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A Robust-filtering Method for Noisy Non-Stationary Time Series with an Application to Japanese Macro-consumption *

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Abstract
We investigate a new filtering method to estimate the hidden states of random variables and to handle multiple non-stationary time series data. It is useful for analyzing small sample non-stationary macro-economic time series in particular and the method is based on the separating information maximum likelihood (SIML) developed by Kunitomo, Sato and Kurisu (2018), Kunitomo and Sato (2017) for estimating the non-stationary errors-in-variables models and Nishimura, Sato and Takahashi (2019) for a financial application. We generalize the previous results to solve the filtering problem of hidden random variables of trend and seasonality, which gives a useful method of handling macro-economic time series. We develop the asymptotic theory based on the frequency domain for non-stationary time series. We apply our filtering method to analyze quarterly and monthly macro-consumption data in Japan.

Key Words

Noisy Non-stationary time series, Errors-variables models, Measurement Error, Macroeconomic time series, trend, seasonality and noise, Robust-filtering, SIML, Fourier-Inversion, Macro-consumption in Japan

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1. Introduction

There has been a vast amount of published research on the use of statistical time series analysis of macro-economic time series. One important feature of macroeconomic time series, which is different from the standard time series analysis, is the fact that the observed time series is an apparent mixture of non-stationary components and stationary components. The second feature is the fact that the measurement errors in economic time series play important roles because macro-economic data are usually constructed from various sources including sample surveys in major official statistics while the statistical time series analysis often ignored measurement errors. There is yet third important issue that the sample size of macro-economic data is rather small and we have 120, say, time series observations for each series when we have quarterly data over 30 years. The quarterly GDP series, which is the most important data in Japanese macro-economy for instance, has been published since 1994 by the cabinet office of Japan. Since the sample size is small, it is important to use an appropriate statistical procedure to extract information on trend and noise (or measurement error) components in a systematic way from data. Since the sample size is small, it is important to use an appropriate statistical procedure to extract information on trend and noise (or measurement error) components in a systematic way from data. Some of these aspects have been discussed by Morgenstern (1950), Granger and Newbold (1977), and Nerlove, Grether and Carvalho (1995) for instance. See https://www.esri.cao.go.jp/index-e.html for the official macro-economic (GDP) data published by Cabinet Office of Japanese Government.

In this study we will investigate a new filtering procedure to estimate the hidden states of random variables, which were non-stationary, and to handle multiple time series data including small sample economic time series of macro-economic variables. Kunitomo and Sato (2017), and Kunitomo, Sato and Kurisu (2018) have developed the separating information maximum likelihood (SIML) method for estimating the non-stationary errors-in-variables models. They have discussed the asymptotic properties and finite sample properties of the estimation of unknown parameters in the statistical models. We utilize their results to solve the filtering problem of hidden random variables and the resulting method leads to a powerful new method of handling macro-economic time series.

Earlier and related literature on the non-stationary economic time series analysis are Engle and Granger (1987) and Johansen (1995), which dealt with multivariate non-stationary and stationary time series and developed the notion of co-integration without measurement errors. The problem of our interest is related to their work, but it has different aspects and our focus is on the non-stationarity and measurement error in the non-stationary errors-in-variable models. Also in econometric literature the issue of identification of parametric models and the issue of estimation when the true parameters are around the boundary points have been discussed. There are some recent econometric studies on time series in the frequency domain such as Baxter and King (1999), and Muller and Watson (2018). In this respect, the present study on the non-stationary errors-in-variables models may give some interpretation on their methods.

In the statistical multivariate analysis, there is also some literature on the errors-in-variables models as Anderson (1984, 2003) and Fuller (1987), but they considered the multivariate cases of independent observations and the underlying situation is different from ours.

Kitagawa (2010) has discussed the standard statistical filtering methods already known including the Kalman-filtering and the particle-filtering methods. Since (i) these methods depend on the underlying distributions such as the Gaussian distributions for the Kalman-filtering and (ii) the procedure essentially depends on the dimension of state variables, there may be some difficulty to extend to the high-dimension cases even when it is fixed, say 10. On the other hand, we expect that our method has some merits when we need to handle small sample economic times series with non-stationarity and seasonality with many variables because our method does not depend on the specific distributions as well as the dimension of the underlying random variables. See Kunitomo, Awaya and Kurisu (2017) for a comparison of small sample properties of the ML and SIML estimation methods for the errors-in-variables models and Nishimura et al. (2019) for an application of financial data smoothing. The most important feature of the present procedure is that it may be applicable to small sample time series data. Also it seems that our new method has a solid mathematical and statistical foundation based on the spectral decomposition of stochastic processes. As an illustrative application, we use our filtering method to analyze quarterly and monthly macro-consumption data in Japan.

In Section 2 we give some macro-economic data, which have motivated the present study. In Section 3 we define the non-stationary errors-in-variables model. and the SIML method. Then in Section 4 we introduce the SIML filtering method. Then in Section 5, we discuss the statistical foundation of the method and in Section 6 we propose a method of choosing the numbers of terms, which is based on prediction, and we give some numerical examples based on a set of simulations. Some empirical examples on macro-consumption data in Japan will be given in Section 7 and concluding remarks are given in Section 8. Some mathematical derivations of our results will be in Appendix.

2. Two Illustrative Examples

As an illustrative example, we plot the graph of two macro-economic time series in Japan : quarterly (real) consumption and quarterly (real) GDP (1994Q1-2018Q2) as Figure 2.1. It looks like a simple example of linear regression appeared in the undergraduate textbooks. However, if we draw the time series sequences of these (original and seasonally unadjusted) macro-data published by the Cabinet office of Japanese

Government as Figure 2.1, then we find that the situation is not so simple as it looks in Figure 2.1. In two time series data, there are clearly trends components, seasonal components, noise components, and possibly business cycle components. Although many economists usually use the seasonally adjusted (published) data, which were constructed by using the X-12-ARIMA program, the effects of filtering of the program are often unknown. The X-12-ARIMA program uses the univariate ARIMA and reg-ARIMA models and the DECOMP program, which was developed by Kitagawa (2010) and it is a possible choice particularly in Japan, uses the univariate AR model and Kalman filtering. Since each time series are handled by using different filtering procedures (that is, different reg-ARIMA models for instance), it may cause a fundamental problem in their interpretation when the focus of interest is on the relationships among different non-stationary data in particular.

Figure 2.3 gives three different macro-consumption data (2002 January - 2016 December), which are observed as monthly time series and widely used by economists in Japan to judge the current macro-business condition in Japan. The first series is *Kakei-Chosa* (the data from monthly consumer-survey), the second one is *Shohi-Douko* (the data from monthly retail), and the third one is *DaiSanji-Sangyo* (the index data on commerce). The data construction processes are different and quite complex, and each data reflect different aspects of macro-consumption. Since they show the similar movements, but there are some different aspects of trends and seasonal at the same time. Then we need to unify the monthly consumption series because we want to judge the business condition by just observing these data for evaluating the state of Japanese macro-economy and making macro-economic policy. Many economists in both central governments and private sectors usually use the seasonally adjusted data, which were constructed from the quarterly or monthly (original) time series and by using the univariate X-12-ARIMA seasonal adjustment program. Thus it is important to construct the monthly consumption index, which is consistent with the published quarterly macro-consumption data.

These two empirical examples and related issues motivated us to develop the multivariate non-stationary errors-in-variables models and the filtering method for the hidden state variables with measurement errors.

3. The Non-Stationary Errors-in-Variables Model and SIML

Let y_{ji} be the $i-$th observation of the $j-$th time series at i for $i = 1, \cdots, n; j = 1, \cdots, p$. We set $\mathbf{y}_i = (y_{1i}, \cdots, y_{pi})'$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i') \ (= (y_{ij}))$ be an $n \times p$ matrix of observations and denote $\mathbf{y}_0$ as the initial $p \times 1$ vector. We estimate the model when the underlying non-stationary trends $\mathbf{x}_i' (= (x_{1i}, \cdots, x_{pi}))$ $(i = 1, \cdots, n)$, but we have the vector of noise component $\mathbf{v}_i' = (v_{1i}, \cdots, v_{pi})$, which are independent of $\mathbf{x}_i$. We use the non-stationary errors-in-variables representation

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i \quad (i = 1, \cdots, n), \tag{3.1}$$

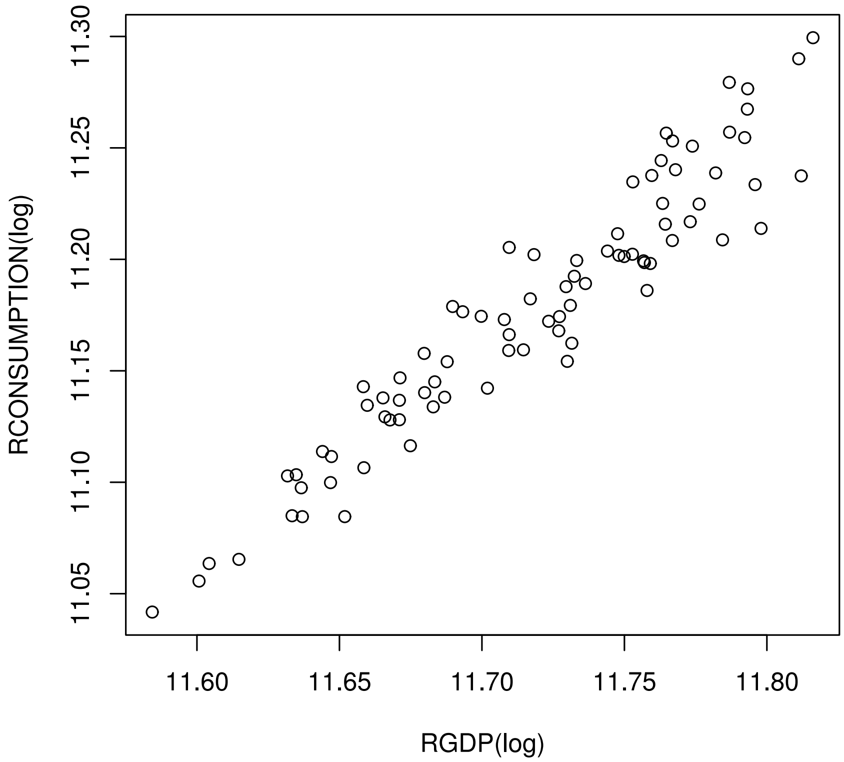

Figure 2.1 : real-GDP vs. real-Consumption

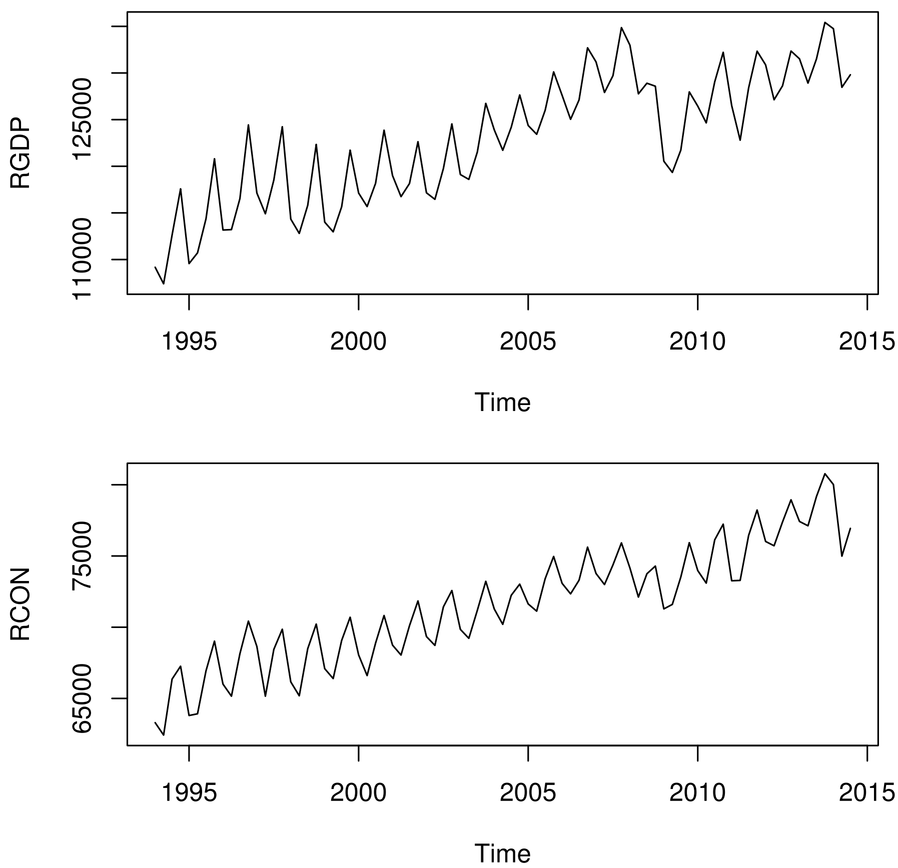

Figure 2.2 : real-GDP and real-Consumption

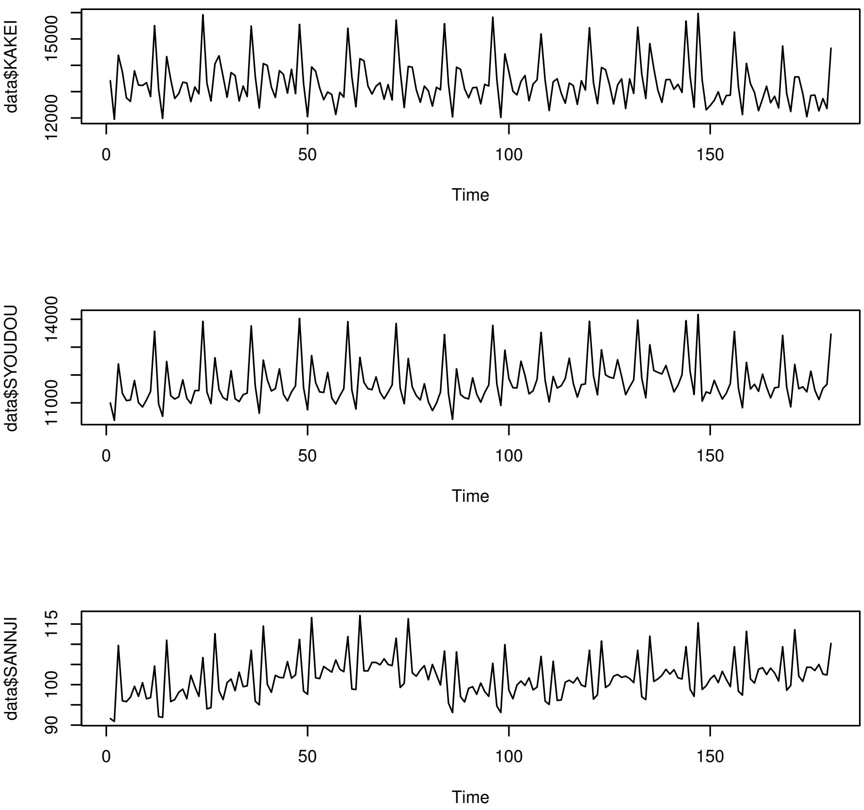

Figure 2.3 : Monthly consumption data

where $\mathbf{x}_i$ $(i = 1, \cdots, n)$ are a sequence of non-stationary I(1) process which satisfy

$$\Delta \mathbf{x}_i = (1 - \mathcal{L})\mathbf{x}_i = \mathbf{v}_i^{(x)} , \tag{3.2}$$

with the lag-operator $\mathcal{L}\mathbf{x}_i = \mathbf{x}_{i-1}$, $\Delta = 1 - \mathcal{L}$,

$$\mathbf{w}_i^{(x)} = \sum_{j=0}^{\infty} \mathbf{C}_j^{(x)} \mathbf{e}_{i-j}^{(x)} , \tag{3.3}$$

and $\mathbf{e}_i^{(x)}$ is a a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(x)}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{e}_i^{(x)}\mathbf{e}_i^{(x)\prime}) = \mathbf{\Sigma}_e^{(x)}$ (positive-semi-definite). The coefficients matrices $\mathbf{C}_j^{(x)}$ $(= c_{kl}^{(x)}(j))$ are absolutely summable such that $\sum_{j=0}^{\infty} \|\mathbf{C}_j^{(x)}\| < \infty$, where $\|\mathbf{C}_j^{(x)}\| = \max_{k,l=1,\cdots,p} |c_{kl}^{(x)}(j)|$ and $\mathbf{C}_j^{(x)} = (c_{kl}^{(x)}(j))$.
The random vectors $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are a sequence of stationary I(0) process with

$$\mathbf{v}_i = \sum_{j=0}^{\infty} \mathbf{C}_j^{(v)} \mathbf{e}_{i-j}^{(v)} , \tag{3.4}$$

where the coefficient matrices $\mathbf{C}_j^{(v)}$ are absolutely summable $(\sum_{j=0}^{\infty} \|\mathbf{C}_j^{(v)}\| < \infty$, where $\|\mathbf{C}_j^{(v)}\| = \max_{k,l=1,\cdots,p} |c_{kl}^{(v)}(j)|$ and $\mathbf{C}_j^{(v)} = (c_{kl}^{(v)}(j)))$ and $\mathbf{e}_i^{(v)}$ are a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(v)}) = \mathbf{0}$, $\mathbf{E}(\mathbf{e}_i^{(v)}\mathbf{e}_i^{(v)\prime}) = \mathbf{\Sigma}_e^{(v)}$ (non-negative definite).

Let $\mathbf{f}_{\Delta x}(\mu)$ and $\mathbf{f}_v(\mu)$ be the spectral density $(p \times p)$ matrices of $\Delta \mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ as

$$\mathbf{f}_{\Delta x}(\mu) = (\sum_{j=0}^{\infty} \mathbf{C}_j^{(x)} e^{2\pi i \mu j}) \mathbf{\Sigma}_e^{(x)} (\sum_{j=0}^{\infty} \mathbf{C}_j^{(x)\prime} e^{-2\pi i \mu j}), \quad (-\frac{1}{2} \leq \mu \leq \frac{1}{2}) \tag{3.5}$$

and

$$\mathbf{f}_v(\mu) = (\sum_{j=0}^{\infty} \mathbf{C}_j^{(v)} e^{2\pi i \mu j}) \mathbf{\Sigma}_e^{(v)} (\sum_{j=0}^{\infty} \mathbf{C}_j^{(v)\prime} e^{-2\pi i \mu j}), \quad (-\frac{1}{2} \leq \mu \leq \frac{1}{2}) \tag{3.6}$$

where we set $\mathbf{C}_0^{(x)} = \mathbf{C}_0^{(v)} = \mathbf{I}_p$ and $i^2 = -1$ (see Chapter 7 of Anderson (1971) for instance.)
Then the spectral density matrix of the transformed vector process $\Delta \mathbf{y}_i$ $(= \mathbf{y}_i - \mathbf{y}_{i-1})$ is

$$\mathbf{f}_{\Delta y}(\mu) = \mathbf{f}_{\Delta x}(\mu) + (1 - e^{2i\mu}) f_v(\mu) (1 - e^{-2i\mu}) \tag{3.7}$$

and we have the key relation at the zero-frequency such that

$$\mathbf{f}_{\Delta y}(0) = \mathbf{f}_{\Delta x}(0) \ . \tag{3.8}$$

We denote the long-run variance-covariance matrices of the trend components and the stationary components for $g,h=1,\cdots,p$ as

$$\boldsymbol{\Omega}_x = \mathbf{f}_{\Delta x}(0)\ (=(\omega_{gh}^{(x)}))\ , \tag{3.9}$$

and

$$\boldsymbol{\Omega}_v = f_v(0)\ = (\omega_{gh}^{(v)})\ . \tag{3.10}$$

When each pair of vectors $\Delta\mathbf{x}_i$ and $\mathbf{v}_i$ are independently, identically, and normally distributed (i.i.d.) as $N_p(\mathbf{0},\boldsymbol{\Sigma}_x)$ and $N_p(\mathbf{0},\boldsymbol{\Sigma}_v)$, respectively, and we have the observations of an $n\times p$ matrix $\mathbf{Y}_n=(\mathbf{y}_i^{'})$ and set the $np\times 1$ random vector $(\mathbf{y}_1^{'},\cdots,\mathbf{y}_n^{'})^{'}$. Given the initial condition $\mathbf{y}_0$, we have

$$\mathrm{vec}(\mathbf{Y}_n) \sim N_{n\times p}\left(\mathbf{1}_n\cdot\mathbf{y}_0^{'},\mathbf{I}_n\otimes\boldsymbol{\Sigma}_v+\mathbf{C}_n\mathbf{C}_n^{'}\otimes\boldsymbol{\Sigma}_x\right)\ , \tag{3.11}$$

where $\mathbf{1}_n^{'}=(1,\cdots,1)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix}_{n\times n} . \tag{3.12}$$

We return to consider the general case of (2.1)-(2.4) and use the $\mathbf{K}_n-$transformation that from $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(=(\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = \mathbf{K}_n\left(\mathbf{Y}_n-\bar{\mathbf{Y}}_0\right)\ , \mathbf{K}_n=\mathbf{P}_n\mathbf{C}_n^{-1}\ , \tag{3.13}$$

where

$$\mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{n\times n} , \tag{3.14}$$

and

$$\mathbf{P}_n=(p_{jk}^{(n)})\ ,\ p_{jk}^{(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}}\cos\left[\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})\right]\ . \tag{3.15}$$

By using the spectral decomposition $\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}=\mathbf{P}_n\mathbf{D}_n\mathbf{P}_n^{'}$ and $\mathbf{D}_n$ is a diagonal matrix with the k-th element $d_k=2[1-\cos(\pi(\frac{2k-1}{2n+1}))]$ $(k=1,\cdots,n)$ and we write

$$a_{kn}^{*}\ (=d_k)\ = 4\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right]\ (k=1,\cdots,n)\ . \tag{3.16}$$

Then the separating information maximum likelihood (SIML) estimator of $\hat{\mathbf{\Sigma}}_x$ when $\mathbf{w}_i^{(x)}$ are i.i.d. vectors can be defined by

$$\mathbf{G}_m = \hat{\mathbf{\Sigma}}_{x,SIML} = \frac{1}{m_n}\sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \ , \tag{3.17}$$

where we set $m = m_n = [n^{\alpha}]$ $(0 < \alpha < 1)$.
The estimation of the variance-covariance matrix $\mathbf{\Sigma}_v$ when $\mathbf{v}_i$ are i.i.d. vectors have been discussed by Kunitomo and Sato (2017). Some consistent estimators of $\mathbf{\Sigma}_v$ have been developed.

4. The SIML Filtering Method

4.1 The Basic Filtering

We consider the general filtering procedure based on the $\mathbf{K}_n-$transformation (3.13). Because the elements of the resulting $n \times p$ random matrix $\mathbf{Z}_n$ by this transformation take real values in the frequency domain, it is easy to interpret their roles. We consider the inversion of a transformation of orthogonal frequency processes. Let an $n \times p$ matrix

$$\hat{\mathbf{X}}_n = \mathbf{C}_n \mathbf{P}_n^{'} \mathbf{Q}_n \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \tag{4.1}$$

and

$$\mathbf{Z}_n = \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \ . \tag{4.2}$$

The stochastic process $\mathbf{Z}_n$ are the orthogonal decomposition of the original time series $\mathbf{Y}_n$ and $\mathbf{Q}_n$ is an $n \times n$ filtering matrix. We give explicit forms of useful examples including the trend filtering procedure and the seasonal filtering procedure for macro-time series. Although we can interpret the existing filtering procedures within our general framework, first it is useful to give linear filtering procedures.

(i) **General Linear Filtering** : Let an $n \times n$ diagonal matrix

$$\mathbf{Q}_n = \sum_{i=1}^{n} w_{i,n} \mathbf{e}_i \mathbf{e}_i^{'} \tag{4.3}$$

and $\mathbf{e}_i = (0, \cdots, 1, \cdots)^{'}$ $(i = 1, \cdots, n)$ are the unit vectors and $w_{i,n}$ $(i = 1, \cdots, n)$ are some non-negative constants. The trivial case is when we take $w_{i,n} = 1$ $(i = 1, \cdots)$, we have the identity matrix. There are several useful linear filtering cases and we will give two cases as the trend filtering and the seasonal filtering by choosing

$w_{i,n} = 1$ or 0 for some $i's$.

(ii) **Trend Filtering** : Let an $m \times n$ choice matrix $\mathbf{J}_m = (\mathbf{I}_m, \mathbf{O})$, and let also $n \times p$ matrix

$$\hat{\mathbf{X}}_n = \mathbf{C}_n \mathbf{P}_n^{'} \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \tag{4.4}$$

and and an $n \times n$ matrix

$$\mathbf{Q}_n = \mathbf{P}_n \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n \ . \tag{4.5}$$

We will construct an estimator of $n \times p$ hidden state matrix $\mathbf{X}_n$ in the lower frequency parts by using the inverse transformation of $\mathbf{Z}_n$ by deleting the estimated noise parts. (See Nishimura, Sato and Takahashi (2019) as an example.) For this purpose, let the $[m + (n - m)] \times [m + (n - m)]$ partitioned matrix

$$\mathbf{P}_n = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

and

$$\mathbf{P}_n \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n = \begin{pmatrix} \mathbf{P}_{11}^{'} \\ \mathbf{P}_{12}^{'} \end{pmatrix} (\mathbf{P}_{11}, \mathbf{P}_{12}) = \mathbf{I}_n - \begin{pmatrix} \mathbf{P}_{21}^{'} \\ \mathbf{P}_{22}^{'} \end{pmatrix} (\mathbf{P}_{21}, \mathbf{P}_{22}) \ . \tag{4.6}$$

After some calculations (see the Appendix), the $(j, j^{'})$-th element of $\mathbf{Q}_n = \mathbf{P}_n \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n$ $(= (q_{j,j^{'}}))$ is given by

$$\begin{aligned} q_{j,j} &= \frac{2m}{2n+1} + \frac{1}{2n+1} \left[\frac{\sin \frac{2m\pi}{2n+1}(2j-1)}{\sin \frac{\pi}{2n+1}(2j-1)} \right] , \\ q_{i,j^{'}} &= \frac{1}{2n+1} \left[\frac{\sin \frac{2m\pi}{2n+1}(j+j^{'}-1)}{\sin \frac{\pi}{2n+1}(j+j^{'}-1)} + \frac{\sin \frac{2m\pi}{2n+1}(j-j^{'})}{\sin \frac{\pi}{2n+1}(j-j^{'})} \right] \quad (j \neq J^{'}) \ . \end{aligned} \tag{4.7}$$

We evaluate MSE of $\hat{\mathbf{X}}_n (n \times p)$ and then in the simple case

$$\begin{aligned} & \mathcal{E}[\mathrm{tr}(\hat{\mathbf{X}}_n - \mathbf{X}_n)^{'} (\hat{\mathbf{X}}_n - \mathbf{X}_n)] \\ = \ & \mathrm{tr}(\mathbf{\Sigma}_v) \mathrm{tr}[\mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{'} \mathbf{C}_n \mathbf{P}_n^{'} \mathbf{J}_m^{'} \times \mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} \mathbf{P}_n^{'} \mathbf{J}_m^{'}] \\ & + \mathrm{tr}(\mathbf{\Sigma}_x) \mathrm{tr}[\mathbf{J}_m^{*} \mathbf{P}_n \mathbf{C}_n^{'} \mathbf{C}_n \mathbf{P}_n^{'} \mathbf{J}_m^{*'} \times \mathbf{J}_m^{*} \mathbf{P}_n \mathbf{C}_n^{-1} \mathbf{C}_n^{'-1} \mathbf{P}_n^{'} \mathbf{J}_m^{*'}] \ , \end{aligned} \tag{4.8}$$

where $\mathbf{J}_m = (\mathbf{I}_m, \mathbf{O})$ $(m \times n$ matrix) and $\mathbf{J}_m^{*} = (\mathbf{O}, \mathbf{I}_{n-m})$ $((n - m) \times n$ matrix).

(iii) **Seasonality Filtering** : We consider the filtering based on the $\mathbf{K}_n$−transformation in (3.13). We consider the inversion of some frequency parts of the random matrix $\mathbf{Z}_n$. The leading example is the seasonal frequency in the discrete time series and we take s (>1) being a positive integer. Let an $m_2 \times [m_1+m_2+(n-m_1-m_2)]$ choice matrix $\mathbf{J}_{m_1,m_2,n} = (\mathbf{O}, \mathbf{I}_{m_2}, \mathbf{O})$, and let also $n \times p$ matrix

$$\hat{\mathbf{X}}_n = \mathbf{C}_n \mathbf{P}_n^{'} \mathbf{J}_{m_1,m_2,n}^{'} \mathbf{J}_{m_1,m_2,n} \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \tag{4.9}$$

and an $n \times n$ matrix

$$\mathbf{Q}_n = \mathbf{P}_n \mathbf{J}_{m_1,m_2,n}^{'} \mathbf{J}_{m_1,m_2,n} \mathbf{P}_n \ . \tag{4.10}$$

As an example, when we have the seasonal frequency s (>1), we can take $m_1 = [2n/s] - [m/2]$ and $m_2 = m$. For instance, we take $s = 4$ for quarterly data and $s = 12$ for monthly data. After some calculations, the $(j, j^{'})$-th elemnt of $\mathbf{Q}_n = \mathbf{P}_n \mathbf{J}_{m_1,m_2,n}^{'} \mathbf{J}_{m_1,m_2,n} \mathbf{P}_n$ $(= (q_{j,j^{'}}))$ is given by

$$
\begin{aligned}
(4.11)\, q_{j,j} &= \frac{2m_2}{2n+1} + \frac{1}{2n+1}\left[\frac{\sin\frac{2(m_1+m_2)\pi}{2n+1}(2j-1) - \sin\frac{2(m_1)\pi}{2n+1}(2j-1)}{\sin\frac{\pi}{2n+1}(2j-1)}\right] , \\
q_{i,j^{'}} &= \frac{1}{2n+1}\left[\frac{\sin\frac{2(m_1 1+m_2)\pi}{2n+1}(j+j^{'}-1) - \sin\frac{2(m_1)\pi}{2n+1}(j+j^{'}-1)}{\sin\frac{\pi}{2n+1}(j+j^{'}-1)}\right. \\
&\qquad \left. + \frac{\sin\frac{2(m_1+m_2)\pi}{2n+1}(j-j^{'}) - \sin\frac{2(m_1)\pi}{2n+1}(j-j^{'})}{\sin\frac{\pi}{2n+1}(j-j^{'})}\right] \quad (j \neq j^{'}) \ .
\end{aligned}
$$

We note that when $m_1 = 0$ and $m_2 = m$, the resulting formula become to those in the basic case.

We evaluate MSE of $\hat{\mathbf{S}}_n$, which is $\hat{\mathbf{X}}$ $(n \times p)$ and then in the simple case

$$
\begin{aligned}
& \mathcal{E}[\mathrm{tr}(\hat{\mathbf{S}}_n - \mathbf{S}_n)^{'}(\hat{\mathbf{S}}_n - \mathbf{S}_n)] \\
= \ & \mathrm{tr}(\boldsymbol{\Sigma}_v)\mathrm{tr}[\mathbf{J}_{m_1,m_2,n}\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_{m_1,m_2,n}^{'} \times \mathbf{J}_{m_1,m_2,n}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'}\mathbf{J}_{m_1,m_2,n}^{'}] \\
& + \mathrm{tr}(\boldsymbol{\Sigma}_x)\mathrm{tr}[\mathbf{J}_{m_1,m_2,n}^{*}\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_{m_1,m_2,n}^{*'} \times \mathbf{J}_{m_1,m_2,n}^{*}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'}\mathbf{J}_{m_1,m_2,n}^{*'}] \ ,
\end{aligned}
$$

where $\mathbf{J}_{m_1,m_2,n} = (\mathbf{O}, \mathbf{I}_{m_2}, \mathbf{O})$ $(m_2 \times n$ matrix) and an $(n - m_2) \times n$ matrix

$$\mathbf{J}_{m_1,m_2,n}^{*} = \begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{n-m_1-m_2} \end{bmatrix} .$$

(iv) **Non-linear Filtering** : The method of filtering procedure based on (4.1) and (4.2) is simple, but very general. There can be several extensions and a combination of trend filtering and seasonal filtering is such an example, which can be written as (4.3) for instance. There can be possible extensions of the linear filtering procedures to non-linear filtering procedures based on (4.1) and (4.2) and other filtering methods in the existing literature can be interpreted as special cases of (4.3).

4.2 An Extended Errors-in-Variables Model

Now we will consider the meaning of the SIML filtering. For this purpose we extend the basic model of (3.1) by adding seasonal components and investigate the additive decomposition model

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{s}_i + \mathbf{v}_i \ \ (i = 1, \cdots, n), \tag{4.12}$$

where we take positive integers s $(s > 1)$, N, and $n = sN$ for the resulting simplicity and arguments and $\mathbf{s}_i$ $(i = 1, \cdots, n)$ are a sequence of non-stationary process which satisfy

$$\Delta \mathbf{s}_i = (1 - \mathcal{L})\mathbf{s}_i = \mathbf{v}_i^{(s)} \ , \tag{4.13}$$

where

$$\mathbf{v}_i^{(s)} = \sum_{j=0}^{\infty} \mathbf{C}_{sj}^{(s)} \mathbf{e}_{i-sj}^{(s)} \ , \tag{4.14}$$

and $\mathbf{e}_i^{(s)}$ is a a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(s)}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{e}_i^{(s)}\mathbf{e}_i^{(s)\prime}) = \mathbf{\Sigma}_e^{(s)}$ (non-negative definite).

Let $\mathbf{f}_{\Delta x}(\mu)$, $\mathbf{f}_{\Delta s}(\mu)$, and $\mathbf{f}_v(\mu)$ be the spectral density $(p \times p)$ matrices of $\Delta \mathbf{x}_i$, $\Delta \mathbf{s}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ as

$$\mathbf{f}_{\Delta x}(\mu) = (\sum_{j=0}^{\infty} \mathbf{C}_j^{(x)} e^{2\pi i \mu j}) \mathbf{\Sigma}_e^{(x)} (\sum_{j=0}^{\infty} \mathbf{C}_j^{(x)\prime} e^{-2\pi i \mu j}) \ \ (-\frac{1}{2} \le \mu \le \frac{1}{2}) \ , \tag{4.15}$$

$$\mathbf{f}_{\Delta s}(\mu) = (\sum_{j=0}^{\infty} \mathbf{C}_{sj}^{(s)} e^{2\pi i \mu s j}) \mathbf{\Sigma}_e^{(s)} (\sum_{j=0}^{\infty} \mathbf{C}_{sj}^{(s)\prime} e^{-2\pi i \mu s j}) \ \ (-\frac{1}{2} \le \mu \le \frac{1}{2}) \ , \tag{4.16}$$

and

$$\mathbf{f}_{v}(\mu) = (\sum_{j=0}^{\infty} \mathbf{C}_j^{(v)} e^{2\pi i \mu j}) \mathbf{\Sigma}_e^{(v)} (\sum_{j=0}^{\infty} \mathbf{C}_j^{(v)\prime} e^{-2\pi i \mu j}) \ \ (-\frac{1}{2} \le \mu \le \frac{1}{2}) \tag{4.17}$$

where we set $\mathbf{C}_0^{(x)} = \mathbf{C}_0^{(s)} = \mathbf{C}_0^{(v)} = \mathbf{I}_p$ as normalizations and $i^2 = -1$ (see Chapter 7 of Anderson (1971) for instance.)

Then the $p \times p$ spectral density matrix of the transformed vector process, which are observable, the spectral density of the difference series $\Delta \mathbf{y}_i$ $(= \mathbf{y}_i - \mathbf{y}_{i-1})$ can be represented as

$$\mathbf{f}_{\Delta y}(\mu) = \mathbf{f}_{\Delta x}(\mu) + \mathbf{f}_{\Delta s}(\mu) + (1 - e^{2\pi i \mu}) f_v(\mu)(1 - e^{-2\pi i \mu}) \ \ . \tag{4.18}$$

We denote the long-run variance-covariance matrices of the trend components and the stationary components for $g, h = 1, \cdots, p$ as

$$\mathbf{\Omega}_x = \mathbf{f}_{\Delta x}(0)\ (= (\omega_{gh}^{(x)}))\ ,\ \mathbf{\Omega}_s = \mathbf{f}_{\Delta s}(\frac{1}{s})\ (= (\omega_{gh}^{(s)}))\ , \tag{4.19}$$

and

$$\mathbf{\Omega}_v = f_v(0)\ = (\omega_{gh}^{(v)})\ . \tag{4.20}$$

5. A Statistical Foundation

At the first glance, the SIML filtering procedure might be seen as an *ad-hoc* statistical procedure without any mathematical foundation. However, on the contrary, there is a rather solid statistical foundation.
Let $\theta_{jk} = \frac{2\pi}{2n+1}(j - \frac{1}{2})(k - \frac{1}{2})$,

$$p_{jk}^{(n)} = \frac{1}{\sqrt{2n+1}}(e^{i\theta_{jk}} + e^{-i\theta_{jk}})$$

and we write

$$\Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)}) = \sum_{j=1}^{n} p_{jk}^{(n)} \mathbf{r}_j^{(n)}\ ,\ \mathbf{r}_j^{(n)} = \mathbf{y}_j - \mathbf{y}_{j-1}\ , \tag{5.1}$$

which is actually (the real-valued) Fourier-transformation. Then $\Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)})$ $(k = 1, \cdots, n)$ are the (real-valued) Fourier-transformation of data at the frequency $\lambda_k^{(n)}$ $(= (k-1/2)/(2n+1))$, which is a (real-part of) estimate of the orthogonal incremental process $\mathbf{z}(\lambda)$.

For the development of statistical inferences, we have the next result by using the CLT (central limit theorem) for discrete and (ergodic) stationary time series. See the Appendix for the derivations.

Theorem 5.1 : Let $\mathbf{r}_j$ $(j = 1, \cdots, n)$ be an ergodic stationary stochastic process with $\mathbf{\Gamma}(h) = \mathcal{E}(\mathbf{r}_j \mathbf{r}_{j-h}^{'})$ and

$$\sum_{h=0}^{\infty} \|\mathbf{\Gamma}(h)\| < \infty\ \ . \tag{5.2}$$

(i) Let $\Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)}) = \sum_{j=1}^{n} p_{jk}^{(n)} \mathbf{r}_j^{(n)}$ and $\mathbf{r}_j^{(n)}$ be an ergodic stationary sequence with $\mathcal{E}[\mathbf{r}_j] = \mathbf{0}$ and the (symmetrized real) spectral density matrix

$$\mathbf{f}_{SR}(\lambda) = \mathbf{\Gamma}(0) + \sum_{h=1}^{\infty} \cos(2\pi h\lambda)[\mathbf{\Gamma}(h) + \mathbf{\Gamma}(-h)]\ , \tag{5.3}$$

is the positive definite and bounded (real-valued and symmetrized) spectral matrix. Also assume that $\lambda_k^{(n)} \to s$, $\lambda_{k'}^{(n)} \to t$ and $0 < s < t < \frac{1}{2}$.
Then as $n \longrightarrow \infty$

$$\left[\begin{array}{c} \Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)}) \\ \Delta_\lambda \mathbf{z}^{(n)}(\lambda_{k'}^{(n)}) \end{array} \right] \xrightarrow{w} N_{2p} \left[\mathbf{0}, \left[\begin{array}{cc} \mathbf{f}_{SR}(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_{SR}(t) \end{array} \right] \right] . \tag{5.4}$$

(ii) Let $\mathbf{Z}_n(t) - \mathbf{Z}_n(s) = \frac{1}{\sqrt{n}} \sum_{k=[sn]}^{[tn]} \sum_{j=1}^{n} p_{jk}^{(n)} \mathbf{r}_j^{(n)}$ for $0 < s < t < 1$. Then as $n \longrightarrow \infty$

$$\mathbf{Z}_n(t) - \mathbf{Z}_n(s) \xrightarrow{w} N_p \left[\mathbf{0}, F_{SR}(t) - F_{SR}(s) \right] , \tag{5.5}$$

where $F_{SR}(t)$ is the $p \times p$ (symetrized real) spectral distribution matrix

$$F_{SR}(t) = \int_0^t f_{SR}(\lambda) d\lambda . \tag{5.6}$$

This theorem covers the basic model with (3.1)-(3.4) and the extended model with (4.12)-(4.14) with the moment conditions because $\Delta \mathbf{y}_i$ are stationary in these cases. The spectral density in the basic model is given by (3.7) while the spectral density in the extended model is given by (4.18). We immediately find that the long-run variance-covariance matrix of hidden states in the basic model can be estimated by (3.17).

Since the asymptotic variance-covariance matrix of the orthogonal random vectors $\Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)})$ is the (symmetrized real) spectral density matrix, it should be estimated consistently. Although there can be many ways to solve the problem, it may be natural to use the estimation method as (3.17) at the zero frequency.
The SIML estimator of $\mathbf{f}_{SR}(t)$ $(0 < \frac{1}{2})$ can be defined by

$$\mathbf{G}_m(t) = \frac{1}{m_n} \sum_{k=[2nt]-[\frac{m_n}{2}]+1}^{[2nt]+[\frac{m_n}{2}]} (\Delta_t \mathbf{z}_k^{(n)}(\lambda_k^{(n)}))(\Delta_t \mathbf{z}_k^{(n)}(\lambda_k^{(n)}))' . \tag{5.7}$$

Then we have the next result.

Theorem 5.2 : Let $\mathbf{r}_j$ $(j = 1, \cdots, n)$ be an ergodic stationary stochastic process with $\mathbf{\Gamma}(h) = \mathcal{E}(\mathbf{r}_j \mathbf{r}_{j-h}^{'})$ with (5.2) and (5.3). Assume the fourth order moment conditions $\sup_{j \geq 1} [\|\mathbf{r}_j\|^4] < +\infty$. Assume (3.1)-(3.4) and in (5.7) we set $m_n = [n^\alpha]$ $(0 < \alpha < 1)$. Then for any $t \in (0, \frac{1}{2})$, as $n \longrightarrow \infty$

$$\mathbf{G}_m(t) \xrightarrow{p} \mathbf{f}_{SR}(t) \ . \tag{5.8}$$

Furthermore, it is possible to show the asymptotic normality of the SIML estimator under the condition that $0 < \alpha < 0.8$. (Kunitomo and Sato (2017) have discussed the case of $t = 0$.) Furthermore, it may be possible to develop the consistent estimation of spectral components of non-stationary time series in the form of (4.7).

In the traditional statistical time series analysis for a stationary discrete (vector) process $\mathbf{r}_k^*$ with the spectral distribution F, it has been known that there exists a right-continuous orthogonal increment (vector, complex-valued) process $\mathbf{z}^*(\lambda)$ $(-1/2 \leq \lambda \leq 1/2)$ such that

$$\mathbf{r}_k^* = \int_{(-1/2,1/2]} e^{i2\pi k\nu} d\mathbf{z}^*(\nu) \; (k = 1, \cdots, n). \tag{5.9}$$

(The topic here goes back to Doob (1953), but see Hannan (1971) or Brockwell and Davis (1990).) The trend component and seasonal component of (real-valued) time series in our setting can be defined by

$$\mathbf{r}_k^{(u)} = \int_{(0,1/2]} \cos(2\pi k\nu) w(u,\nu) \mathbf{z}(\nu) \; (k = 1, \cdots, n) \tag{5.10}$$

for $u = x$ or $u = s$, where $w(u,\nu)$ is the indicator function of some frequencies around zero (for trend) and seasonal frequency (for seasonality), respectively, and $\mathbf{z}(\nu)$ $0 < \nu \leq 1/2$ is the right-continuous orthogonal increment (real-valued) process, which is the limiting continuous process of discrete time series.

Since $\mathbf{z}(\nu)$ is not observed with finite data, the (real-valued) estimate of $\mathbf{r}_k^{(u)}$ (i.e. the hidden components of $\mathbf{r}_k$) from data can be represented as

$$\mathbf{r}_k^{(u,n)} = \int_{(0,1/2]} \cos(2\pi k\nu) w_n(u,\nu) d\mathbf{z}^{(n)}(\nu) \; (\frac{k-1}{2n} < \nu \leq \frac{k}{2n}, k = 1, \cdots, n), \tag{5.11}$$

where $w_n(u,\nu)$ is a measurable function and $\mathbf{z}^{(n)}(\nu)$ is the estimated orthogonal process at the frequency $\nu \in [0, 1/2]$ from data where we have abused some notations such as $\mathbf{z}^{(n)}(0) = \mathbf{0}$. Hence the l_n-th component of (4.1) with (4.2) and (4.3) could be approximated and written as

$$\hat{\mathbf{x}}_{l_n}^{(u,n)} = \sum_{k=1}^{l_n} \left[\int_{(0,1/2]} \cos(2\pi k\nu) w_n(u,\nu) \, d\mathbf{z}^{(n)}(\nu) \right], \tag{5.12}$$

when we take $l_n = [t\, n]$ $(0 < t < 1)$ and choose the kernel function $w_(u,\nu)$ appropriately in the frequency domain. Hence these representations could be interpreted as the (real) Fourier inverse of $\mathbf{K}_n-$ transformation of finite observations of time series in (5.1).

From this interpretation, we find that there may be an interesting representation problem of (discrete time and continuous time) stationary processes and orthogonal incremental stochastic processes, which is closely related to the method of data analysis we are investigating.

6 Prediction and Model Selection

6.1 Prediction

The prediction problem can be solved as follows. For the simplicity, we consider the $h-$period ahead prediction of $\mathbf{y}_j$ $(j = n+h, h \geq 1)$ given the information available at $j = n$. It is natural to use the incremental vectors $\mathbf{r}_{n+s}^{(n)}$ $(s \geq 1)$ available at $j = n$. Then by using the orthogonal processes and ignoring the sampling errors of estimating them (that is n should be large enough), we have an expression of the prediction error as

$$\mathbf{e}(h) = \sum_{j=1}^{h} \mathbf{r}_j = \int_{-1/2}^{1/2} [e^{-i(n+1)\nu} + \cdots + e^{-i(n+h)\nu}] w(u,\nu) \tilde{\mathbf{z}}(d\nu) \ . \tag{6.13}$$

Hence the prediction MSE is given as

$$\mathcal{E}[\mathbf{e}(h)\mathbf{e}^{'}(h)] = \int_{-1/2}^{1/2} \left[\frac{\sin \frac{h\nu}{s}}{\sin \frac{\nu}{2}} \right]^2 w(u,\nu) \mathbf{f}(\nu) d\nu \ , \tag{6.14}$$

where $\mathbf{f}(\nu)$ is the $p \times p$ spectral density matrix for $\mathbf{r}_j$. When $h = 1$ and $w(u,\nu) = 1$ for $\nu \in (-1/2, 1/2)$, we have the familiar expression of one-period prediction MSE in time series analysis as

$$\mathcal{E}[\mathbf{e}(1)\mathbf{e}^{'}(1)] = \int_{-1/2}^{1/2} \mathbf{f}(\nu) d\nu = \mathcal{E}[\mathbf{r}_{n+1}\mathbf{r}_{n+1}^{'}] \ , \tag{6.15}$$

which corresponds the variance-covariance matrix of innovation vector at $j = n$. When $p = 1$, we find that

$$\int_{-1/2}^{1/2} \mathbf{f}(\nu) d\nu \geq \exp\left[\int_{-1/2}^{1/2} \log[\mathbf{f}(\nu)] d\nu \right] \ , \tag{6.16}$$

which is the lower bound of the one-step ahead prediction. (See Chapter 3 of Hannan (1070).) Therefore the prediction MSE of our predictor is slightly greater than the lower bond of prediction MSE. When the spectral density is flat, two prediction MSE are the same.

6.2 Model Selection

When we have estimates of the state variables $\mathbf{x}_i$ $(i = 1, \cdots, n)$, the estimates of noise components are $\hat{\mathbf{v}}_i = \mathbf{y}_i - \hat{\mathbf{x}}_i$ $(i = 1, \ldots, n)$. Then an estimated MSE of the one-step ahead prediction errors based on the SIML-smoothing or filtering is given by

$$\text{(6.17)} \qquad \text{PMSE}_n(h) = \mathcal{E}[(\mathbf{y}_{n+h} - \mathbf{y}_n)(\mathbf{y}_{n+h} - \mathbf{y}_n)^{'}|\mathcal{F}_n] \ ,$$

where $\mathcal{F}_n$ is the $\sigma-$field (information) available at n.
Then one may try to minimize the estimated h-step prediction MSE by choosing an appropriate m. It may be reasonable to choose $h = 2, 3$ for the estimation of trend while $h = 4, 8$ for the estimation of seasonal from our limited experiments.

6.3 Numerical Experiments

We have done several Monte Carlo experiments on the problem of choosing an appropriate m. For this purpose, we first use the simple model of trend plus noise model, $x_i = x_{i-1} + u_t$ and $y_i = x_i + v_i$ $(i = 1, \cdots, n)$. The criterion function is the prediction MSE given by

$$\text{(6.18)} \qquad \text{PMSE}_n^*(h) = \frac{1}{h} \sum_{i=n-h+1}^{n} (y_i - \hat{x}_i)^2 \ .$$

We give some results on the trend filtering as Tables 6.1-6.3 by taking $h = 2, \cdots, 8$ and $n = 80, 120, 200, 300$. In our simulations, as n increases, we have larger optimal choice of m. Also as h increases, the optimal choice m decreases. When we have long-horizon with h, it may be natural to use small number of lower frequencies. σ_x and σ_v, the optimal choice of m could be stable.

Table 6.1 : Optimal Choice of m
($\sigma_x = 0.3$, $\sigma_v = 0.05$)

n	80	120	200	400
h=2	12	19	32	65
h=3	8	12	21	42
h=4	6	9	15	32
h=5	5	7	13	26
h=6	4	6	10	21
h=7	4	5	9	18
h=8	3	4	8	16

Table 6.2 : Optimal Choice of m
($\sigma_x = 0.3$, $\sigma_v = 0.4$)

n	80	120	200	400
h=2	13	19	33	66
h=3	8	13	21	43
h=4	6	9	16	32
h=5	5	8	13	26
h=6	4	6	11	21
h=7	4	5	9	18
h=8	3	5	8	16

Table 6.3 : Optimal Choice of m
($\sigma_x = 0.3$, $\sigma_v = 1.0$)

n	80	120	200	400
h=2	13	20	33	67
h=3	9	13	22	44
h=4	6	10	16	33
h=5	5	8	13	26
h=6	4	7	11	22
h=7	4	6	9	19
h=8	4	5	8	16

As the second example, we use the simple model of seasonal-plus-noise model. $x_i = x_{i-1} + u_t$, $s_i = x_i \times SA_i$, $SA_i = SA_{i-4}, SA_j = U(-0.75, 0.75)$ $(j = 1, 2, 3), SA_4 = -(SA_1 + SA_2 + SA_3)$ and $y_i = s_i + v_i$ $(i = 1, \cdots, n)$. We have used the filters on $[n/2 - m_1^*, n/2 + m_1^*]$ and $[n - m_2^*, n]$. (The notations are slightly different from the ones in Section 5.) The criterion function is the prediction MSE given by

$$\text{MSE}_n = \sum_{i=n+1-h}^{n} (y_i - \hat{s}_i)^2 \ . \tag{6.19}$$

Table 6.4 : Optimal Choice of m_1^* and m_2^*
($\sigma_x = 0.001$, $\sigma_v = 0.01$)

n	$80(m_1^*, m_2^*)$	$120(m_1^*, m_2^*)$	$240(m_1^*, m_2^*)$
h=4	14,17	22,20	34,22
h=8	11,19	17,20	27,21
h=12	16,18	20,21	24,21

Table 6.5 : Optimal Choice of m_1^* and m_2^*
($\sigma_x = 0.001$, $\sigma_v = 0.5$)

n	$80(m_1^*, m_2^*)$	$120(m_1^*, m_2^*)$	$240(m_1^*, m_2^*)$
h=4	6,6	10,9	20,17
h=8	4,3	5,5	10,10
h=12	4,3	4,3	7,7

Table 6.6 : Optimal Choice of m_1^* and m_2^*
($\sigma_x = 0.001$, $\sigma_v = 1.2$)

n	$80(m_1^*, m_2^*)$	$120(m_1^*, m_2^*)$	$240(m_1^*, m_2^*)$
h=4	6,5	10,9	20,17
h=8	4,3	5,4	10,4
h=12	4,2	4,3	6,6

We give some results on the seasonal filtering as Tables 6.4-6.6 by taking $h = 4, 8, 12$ and $n = 80, 120, 240$ because we investigate the non-stationary seasonal components. In our simulations, as n increases, we have larger optimal choice of m_1 and m_2, but they are not large in Tables 6.5 and 6.6. As h increases, the optimal choice m may be gradually decreasing.

7. An Application to Japanese Macro-consumption

We have applied our filtering method to the analysis of Japanese quarterly (real) consumption-GDP data as the first example and three monthly consumption data as the second example, which have been discussed in Section 2. All figures in this section are gathered in the Appendix B.

First, we calculate the transformation of the original quarterly consumption data, which show the non-stationarity and it may be a typical macro-economic variable. Then we calculate the realized $\mathbf{Z}_n$ as Figure 7.1 from the differenced consumption data. In this case the $\mathbf{Z}_n$ series gives a strange form mainly because the non-stationarity and seasonality. Since there are clear seasonal components in the original series, we calculated the realized $\mathbf{Z}_n$ (Figure 7.2) and the empirical cumulative distribution of Z_n^2 (Figure 7.3), which correspond to the normalized sample spectral distribution. Because we have quarterly macro-data, we have a large up and down around 50, which corresponds to the seasonal frequency at $s = 4$. The empirical spectral density has a abrupt change at this frequency. Since it has been a practice in time series data analysis to use seasonal differencing in the Box-Jenkins method, we calculated the realized $\mathbf{Z}_n$ (Figure 7.4) after seasonal differencing. Although the spectral contribution around the seasonal frequency, there are some rather wild fluctuations at many other frequencies. Because we have some difficulty to interpret the resulting time series, it may not be possible to justify the seasonal differencing procedure. In our analysis we simply use the differencing and then use the frequency domain analysis.

In Figure 7.5, we have investigated the analysis of real GDP. We have chosen $m = [n^{.99}]$ and delete the seasonal frequency around 48-52 and some high-frequency part. It means that we delete 5 data around the seasonal frequency and several high frequencies, which correspond to the aliasing effects frequency and this procedure was necessary to obtain stable empirical results. Then we compared the filtered time series by our method and the official (published) seasonally adjusted time series. We have found that the differences of these two time series are rather small and they are often of negligible magnitude. Although our filtering procedure is quite simple, this empirical example shows the usefulness of our method developed in the present study.

As the second example, we have analyzed three consumption (monthly) time series and the quarterly consumption time series. As we have seen in Section 2, three macro-consumption series have similarities and some differences. In our example, our goal is to construct the monthly consumption index, which is close to the observed quarterly consumption time series. Because there are non-stationary trend, seasonal and measurement errors, it may not be obvious to construct such consumption index by the existing statistical tool.
Let Y_i $(i = 1, \cdots, n)$ be the target (quarterly) time series and Z_{kt} $(k = 1, 2, 3; t = 3(i-1)+j, j = 0, 1, 2)$ be the k-th monthly time series. ($t = 0$ is the initial period.) Then the criterion function is

$$\text{MSE}(m, m_1, m_2, m_3, w_1, w_2, w_3) = \sum_{i=1}^{n} \left[\Delta\hat{Y}_i^{(T)} - \sum_{j=1}^{3} w_j \Delta Z_{ji}^{(T)} \right]^2 , \tag{7.1}$$

where $\Delta\hat{Y}_i^{(T)} = \hat{Y}_i^{(T)} - \hat{Y}_{i-1}^{(T)}$, (the trend part of the estimated ΔY_i because we observe the quarterly data on Y_i) $\Delta Z_{ji}^{(T)} = Z_{ji}^{(T)} - Z_{j,i-1}^{(T)}$ (the trend parts of ΔZ_{ji}), and w_j $(j = 1, 2, 3)$ are (unknown) weight coefficients and m, m_j $(j = 1, 2, 3)$ are the numbers of trend filtering. In the above formulation we need to measure the prediction errors based on differenced data because we have non-stationary trends and seasonal.

By using the least squares method, we minimized the MSE criterion with respect to the underlying parameters. The estimated w_j $(j = 1, 2, 3)$ are 3.69, 5.19 and 1.64 (while the measurement units are different), but their magnitude are about comparable to the published quarterly consumption level at 2002Q1), which are statistically significant with 1%. The optimal choice of $m = 29$ while $m_1 = 36, m_2 = 23$ and $m_3 = 33$. In our limited experiments, we have found some improvements of prediction errors by choosing different m and m_j $(j = 1, 2, 3)$.

The black curves are the original series and the red curves are estimated trend curves in Figures 7.6, 7.7 and 7.8. By taking relatively large m_j $(j = 1, 2, 3)$ we can recover the cycle components of each series, which are the key role to be used as the indicators of macro-business condition. In Figure 7.9, the green curve shows the predicted value calculated from the latest observed (quarterly) data plus the predicted monthly part based on the estimated parameters. Since there is no monthly observation of quarterly published consumption, we draw their latest (quarterly) level by the black curve and the estimated SIML (filtered) values by the red curve. One notable problem may be the introduction of consumption tax in 2014 April and there was a sharp deviation of trend mainly because the black curve is the quarterly observed macro-consumption. In the present study we did not have a focus on this event, but it could be handled by some additional complication such as using a dummy variable. Overall, we have found that while our procedure is relatively simple in comparison to the X-12-ARIMA seasonal adjustment with reg-ARIMA model, the predictive results are satisfactory. In Figure 7.10, we have drawn the prediction errors in terms of the differenced value Y_i $(i = 1, \cdots, n)$ based on our procedure. This figure illustrates the usefulness of the procedure because the macro-economic time series are non-stationary with measurement errors.

8. Concluding Remarks

When the observed non-stationary time series contain noises, it may be difficult to disentangle the effects of trends and the noises. For instance, in many macro-times series we observe non-stationary trend, non-stationary seasonality and stationary cycles with measurement errors. In this paper we investigate a new procedure to decompose of time series into non-stationary trend components, seasonal components and stationary noise (or measurement errors) components. One important conclusion is that it is useful to transform the observed time series by $\mathbf{K}_n-$transformation and investigate the transformed $\mathbf{Z}_n$ series, which is based on Kunitomo and Sato (2017), Kunitomo et al. (2018) and Nishimura et al. (2019). We can investigate the information of noisy-time series such as macro-economic variable by looking at their frequency and re-cover the traditional spectral distribution by the squared $\mathbf{Z}_n-$variables.

As an illustrative empirical example, we have used our filtering method proposed in this paper to analyze quarterly and monthly macro-consumption data in Japan.

We have applied our method to construct the monthly consumption index, which is consistent with the published or official (GDP-)consumption quarterly data. Although the problem is practically complicated, we have shown that our method gives a useful result for practical purposes.

There can be several interesting problems developed by our approach in this paper. Since it is easy to handle the $\mathbf{K}_n-$transformation and the transformed $\mathbf{Z}_n-$data, it may be straight-forward to develop a new way to determine the number of trend factors and seasonal factors. Since there are many important empirical applications such as the examples mentioned by Baxter and King (1999), and Müller and Watson (2018), we have currently investigating several empirical examples with trends and seasonality.

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APPENDIX A : Mathematical Derivations

In this Appendix, we give some details of derivations which we have omitted in the previous sections.

(i) **On (4.7) and (4.11)** :
Let $\theta_{jk} = \frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})$ $(j,k=1,\cdots,n)$. We use the relation that

$$\theta_{jk} + \theta_{j',k} = \frac{2\pi}{2n+1}(j+j^{'}-1)(k-\frac{1}{2}) \ , \ \theta_{jk} - \theta_{j',k} = \frac{2\pi}{2n+1}(j-j^{'})(k-\frac{1}{2}) \ .$$

Then we have

$$\text{(A.1)} \qquad 4\sum_{k\in I_n}[\cos\theta_{jk}\cos\theta_{j',k}]$$
$$= \sum_{k\in I_n}[e^{i(\theta_{jk}+\theta_{j',k})} + e^{-i(\theta_{jk}+\theta_{j',k})}] + \sum_{k\in I_n}[e^{i(\theta_{jk}-\theta_{j',k})} + e^{-i(\theta_{jk}-\theta_{j',k})}] \ ,$$

where $\mathbf{I}_n = [1,\cdots,m]$ (or $\mathbf{I}_n = [m_1+1,\cdots,m_1+m_2]$) is the index set for j and k. For $\mathbf{I}_n = [m_1+1,\cdots,m_1+m_2]$, by re-writing

$$\theta_{jk} + \theta_{j',k} = (m_1-\frac{1}{2})\frac{2\pi}{2n+1}(j+j^{'}-1) + \frac{2\pi}{2n+1}(j+j^{'}-1)(k-m_1) \ ,$$

and

$$\theta_{jk} - \theta_{j',k} = (m_1-\frac{1}{2})\frac{2\pi}{2n+1}(j-j^{'}) + \frac{2\pi}{2n+1}(j-j^{'})(k-m_1) \ ,$$

the summation of the first two terms becomes

$$e^{i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j+j^{'}-1)} \times \frac{1-e^{i\frac{2\pi}{2n+1}(j+j^{'}-1)m_2}}{1-e^{i\frac{2\pi}{2n+1}(j+j^{'}-1)}} + e^{-i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j+j^{'}-1)} \times \frac{1-e^{-i\frac{2\pi}{2n+1}(j+j^{'}-1)m_2}}{1-e^{-i\frac{2\pi}{2n+1}(j+j^{'}-1)}} \ .$$

For the last two terms, we need to evaluate each terms when (i) $j=j^{'}$ and (ii) $j \neq j^{'}$, separately. By using the similar calculations, when $j \neq j^{'}$ the summation of last two terms becomes

$$e^{i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j-j^{'})} \times \frac{1-e^{i\frac{2\pi}{2n+1}(j-j^{'})m_2}}{1-e^{i\frac{2\pi}{2n+1}(j-j^{'})}} + e^{-i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j-j^{'})} \times \frac{1-e^{-i\frac{2\pi}{2n+1}(j-j^{'})m_2}}{1-e^{-i\frac{2\pi}{2n+1}(j-j^{'})}} \ .$$

When $j=j^{'}$, $\theta_{jk}-\theta_{j',k}=0$ and the summation of last two terms become m_2. Hence by using the relation

$$e^{i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j+j^{'}-1)} \times \frac{1-e^{i\frac{2\pi}{2n+1}(j+j^{'}-1)m_2}}{1-e^{i\frac{2\pi}{2n+1}(j+j^{'}-1)}} + e^{-i(m_1+\frac{1}{2})\frac{2\pi}{2n+1}(j+j^{'}-1)} \times \frac{1-e^{-i\frac{2\pi}{2n+1}(j+j^{'}-1)m_2}}{1-e^{-i\frac{2\pi}{2n+1}(j+j^{'}-1)}} .$$

$$= \frac{e^{i\frac{2\pi}{2n+1}\frac{1}{2}(j+j^{'}-1)(m_1)} - e^{i\frac{2\pi}{2n+1}\frac{1}{2}(j+j^{'}-1)(m_1+m_2)}}{e^{i\frac{2\pi}{2n+1}(-\frac{1}{2})(j+j^{'}-1)} - e^{i\frac{2\pi}{2n+1}(\frac{1}{2})(j+j^{'}-1)}} + \frac{e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(j+j^{'}-1)(m_1)} - e^{-i\frac{2\pi}{2n+1}\frac{1}{2}(j+j^{'}-1)(m_1+m_2)}}{e^{-i\frac{2\pi}{2n+1}(-\frac{1}{2})(j+j^{'}-1)} - e^{-i\frac{2\pi}{2n+1}(\frac{1}{2})(j+j^{'}-1)}}$$

and the corresponding results for $j-j^{'}$ (there are two cases when (a)$j=j^{'}$ and (b)$j \neq j^{'}$), we have the result.

(ii) **On (4.8)** : When $\mathbf{Y}_n - \bar{\mathbf{Y}}_0 = \mathbf{X}_n + \mathbf{V}_n$, we re-write

(A.2)
$$\begin{aligned} \hat{\mathbf{X}}_n - \mathbf{X}_n &= \mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{X}_n+\mathbf{V}_n) - \mathbf{C}_n\mathbf{P}_n^{'}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{X}_n \\ &= \mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{V}_n + \mathbf{C}_n\mathbf{P}_n^{'}[\mathbf{I}_n - \mathbf{J}_m^{'}\mathbf{J}_m]\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{X}_n . \end{aligned}$$

In the simple case, we have

(A.3)
$$\begin{aligned} & \mathcal{E}[\mathrm{tr}(\hat{\mathbf{X}}_n - \mathbf{X}_n)^{'}\mathbf{H}^{'}\mathbf{H}(\hat{\mathbf{X}}_n - \mathbf{X}_n)] \\ =\; & \mathrm{tr}\mathbf{H}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}[\mathrm{tr}(\mathbf{\Sigma}_v)\mathbf{I}_n]\mathbf{C}_n^{-1'}\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{H}^{'} \\ & +\mathrm{tr}\mathbf{H}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{*'}\mathbf{J}_m^{*}\mathbf{P}_n\mathbf{C}_n^{-1}[\mathrm{tr}(\mathbf{\Sigma}_x)\mathbf{I}_n]\mathbf{C}_n^{-1'}\mathbf{P}_n^{'}\mathbf{J}_m^{*'}\mathbf{J}_m^{*}\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{H}^{'} \\ =\; & \mathrm{tr}(\mathbf{\Sigma}_v)\mathrm{tr}\mathbf{H}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{-1'}\mathbf{P}_n^{'}\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{H}^{'} \\ & +\mathrm{tr}(\mathbf{\Sigma}_x)\mathrm{tr}\mathbf{H}\mathbf{C}_n\mathbf{P}_n^{'}\mathbf{J}_m^{*'}\mathbf{J}_m^{*}\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{-1'}\mathbf{P}_n^{'}\mathbf{J}_m^{*'}\mathbf{J}_m^{*}\mathbf{P}_n\mathbf{C}_n^{'}\mathbf{H}^{'} , \end{aligned}$$

where $\mathbf{J}_m^{*} = (\mathbf{O}_m, \mathbf{I}_{n-m})$ and $\mathbf{I}_n = \mathbf{J}_m^{'}\mathbf{J}_m + \mathbf{J}_m^{*'}\mathbf{J}_m^{*}$.

(iii) **On Theorems 5.1 and 5.2** : Basically, we apply CLT (Theorem 7.6 of Durrett (1991) for instance) on the sequence of ergodic stationary (discrete) time series Δy_i and we give an outline of our derivations. Hence we need to confirm that the resulting variance-covariance terms correspond to those of the limiting Gaussian random variables.
For this purpose, we evaluate
(A.4)

$$\mathcal{E}\left[\Delta_\lambda\mathbf{z}^{(n)}(\lambda_k^{(n)})\Delta_\lambda\mathbf{z}^{(n)}(\lambda_k^{(n)})\right] = \left[\frac{1}{2n+1}\right]\sum_{j,j^{'}=1}^{n}(e^{i\theta_{jk}}+e^{-i\theta_{jk}})(e^{i\theta_{j^{'}k^{'}}}+e^{-i\theta_{j^{'}k^{'}}})\mathcal{E}[\mathbf{r}_j\mathbf{r}_{j^{'}}^{'}] .$$

When $k \neq k^{'}$, it can be shown that the right-hand side terms are bounded by using the straight-forward and tedious calculations. When $k = k^{'}$, the right-hand side consists of four terms which are associated with

$$
\begin{aligned}
(e^{i\theta_{jk}} + e^{-i\theta_{jk}})(e^{i\theta_{j'k}} + e^{-i\theta_{j'}}) &= e^{i(\theta_{jk}+\theta_{j'k}} + e^{-i(\theta_{jk}+\theta_{j'k})} + e^{i(\theta_{jk}-\theta_{j'k}} + e^{-i(\theta_{jk}-\theta_{j'k})} \\
&= (1) + (2) + (3) + (4) \ (\text{, say}) \ .
\end{aligned}
$$

The sums of each terms of (1) and (2) are bounded. Under the assumption of stationarity of $\mathbf{r}_j$, The dominant terms of (A.2) are (3) and (4), and then they are approximately to

(A.5)

$$
n\mathbf{\Gamma}(0) + n\cos 2\pi(\frac{k}{2n+1})[\Gamma(1)+\Gamma(-1)] + n\cos 2\pi(\frac{k}{2n+1})[\Gamma(2)+\Gamma(-2)] + \cdots \ .
$$

We take $k/(2n+1) \longrightarrow t \ (0 \leq t < 1/2)$. Then we have (5.3) and (5.4).
Next, we consider

(A.6)

$$
\frac{1}{m}\sum_{k\in I_m} \Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)})\Delta_\lambda \mathbf{z}^{(n)}(\lambda_k^{(n)}) = \frac{1}{m}\sum_{k\in I_m}\frac{1}{2n+1}\sum_{j,j'=1}^{n}(e^{i\theta_{jk}}+e^{-i\theta_{jk}})(e^{i\theta_{j'k'}}+e^{-i\theta_{j'k'}})\mathbf{r}_j\mathbf{r}_{j'}^{'} .
$$

There are four terms, but the first two terms are stochastically bounded. We find that

(A.7)

$$
\sum_{j,j'=1}^{n}\left[e^{i(\theta_{jk}-\theta_{j'k})} + e^{-i(\theta_{jk}-\theta_{j'k})}\right]\mathbf{r}_j\mathbf{r}_{j'}^{'} = \sum_s \left[2\cos 2\pi s(\frac{k-1/2}{2n+1})\right][\sum_{j'=1}^{n}\mathbf{r}_{s+j'}\mathbf{r}_{j'}^{'}] \ .
$$

Under the assumptions we have made, we find that for any s

(A.8)
$$
\frac{1}{n}\sum_{j'=1}^{n}\mathbf{r}_{s+j'}\mathbf{r}_{j'}^{'} \xrightarrow{p} \Gamma(s) \ .
$$

(See Chapter 8 of Anderson (1971) and Brockwell and Davis (1990) for instance.) Therefore, we can show that (A.4) devided by n is approximately as

(A.9)
$$
\mathbf{\Gamma}(0) + \cos 2\pi(\frac{k}{2n+1})[\Gamma(1)+\Gamma(-1)] + \cos 2\pi(\frac{k}{2n+1})[\Gamma(2)+\Gamma(-2)] + \cdots \ .
$$

By taking $k/(2n+1) \longrightarrow t \ (0 \leq t < 1/2)$ as $n \longrightarrow \infty$, we have (5.8). The rigorous arguments of derivations could be tedious, but they are straight-forward and we have omitted them.

APPENDIX B : Some Figures

In this Appendix B, we give figures used in Section 7. As we have explained in Section 2, all data are official data published by ESRI (Economic and Social Research Institute), Cabinet Office of Japan and Statistics Bureau, Ministry of Internal Affairs and Communications. They are available from the government official Web-cite : e.stat.

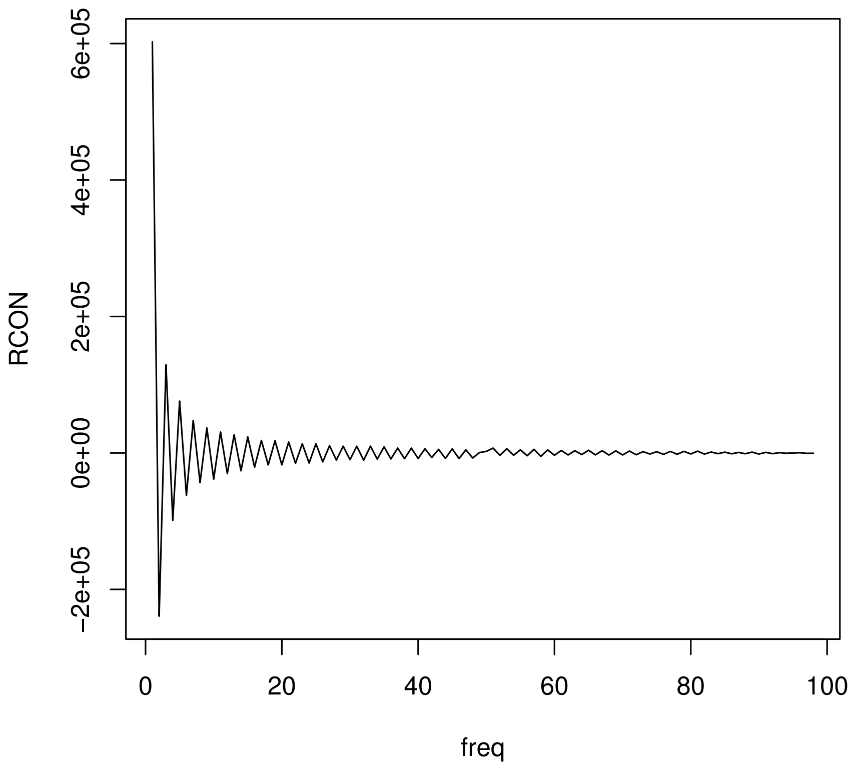

Figure 7.1 : Real-Consumption (original series)

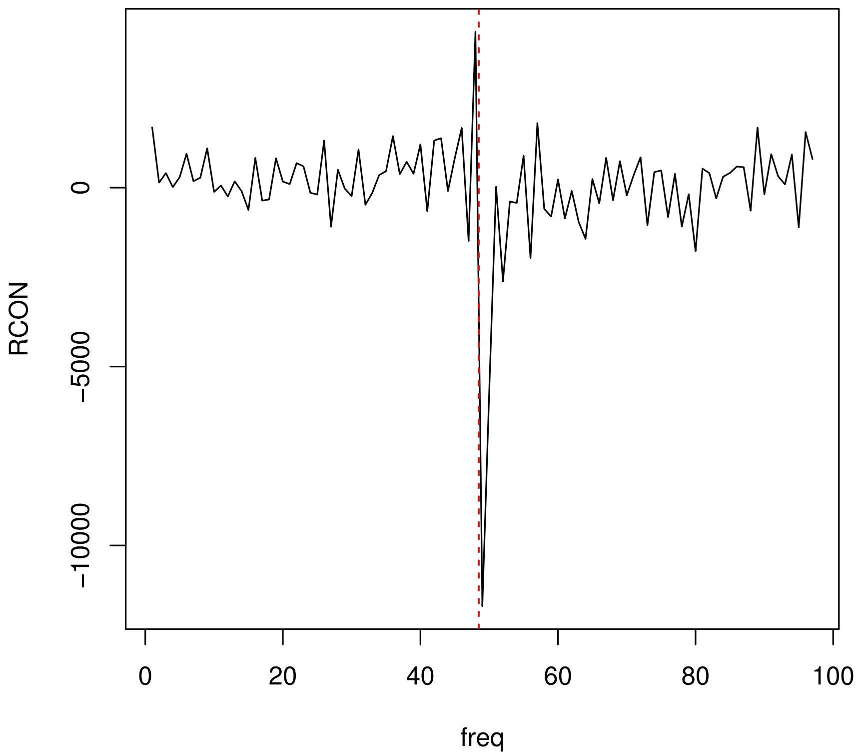

Figure 7.2 : Real-Consumption (differencing series)

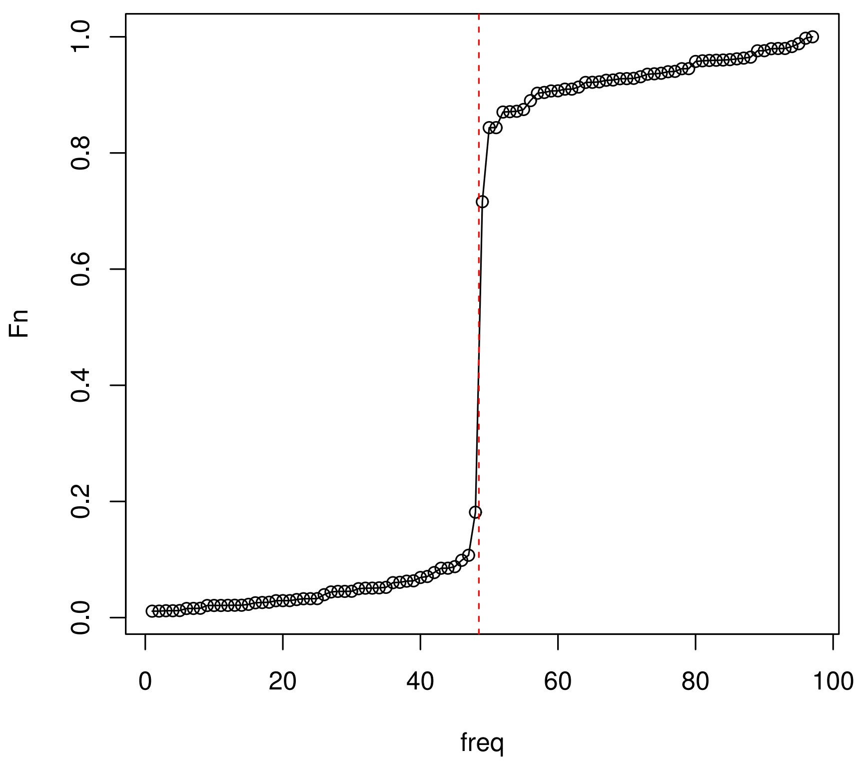

Figure 7.3 : Real-Consumption (empirical spectral distribution)

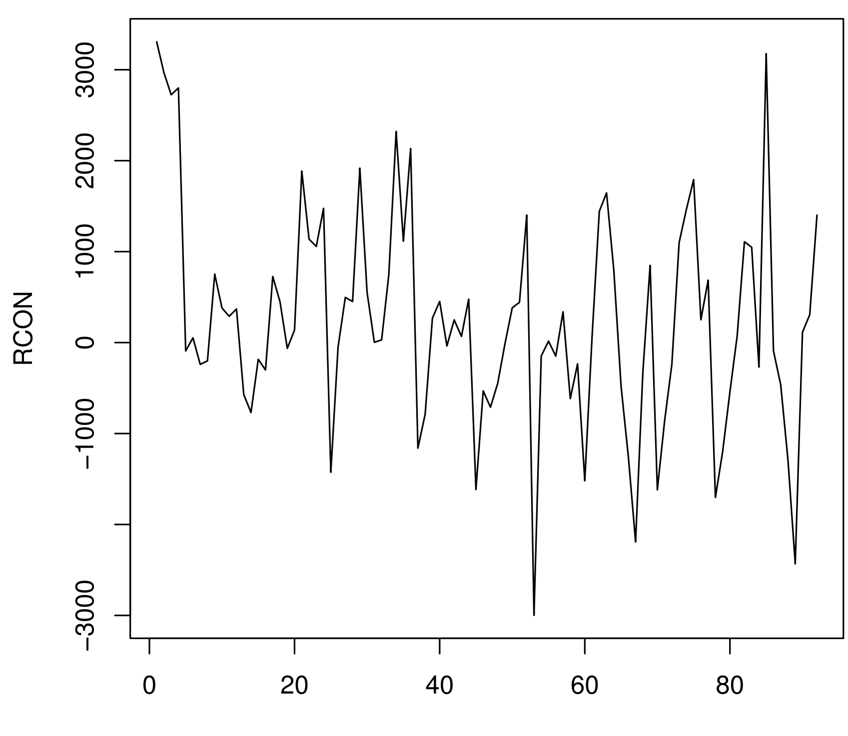

Figure 7.4 : Real-Consumption (seasonal differencing)

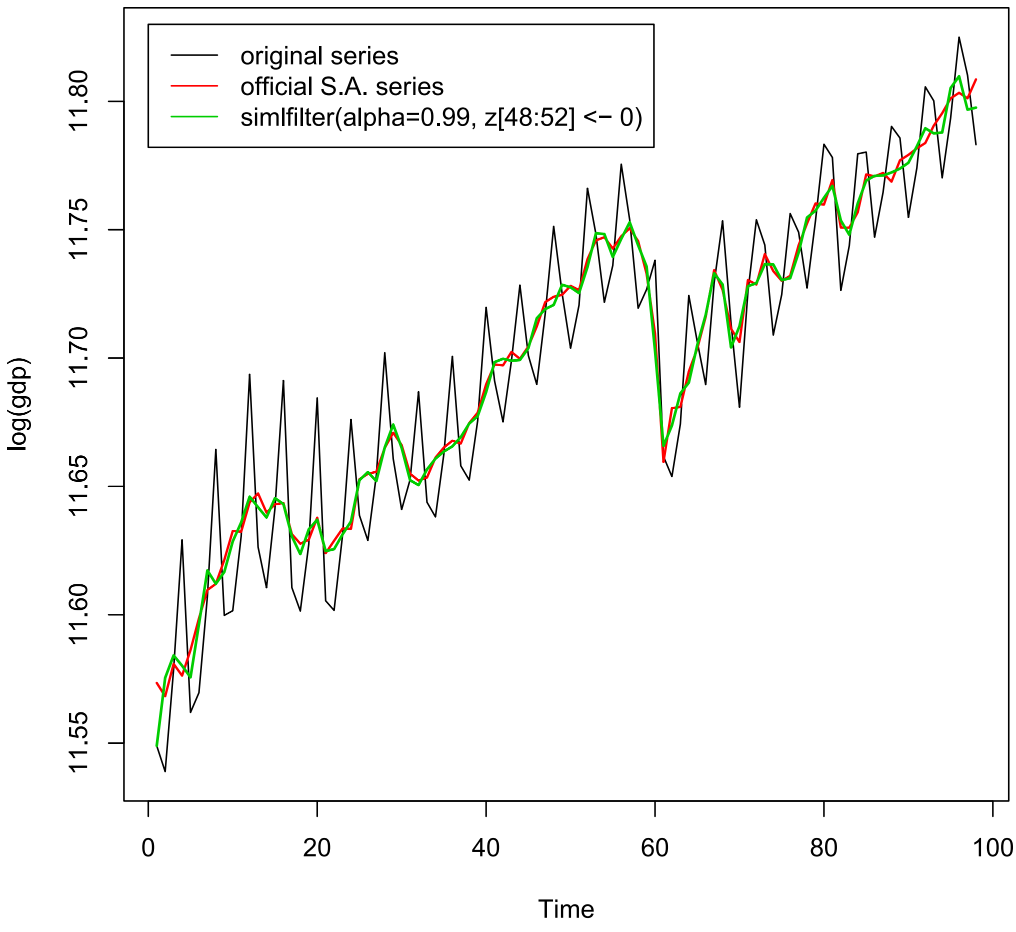

Figure 7.5 : Quarterly real-GDP

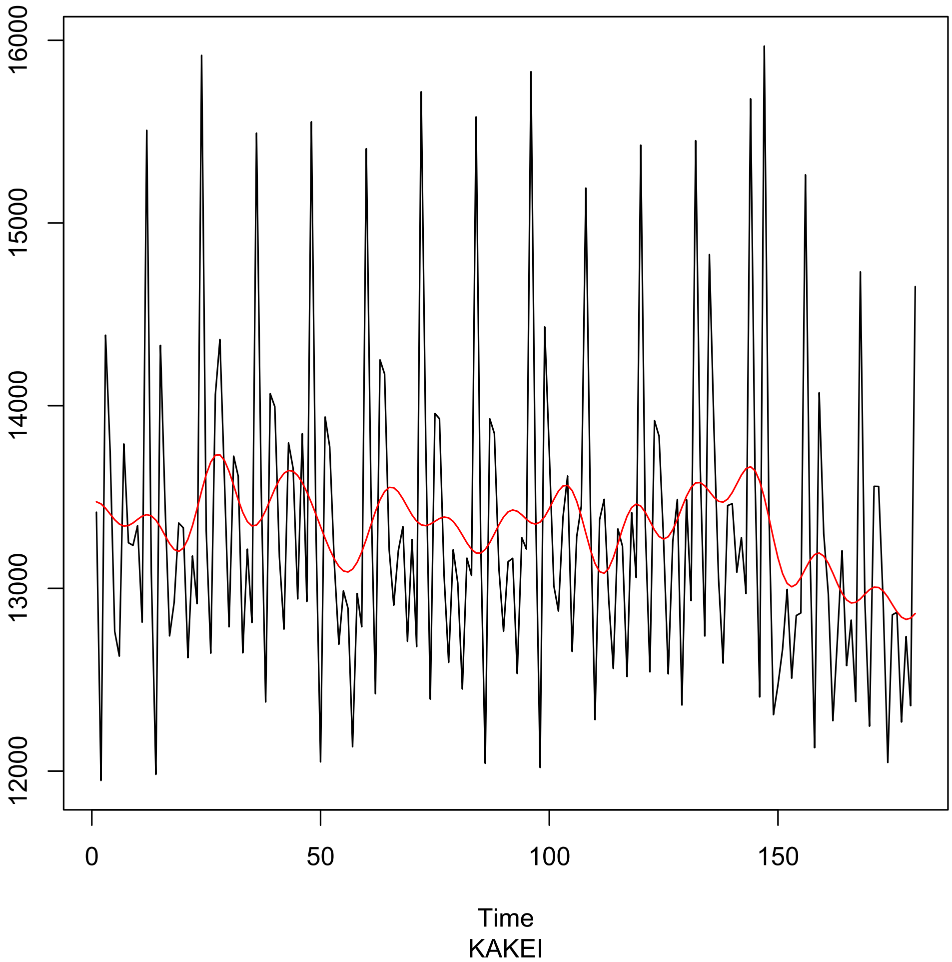

Figure 7.6 : Kakei-Chosa Series

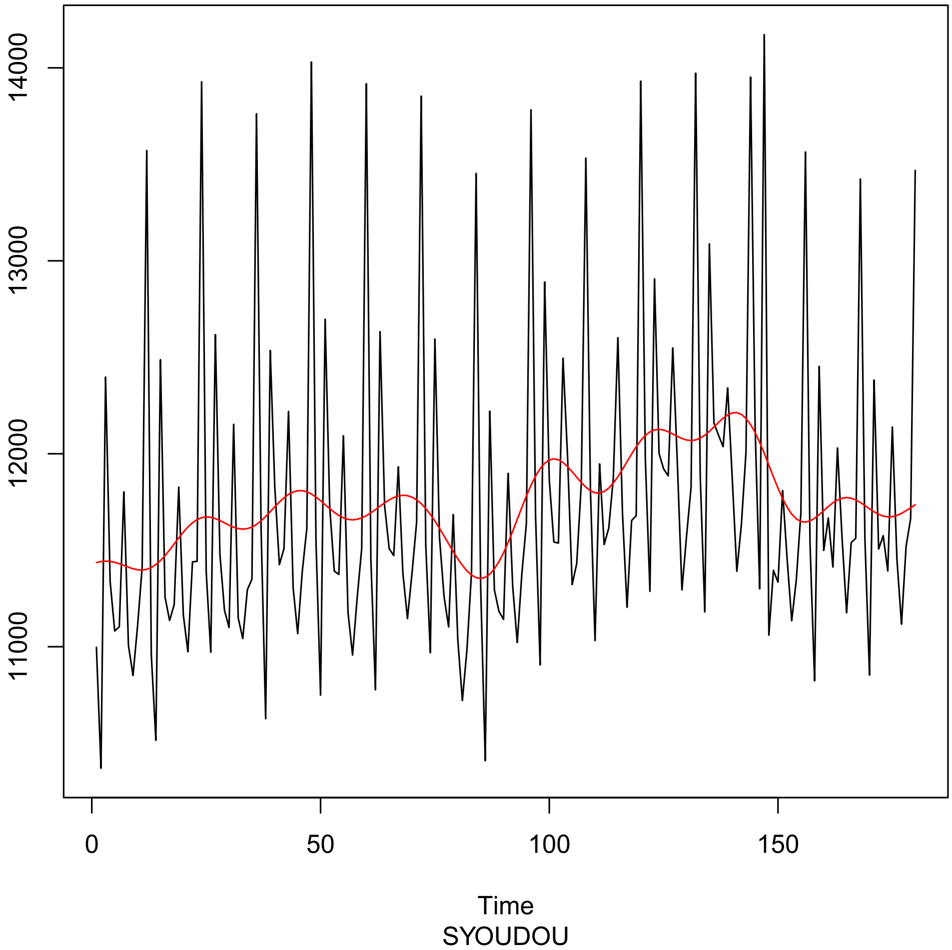

Figure 7.7 : Shoudou Series

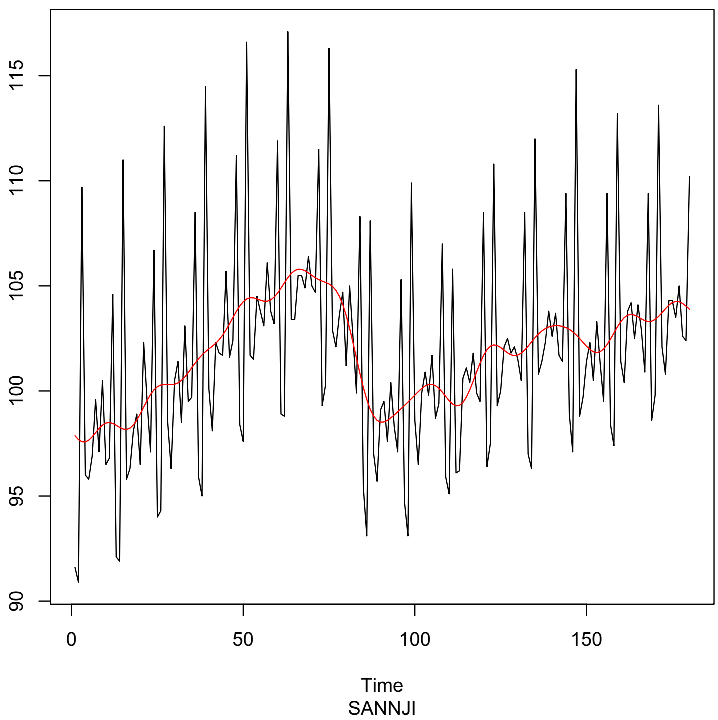

Figure 7.8 : Sanji Series

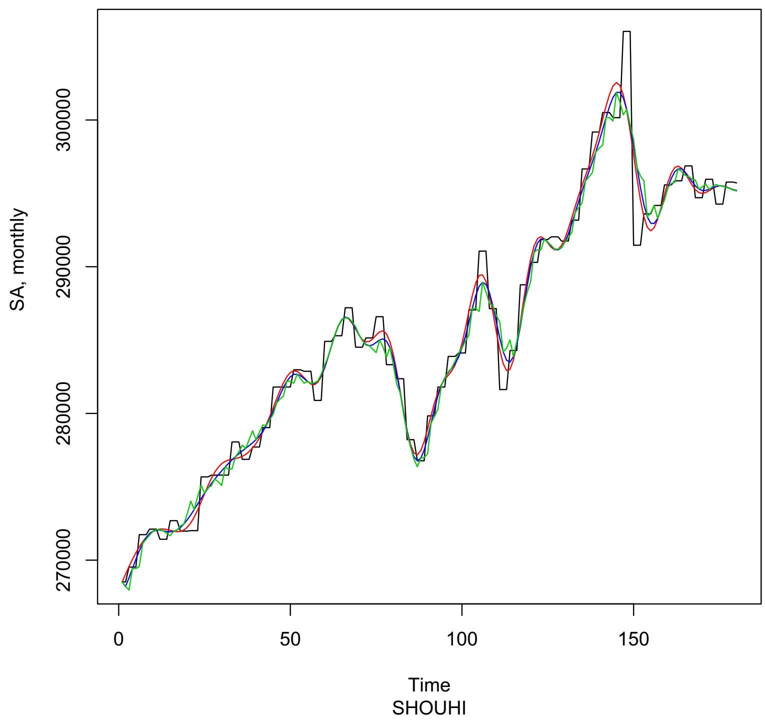

Figure 7.9 : Consumption Series

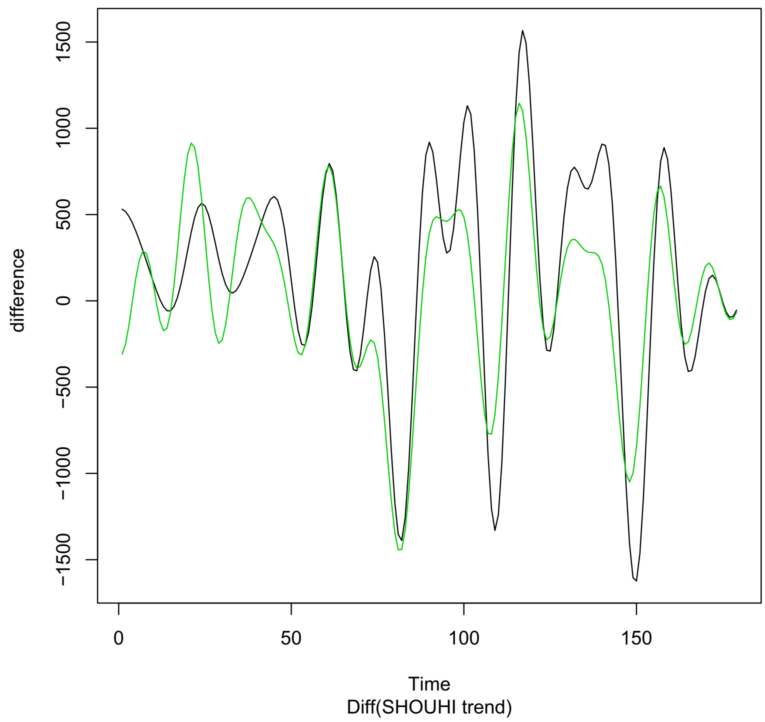

Figure 7.10 : Predicted and realized Consumption Series (in differencing)