt{-1}h(r)s_0)\right\} \\ - 2\sqrt{-1}\partial_r\tau(r)s_1 - \sqrt{-1}\tau(r)\partial_r s_1 - (\partial_\theta s_1 - 2\pi\sqrt{-1}h(r)s_1) \end{aligned}$$

for $s = s_0 + \partial_\theta \otimes s_1 \in \Gamma(W)$, where $s_0 \in \Gamma(L)$ and $\partial_\theta \otimes s_1 \in \Gamma((TM, J) \otimes_{\mathbb{C}} L)$ are even and odd parts of s, respectively.

We consider the map $\pi\colon M \to \mathbb{R}_{\geq 0}$ defined by

$$\pi(z) = h(|z|).$$

π has the unique singular fiber at $0 \in \mathbb{R}_{\geq 0}$ and except on $\pi^{-1}(0)$ π is a S^1-bundle. We put $V := M \setminus \pi^{-1}(0)$. Then, π induces a compatible fibration consisting of a single S^1-bundle $\pi|_V\colon \to \pi(V)$. Let $T[\pi]$ be the tangent bundle along fibers of $\pi|_V$. Then, as complex vector bundles, $T[\pi] \otimes_{\mathbb{R}} \mathbb{C}$ is identified with $(TM, J)|_V$ by (5.5). Under the identification (5.5) let D_{fiber} be the de Rham operators along fibers of $\pi|_V$ which is defined by

$$D_{\text{fiber}}s = \partial_\theta \otimes \tau(r)^{-2}(\partial_\theta s_0 - 2\pi\sqrt{-1}h(r)s_0) - (\partial_\theta s_1 - 2\pi\sqrt{-1}h(r)s_1).$$

It is easy to see that on V D_{fiber} is an acyclic compatible system. See also Proposition 6.4.

We compute the local index $\mathrm{ind}(M,V)$. Let $\rho(r)$ be an increasing function on $\mathbb{R}_{\geq 0}$ that satisfies $\rho(r)\equiv 0$ for $0\leq r<\delta/3$ and $\rho(r)\equiv 1$ for $2\delta/3\leq r$. Then, the deformed operator $D_t = D + t\rho(r)D_{\mathrm{fiber}}$ is written in the following way

$$\begin{aligned} D_t s =& \partial_\theta \otimes \left\{-\sqrt{-1}\tau(r)^{-1}\partial_r s_0 + (1+t\rho(r))\,\tau(r)^{-2}\left(\partial_\theta s_0 - 2\pi\sqrt{-1}h(r)s_0\right)\right\} \\ &- \left\{2\sqrt{-1}\partial_r\tau(r)s_1 + \sqrt{-1}\tau(r)\partial_r s_1 + (1+t\rho(r))\left(\partial_\theta s_1 - 2\pi\sqrt{-1}h(r)s_1\right)\right\}. \end{aligned}$$

First let us compute $\ker D_t^0 \cap L^2$. By using the Fourier expansion (5.7) of s_0 with respect to θ, s_0 satisfies $0 = D_t^0 s_0$ if and only if each a_n is of the form

$$a_n(r) = c_n \exp\left\{2\pi\int_{r_0}^{r}(1+t\rho(r))\,\tau(r)^{-1}(n-h(r))\,dr\right\}$$

for some sufficiently small positive number $0<r_0$.

We investigate the condition for s_0 to be L^2-bound. Put

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \tau(r)^2 \end{pmatrix}.$$

The norm of s_0 is written as

$$\|s_0\|^2 = \sum_n \int_0^1\int_0^\infty |a_n|^2\sqrt{\det G}\,dr\,d\theta = \sum_n \int_0^\infty |a_n|^2\tau(r)dr.$$

Since $\tau(r)=2\pi r$, $h(r)=r^2$, and $\rho(r)=0$ for $0\leq r<\delta/3$ and $\tau(r)=1$, $h(r)=\delta^2$, and $\rho(r)=1$ for $2\delta/3\leq r$, we have

$$\int_0^\infty |a_n|^2\tau(r)dr = \left(\int_0^{\delta/3} + \int_{\delta/3}^{2\delta/3} + \int_{2\delta/3}^{\infty}\right)|a_n|^2\tau(r)dr$$

$$= 2\pi|c_n|^2 e^{r_0^2} r_0^{-2n}\int_0^{\delta/3} r^{2n+1}e^{-r^2}dr + \int_{\delta/3}^{2\delta/3}|a_n|^2\tau(r)dr \tag{5.8}$$

$$+ |c_n|^2|d_n|^2\int_{2\delta/3}^{\infty}\exp\left\{4\pi(1+t)(n-\delta^2)(r-2\delta/3)\right\}dr, \tag{5.9}$$

where

$$d_n = c_n\left(\frac{\delta}{3r_0}\right)^n \exp\left(\frac{r_0^2}{2}-\frac{\delta^2}{18}\right)\exp\left(2\pi\int_{\delta/3}^{2\delta/3}(1+t\rho(r))\,\tau(r)^{-1}(n-h(r))\,dr\right).$$

We can estimate the integral of the first term in (5.8) as follows

$$\begin{aligned} \text{the integral of the first term in (5.8)} &\geq e^{-\left(\frac{\delta}{3}\right)^2}\int_0^{\delta/3} r^{2n+1}dr \\ &= \lim_{\varepsilon\to+0} e^{-\left(\frac{\delta}{3}\right)^2}\begin{cases} \dfrac{\left(\frac{\delta}{3}\right)^{2n+2}-\varepsilon^{2n+2}}{2n+2} & n\neq -1 \\ \log\left(\dfrac{\delta}{3\varepsilon}\right) & n=-1 \end{cases} \\ &= \begin{cases} \dfrac{e^{-\left(\frac{\delta}{3}\right)^2}}{2n+2}\left(\frac{\delta}{3}\right)^{2n+2} & 2n+2>0 \\ +\infty & 2n+2\leq 0. \end{cases} \end{aligned}$$

Hence, if s_0 is L^2-bounded, then, $c_n = 0$ for all $n \leq -1$. Moreover, the integral in (5.9) is bounded if and only if $c_n = 0$ for all $n \geq \delta^2$. This implies

$$\dim_{\mathbb{C}} \ker D_t^0 \cap L^2 = \#\mathbb{Z} \cap (-1, \delta^2) = 1. \tag{5.10}$$

Next, let us compute $\ker D_t^1 \cap L^2$. By using the Fourier expansion (5.7) for s_1 with respect to θ, $\partial_\theta \otimes s_1$ satisfies $0 = D_t^1(\partial_\theta \otimes s_1)$ if and only if each a_n is of the form

$$a_n(r) = c_n \exp\left\{-\int_{r_0}^{r} \frac{2\partial_r \tau(r) + 2\pi\left(1 + t\rho(r)\right)\left(n - h(r)\right)}{\tau(r)} dr\right\}.$$

We investigate the condition for $\partial_\theta \otimes s_1$ to be L^2-bound. The norm of $\partial_\theta \otimes s_1$ is written as

$$\|\partial_\theta \otimes s_1\|^2 = \sum_n \int_0^1 \int_0^\infty \tau(r)^2 |a_n|^2 \sqrt{\det G} dr d\theta = \sum_n \int_0^\infty \tau(r)^3 |a_n|^2 dr.$$

By the same way as before, we have

$$\begin{aligned}
&\int_0^\infty \tau(r)^3 |a_n|^2 dr \\
&= (2\pi)^3 |c_n|^2 e^{-r_0^2} r_0^{2(n+2)} \int_0^{\delta/3} r^{-2n-1} e^{r^2} dr + \int_{\delta/3}^{2\delta/3} \tau(r)^3 |a_n|^2 dr \qquad (5.11)\\
&+ |c_n|^2 |d_n'|^2 \int_{2\delta/3}^\infty \exp\left\{-4\pi(1+t)(n - \delta^2)(r - 2\delta/3)\right\} dr, \qquad (5.12)
\end{aligned}$$

where

$$\begin{aligned}
d_n' =& c_n \left(\frac{\delta}{3r_0}\right)^{-(n+2)} \exp\left(-\frac{r_0^2}{2} + \frac{\delta^2}{18}\right) \\
&\times \exp\left(-\int_{\delta/3}^{2\delta/3} 2\tau(r)^{-1}\partial_r \tau(r) + 2\pi\left(1 + t\rho(r)\right)\tau(r)^{-1}\left(n - h(r)\right) dr\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\text{the integral of the first term in (5.11)} &\geq \int_0^{\delta/3} r^{-2n-1} dr \\
&= \lim_{\varepsilon \to +0} \begin{cases} \dfrac{-\left(\frac{\delta}{3}\right)^{-2n} + \varepsilon^{-2n}}{2n} & n \neq 0 \\ \log\left(\dfrac{\delta}{3\varepsilon}\right) & n = 0 \end{cases} \\
&= \begin{cases} \dfrac{-1}{2n}\left(\frac{\delta}{3}\right)^{-2n} & 2n < 0 \\ +\infty & 2n \geq 0. \end{cases}
\end{aligned}$$

Hence, if $\partial_\theta \otimes s_1$ is L^2-bounded, then, $c_n = 0$ for all $n \geq 0$. Moreover, the integral in (5.12) is bounded if and only if $c_n = 0$ for all $n \leq \delta^2$. This implies

$$\dim_{\mathbb{C}} \ker D_t^1 \cap L^2 = 0. \tag{5.13}$$

Therefore, by (5.10) and (5.13), we obtain $\operatorname{ind}(M, V) = 1$.

6. Applications

6.1. **Nonsingular case.** The purpose of this section is to show the following theorem as an equality between virtual vector spaces.

Theorem 6.1 ([1, 9])**.** *Let (M,ω) be a closed symplectic manifold, $\pi\colon (M,\omega)\to B$ a Lagrangian fibration with compact connected fibers, $(L, \langle\ ,\ \rangle, \nabla)\to (M,\omega)$ a prequantization line bundle. Then, the Riemann-Roch number of (M,ω) and the number of Bohr-Sommerfeld fibers agree with each other.*

The following proposition tells us what we should take as D_α.

Proposition 6.2. *The fiber $\pi^{-1}(b)$ at $b\in B$ is Bohr-Sommerfeld if and only if the cohomology $H^\bullet\left(\pi^{-1}(b), (L,\nabla)_{\pi^{-1}(b)}\right)$ with coefficients in the local system $(L,\nabla)_{\pi^{-1}(b)}$ is non-trivial.*

The proposition follows from the next lemma.

Lemma 6.3. *Let T be a torus, (L,∇) a flat complex line bundle on T. Then, the cohomology $H^\bullet(T;(L,\nabla))$ with coefficients in (L,∇) is trivial if and only if the degree zero cohomology $H^0(T;(L,\nabla))$ is trivial.*

Proof of Proposition 6.2. By definition, $\pi^{-1}(b)$ is Bohr-Sommerfeld if and only if $H^0\left(\pi^{-1}(b), (L,\nabla)_{\pi^{-1}(b)}\right)$ is non-trivial. Since a fiber of π is compact and connected, by Arnold-Liouville's theorem the fiber is a torus. Hence, by Lemma 6.3, we obtain Proposition 6.2. □

Moreover, by the local coefficient version of Theorem 2.30, we can put the Bohr-Sommerfeld condition in the following analytic condition.

Proposition 6.4. *The fiber $\pi^{-1}(b)$ at $b\in B$ is Bohr-Sommerfeld if and only if the de Rham operator on $\pi^{-1}(b)$ with coefficients in $(L,\nabla)_{\pi^{-1}(b)}$ has the non-trivial kernel.*

The next technical lemma is necessary to show the theorem.

Lemma 6.5. *There exists an almost Hermitian structure (J,g) compatible with ω such that (J,g) is invariant under the fiberwise action* (3.3) *of the canonical model T^*B_T on (M,ω).*

Proof of Theorem 6.1. We take and fix an almost Hermitian structure (J,g) on (M,ω) as in Lemma 6.5. Then, the Riemannian metric g induces a Riemannian metric on B so that π is a Riemannian submersion.

Let $T[\pi]\to M$ be the tangent bundle along fibers of π, $\iota\colon T[\pi]\to TM$ the natural inclusion, and $q\colon TM\to TM$ the orthogonal projection to $T[\pi]$ with respect to g. We define the Levi-Civita connection $\nabla^{T[\pi]}\colon \Gamma(TM)\to\Gamma(T^*[\pi]\otimes TM)$ along the fibers of π by

$$\nabla^{T[\pi]} = \iota^*\otimes q\circ\nabla^{LC}\circ q.$$

$\nabla^{T[\pi]}$ induces the unique connection $\nabla^{\mathrm{fiber}}\colon\Gamma(W)\to\Gamma(T^*[\pi]\otimes W)$ on W along the fibers of π that satisfies

$$\nabla^{\mathrm{fiber}}(\alpha\wedge\beta) = (\nabla^{\mathrm{fiber}}\alpha)\wedge\beta + \alpha\wedge(\nabla^{\mathrm{fiber}}\beta).$$

By construction for each $b\in B$, with the natural isomorphism

$$T\pi^{-1}(b)\cong T[\pi]|_{\pi^{-1}(b)}, \tag{6.1}$$

the restriction of ∇^{fiber} to $\pi^{-1}(b)$ agrees with the Levi-Civita connection on $\pi^{-1}(b)$ with respect to $g|_{\pi^{-1}(b)}$. Then, we define D_{fiber} by

$$D_{\text{fiber}} = c \circ \nabla^{\text{fiber}} \otimes \mathrm{id}_L + \mathrm{id} \otimes \nabla^L : \Gamma(W) \to \Gamma(W).$$

There is an isomorphism

$$T[\pi] \otimes_{\mathbb{R}} \mathbb{C} \ni u \otimes (x + \sqrt{-1}y) \mapsto xu + yJu \in TM \tag{6.2}$$

between $T[\pi] \otimes_{\mathbb{R}} \mathbb{C}$ and (TM, J) as complex vector bundles. (6.2) induces the isomorphism

$$\wedge^\bullet T^*[\pi] \otimes_{\mathbb{R}} \mathbb{C} \to \wedge^\bullet T^* M.$$

The direct computation shows the map

$$c \circ \iota : T\pi^{-1}(b) \cong T[\pi]|_{\pi^{-1}(b)} \to \mathrm{End}\left((\wedge^\bullet T^*[\pi])|_{\pi^{-1}(b)}\right) \cong \mathrm{End}(\wedge^\bullet T^* \pi^{-1}(b))$$

agrees with the Clifford multiplication defined in Example 2.14 for $\pi^{-1}(b)$. This implies that the restriction of D_{fiber} to $\pi^{-1}(b)$ agrees with the de Rham operator on $\pi^{-1}(b)$ with coefficients in L.

Moreover, by Lemma 6.5, the restriction of g to each fiber is a flat metric. From these fact, we can show that D_{fiber} satisfies the condition (iii) in Definition 5.4.

Let $V \subset M$ be the complement of all Bohr-Sommerfeld fibers. V is an open set. We define the structure of a compatible fibration on V consisting of a single torus bundle by

$$\pi_{\text{fiber}} = \pi|_V : V \to U = \pi(V).$$

Since V is the complement of all Bohr-Sommerfeld fibers Proposition 6.4 shows that the restriction of D_{fiber} to V is an acyclic compatible system.

Now suppose that there are exactly k Bohr-Sommerfeld fibers, say, $\pi^{-1}(b_1)$, ..., $\pi^{-1}(b_k)$. Let O_1, ..., O_k be mutually disjoint sufficiently small open neighborhoods of $\pi^{-1}(b_1), \ldots \pi^{-1}(b_k)$, respectively. Applying Corollary 5.8 to this situation, the equality

$$RR(M, \omega) = \sum_{i=1}^{k} \mathrm{ind}(O_i, O_i \cap V)$$

holds. Moreover, by taking each O_i sufficiently small if necessary, O_i can be identified with a neighborhood of the fiber $(S^1 \times \{0\})^n$ of $(S^1 \times \mathbb{R})^n \to \mathbb{R}^n$ at the origin. Then, by using the result in Section 5.3.1 and the product formula, we can show

$$\mathrm{ind}(O_i, O_i \cap V) = 1$$

for all i. □

6.2. **Singular case.** For a general Lagrangian fibration with singular fibers, we have the partial answer of Question 4.31 which are as follows.

Theorem 6.6 ([9, 10])**.** *Let (M, ω) be a closed symplectic manifold with prequatization line bundle and $\pi : (M, \omega) \to B$ a singular Lagrangian fibration with compact connected fibers. Then, the Riemann-Roch number is described as the sum of the number of nonsingular Bohr-Sommerfeld fibers and the contributions of singular fibers.*

Remark 6.7. Up to now we have not established the general method to compute the contributions of singular fibers yet.

In the case of locally toric Lagrangian fibrations, since they are locally identified with the moment map of a toric manifold, even the singular fibers are smooth as manifolds. In particular, the Bohr-Sommerfeld condition (4.25) makes sense. In the following special case we can show the following theorem.

Theorem 6.8 ([10])**.** *Let (M,ω) be a closed four-dimensional symplectic manifold with prequatization line bundle and $\pi\colon (M,\omega)\to B$ a locally toric Lagrangian fibration. Then, the Riemann-Roch number is equal to the number of both singular and nonsingular Bohr-Sommerfeld fibers.*

For the proof, see [9, 10].

6.3. **Application to Hamiltonian torus actions.** Let G be a torus and $\mathfrak{g}$ be the Lie algebra of G. We denote by $\mathfrak{g}_{\mathbb{Z}}$ and $\mathfrak{g}^*_{\mathbb{Z}}$ the integral lattice and the weight lattice which are defined by

$$\mathfrak{g}_{\mathbb{Z}} = \{\xi\in\mathfrak{g} \mid \exp\xi = e\in G\},$$
$$\mathfrak{g}^*_{\mathbb{Z}} = \{\eta^*\in\mathfrak{g}^* \mid \langle \xi,\eta^*\rangle\in\mathbb{Z}\ \forall \xi\in\mathfrak{g}_{\mathbb{Z}}\}.$$

Let (M,ω) be a closed symplectic manifold with prequantization line bundle $(L,\langle\ ,\ \rangle,\nabla)\to(M,\omega)$. Suppose that G acts effectively on M and the G-action lifts to that on L preserving all the data. Then, the G-action is Hamiltonian. In fact, the moment map is defined by the following formula

$$\frac{d}{dt}\Big|_{t=0}\psi_{\exp -t\xi}\circ s\circ\varphi_{\exp t\xi} = \nabla_{X_\xi}s + 2\pi\sqrt{-1}\,\langle \mu,\xi\rangle\, s \tag{6.3}$$

for $\xi\in\mathfrak{g}$ and $s\in\Gamma(L)$, where φ_g denotes the G-action on M and ψ_g denotes the lift of φ_g to L for $g\in G$. Moreover, each orbit of the G-action on M is an isotropic torus, where isotropic means that the restriction of ω to the orbit vanishes. In particular, the restriction of (L,∇) to $\mathcal{O}$ is flat.

Proposition 6.9. *Let $\mathcal{O}$ be an orbit of the G-action on M. If the restriction of (L,∇) to $\mathcal{O}$ has a nontrivial global parallel section, then, the image of $\mathcal{O}$ by μ lies in the weight lattice $\mathfrak{g}^*_{\mathbb{Z}}$. Moreover, if all isotropic subgroups appeared in the G-action on M are connected, then, the converse is true.*

Proof. By Definition 3.19 (ii) every orbit lies in a level set of μ. Let $\eta^*\in\mathfrak{g}^*$ be the image $\mu(\mathcal{O})$ of $\mathcal{O}$ by μ. Let $s\in\Gamma(L|_{\mathcal{O}})$ be a non-trivial parallel section. For arbitrary element $\xi\in\mathfrak{g}_{\mathbb{Z}}$ in the integral lattice $\mathfrak{g}_{\mathbb{Z}}$ and $x\in\mathcal{O}$, by (6.3) we have

$$\frac{d}{dt}\psi_{\exp -t\xi}\circ s\circ\varphi_{\exp t\xi}(x) = 2\pi\sqrt{-1}\,\langle\eta^*,\xi\rangle\,\psi_{\exp -t\xi}\circ s\circ\varphi_{\exp t\xi}(x).$$

This implies that

$$\psi_{\exp -t\xi}\circ s\circ\varphi_{\exp t\xi}(x) = e^{2\pi\sqrt{-1}t\langle\eta^*,\xi\rangle}s(x).$$

Since $\xi\in\mathfrak{g}_{\mathbb{Z}}$, by putting $t=1$, we have

$$s(x) = \psi_{\exp -\xi}\circ s\circ\varphi_{\exp \xi}(x) = e^{2\pi\sqrt{-1}\langle\eta^*,\xi\rangle}s(x).$$

Then, $\langle\eta^*,\xi\rangle$ must be in $\mathbb{Z}$.

Assume that all isotropic subgroups appeared in the G-action on M are connected. Let $\mathcal{O}$ be an orbit such that $\mu(\mathcal{O})=\eta^*\in\mathfrak{g}^*_{\mathbb{Z}}$. We take arbitrary elements $x_0\in\mathcal{O}$ and $s_0\in L_{x_0}$ and fix them. Since G is a torus, for arbitrary point $x\in\mathcal{O}$

there exists $\xi \in \mathfrak{g}$ such that $x = \varphi_{\exp\xi}(x_0)$. Then, we define the section $s \in \Gamma(L|_{\mathcal{O}})$ by

$$s(x) := e^{2\pi\sqrt{-1}\langle \eta^*, \xi\rangle}\psi_{\exp\xi}(s_0). \tag{6.4}$$

We show s is well-defined. In order to show the well-definedness of (6.4), it is sufficient to show that

$$s\left(\varphi_g(x_0)\right) = s_0 \tag{6.5}$$

for $g \in G_{x_0}$, where G_{x_0} is the isotropic subgroup of x_0. By assumption G_{x_0} is connected. So there exists an element ξ in the Lie algebra $\mathfrak{g}_{x_0}$ of G_{x_0} such that $g = \exp\xi$. Let $s' \in \Gamma(L|_{\mathcal{O}})$ be a section with $s'(x_0) = s_0$. Since $X_\xi(x_0) = 0$, by (6.3) for s' we have

$$\psi_{\exp -t\xi} \circ s' \circ \varphi_{\exp t\xi}(x_0) = e^{2\pi\sqrt{-1}t\langle \eta^*, \xi\rangle} s_0.$$

Since $\varphi_{\exp\xi}(x_0) = x_0$, by putting $t = 1$,

$$\begin{aligned} s_0 &= s'(x_0) \\ &= s' \circ \varphi_{\exp\xi}(x_0) \\ &= e^{2\pi\sqrt{-1}\langle \eta^*, \xi\rangle}\psi_{\exp\xi}(s_0) \\ &= s\left(\varphi_g(x_0)\right). \end{aligned}$$

This shows (6.5). By the construction of s, it is clear that s is a nontrivial global parallel section. □

Since Proposition 6.4 holds not only for $\pi^{-1}(b)$ but also for $\mathcal{O}$, we can construct an acyclic compatible system on $M \setminus \mu^{-1}(\mathfrak{g}^*_{\mathbb{Z}})$. Thus, by applying Corollary 5.8 to this case, we have the following result.

Theorem 6.10 ([10])**.** *In this case the Riemann-Roch number is described in terms of the data near the inverse image of the weight lattice by μ.*

For more details, see [10].

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