Topological Crystallography

—in view of Discrete Geometric Analysis—

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PREFACE

Russian mathematician P. L. Chebyshev (1815-1897) once said in a general context that the agreement of theory and practice brings most beneficial results in sciences. His words pertain to what this monograph intends to convey to the reader. That is, the author wishes primarily to provide the reader with a mathematical insight into modern crystallography, a typical practical science that originated in the classification of the observed shapes of crystals. However the tools we shall employ are not adopted from the traditional theory of crystallographic groups, but from algebraic topology, a field in pure mathematics cultivated during the last century. More specifically the theory of covering spaces and homology theory are effectively used in the discussion on the 3D networks associated with crystals. This explains the reason why this monograph is entitled Topological Crystallography. Further we formulate a minimum principle for crystals in the framework of discrete geometric analysis, which, in spite of its pure-mathematical nature, turns out to fit with a systematic enumeration of crystal structures, an area of considerable scientific interest for many years.

The objects that topological crystallography concerns are not necessarily restricted to crystals. Ornamental patterns having crystallographic symmetry in art, nature and architectures are the objects falling within the scope of this monograph. Indeed, many interesting forms (Katachi in Japanese) which are potentially useful for artistic designs in various areas are generated from canonical placements characterized by the minimum principle. Therefore, the main target of this monograph is, naturally enough, both mathematicians (including graduate and even undergraduate students) and a wide circle of practical scientists (especially crystallographers and design scientists in art and architecture as well) who want to know how ideas and theories developed in pure mathematics are applied to a practical problem.

This monograph has developed out of the note that I prepared for my
lectures at Meiji University during the academic year 2011-12. My thanks are due to Davide M. Proserpio who provided me with relevant references in chemical crystallography. I also thank Hisashi Naito and my daughter Kayo for producing the beautiful CG images of several hypothetical crystals. This work could not have been done without the friendly help and advice of several people, especially Polly Wee Sy. I have great pleasure in thanking her.

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