SDS-10

Detecting Factors of Quadratic Variation in the Presence of Market Microstructure Noise

Naoto Kunitomo and Daisuke Kurisu

April 2019

Statistics & Data Science Series back numbers:
http://www.mims.meiji.ac.jp/publications/datascience.html

Detecting Factors of Quadratic Variation in the Presence of Market Microstructure Noise *

Naoto Kunitomo †
and
Daisuke Kurisu ‡

March 30, 2019

Abstract

We develop a new method of detecting hidden factors of Quadratic Variation (QV) of Itô semimartingales from a set of discrete observations when the market microstructure noise is present. We propose a statistical way to determine the number of factors of quadratic co-variations of asset prices based on the SIML (separating information maximum likelihood) method developed by Kunitomo, Sato and Kurisu (2018). In high-frequency financial data, it is important to disentangle the effects of the possible jumps and the market microstructure noise existed in financial markets. We explore the variance-covariance matrix of hidden returns of the underlying Itô semimartingales and investigate its characteristic roots and vectors of the estimated quadratic variation. We give some simulation results to see the finite sample properties of the proposed method and illustrate an empirical data analysis on the Tokyo stock market.

Key Words

Itô-Semimartingales, High-Frequency Financial Data, Market Microstructure Noise, Quadratic Variation, Hidden Factors, SIML Estimation, Characteristic Roots and Vectors, Limiting Distributions.

---

*2019-11. We thank Hiroumi Misaki (Tsukuba University) for giving useful information on high-frequency financial data at JXP. This is a preliminary memorandum. The research was supported by Grant-in-Aid for Scientific Research (B) (17H02513) from the JSPS, and Daisuke Kurisu is supported by Grant-in-Aid for Research Activity Start-up (18H05679) from the JSPS.

†School of Political Science and Economics, Meiji University, Kanda-Surugadai 1-1, Chiyoda-ku 101-8301, Tokyo, JAPAN, naoto.kunitomo@gmail.com

‡Tokyo Institute of Technology, School of Engineering, Meguro-ku, Ôokayama 2-12-1 W9-96, Tokyo 152-8552, JAPAN, kurisu.d.aa@m.titech.ac.jp

1. Introduction

In financial econometrics, several statistical methods have been proposed to estimate the integrated volatility and co-volatility from high-frequency data. The integrated volatility is one type of Brownian functionals and the realized volatility (RV) estimate has been often used when there does not exist any market microstructure noise and the underlying diffusion process is directly observed. It has been known that the RV estimator is quite sensitive to the presence of market microstructure noise in high-frequency financial data. Then several statistical methods have been proposed to estimate the integrated volatility and co-volatility. See Aït-Sahalia and Jacod (2014) for the detail of recent developments of financial econometrics. In particular, Malliavin and Mancino (2002, 2009) have developed the Fourier series method, which is related to the SIML (separating information maximum likelihood) estimation by Kunitomo, Sato and Kurisu (2018) used in this paper. See Mancino and Stanfelici (2008), and Mancino, Recchioni and Sanfelici (2017) on the recent development of the Fourier series method.

In this paper we develop a new statistical way of detecting hidden factors of Quadratic Variation of Itô-semimartingales from a set of discrete observations when the market microstructure noise is present. We will use the high-frequency asymptotic method such that the length of observation intervals becomes small as the number of observations grows, which has been often used in recent financial econometrics. In finance it is important to find several hidden factors among many financial prices such as stocks, bonds and other financial products. It might be a practice to find hidden factors after calculating various returns from price data and apply statistical tools such as principal component analysis, factor analysis and other statistical multivariate techniques. However, it should be noted that the standard statistical analysis has been developed to analyze independent (or stationary) observations and most financial prices are classified neither as independent nor stationary observations. In addition to this fact, it is important to notice that when we have market microstructure noises or measurement errors for prices, we have another statistical problem when we use high-frequency financial data. Although the multivariate statistical analyses such as principal components and factor models have been applied to financial data, these statistical methods do not necessarily give the right answers when we have market microstructure noise in high-frequency data. The standard statistical procedures could be a misleading way to analyze high-frequency financial data. There have been several attempts to find the structure of volatilities and the related issues. See Aït-Sahalia and Xiu (2017a, b), Fissler and Podolskij (2017), Jacod and Podolskij (2013), for instance. It seems that our approach is different from other methods and there are some merits as statistical method.

In this paper, we develop a new way to determine the number of factors of quadratic covariation or the integrated volatility of asset prices based on the SIML method, which was originally developed by Kunitomo et al. (2018). In high-frequency financial data it is important to disentangle the effects of the possible jumps and the market microstructure noise existed in financial markets. We explore the estimation problem of the variance-covariance matrix of the underlying Itô semimartingales, that is, the quadratic variation (QV). We shall show that it is possible to derive the asymptotic properties of the characteristic vectors and roots of the estimated QV, and then develop some test statistic for the rank condition. Our estimators of characteristic vectors and roots are consistent and they have the asymptotic normality. We develop some test statistics based on the characteristic roots and vectors to detect the number of factors of QV. We also give a real data analysis on the Tokyo stock market as an illustration.

In Section 2 we define the Itô semimartigale and Quadratic Variation, which is an extension of the integrated volatility with jump parts. Then we define the SIML estimation and its asymptotic property for Itô semimartingales. In Section 3, we consider the characteristic equations of the estimated hidden and conditional variance-covariance matrix] and give the theoretical results on the asymptotic properties of the associated characteristic roots and vectors when the true process is an Itô semimartingale and there are market microstructure noises. We also give some test statistics for the rank condition of the Quadratic Variation, which can be applied to detect the number of factors of integrated volatilities for the continuous diffusion case as a special case. In Section 4, we give some results on the Monte Carlo simulations of our procedures and in Section 5 we illustrate an empirical data analysis on the Tokyo stock market. Then in Section 6, we give concluding remarks. Some mathematical details are given in the Appendix.

2. Estimation of Quadratic Variation

We consider a continuous-time financial market in a fixed terminal time T and we set $T = 1$ without loss of generality. The underlying log-price is a p-dimensional Itô semimartingale, but we focus on the fact that we observe the log-price process in high-frequency financial prices and they are contaminated by the market microstructure noise. We define the filtered probability space on which the prices follow the Itô semimartingale in the presence of market microstructure noise.

Let a first filtered probability space be $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t^{(0)})_{t\geq 0}, P^{(0)})$ on which the p-dimensional Itô semimartingale $\mathbf{X} = (\mathbf{X}(t))_{0\leq t\leq 1}$ is defined. We adopt the construction of the whole filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$, where both the process $\mathbf{X}$ and the noiseare are defined (see Christensen, Podolskij and Vetter (2013)). Let $\mathcal{B}^p$ be the Borel σ-field of $\mathbf{R}^p$ and Q be a probability measure on $(\mathbf{R}^p, \mathcal{B}^p)$. We consider a second filtered probability space $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\in[0,1]}, P^{(1)})$, where $\Omega^{(1)}$ is the set of functions from $[0, 1]$ to $\mathbf{R}^p$, $\mathcal{F}^{(1)}$ is the Borel σ-field on $\Omega^{(1)}$, and $P^{(1)} = \otimes_{t\in[0,1]} P_t$ with $P_t = Q$. Define the market microstructure noise process $\mathbf{v} = (\mathbf{v}(t))_{t\in[0,1]}$ as the canonical process on $(\Omega^{(1)}, \mathcal{F}^{(1)}, (\mathcal{F}_t^{(1)})_{t\in[0,1]}, P^{(1)})$ with the canonical filtration $\mathcal{F}_t^{(1)} = \sigma(\mathbf{v}(s) : s \leq t)$ for $0 \leq t \leq 1$. From the definition of $\mathbf{v}$, Q is the marginal distribution of $(\mathbf{v}(t))_{t\in[0,1]}$. We shll use the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$, where $\Omega = \Omega^{(0)} \times \Omega^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}$, $\mathcal{F}_t = \mathcal{F}_t^{(0)} \otimes \mathcal{F}_t^{(1)}$ and $P = P^{(0)} \times P^{(1)}$.

When we consider the continuous time stochastic processes, the class of Itô semimartingales is a fundamental one and it includes the diffusion processes and jump processes as special cases. In their applications to high-frequency financial data, it has been known in financial econometrics that the role of market microstructure noise is important. However, it is not straight-forward to estimate the volatility and co-volatilities or quadratic variation in the general case in the presence of market microstructure noise.

2.1 Itô semimartingale and Quadratic Variation

In this section, we describe the statistical model of the present paper. Let $\mathbf{Y}(t_i^n) = (Y_j(t_i^n))(j = 1, \cdots, p)$ be the (p-dimensional) observed (log-)prices at $t_i \in [0,1]$ and $i = 1, \cdots, n$, which satisfyies

$$\mathbf{Y}(t_i^n) = \mathbf{X}(t_i^n) + \mathbf{v}(t_i^n) \;\; (i = 1, \cdots, n) \; , \tag{2.1}$$

where $\mathbf{X}(t_i^n) = (X_j(t_i^n))$ is the $p \times 1$ hidden stochastic vector process and $\mathbf{v}(t_i^n)$ $(= (v_j(t_i^n)))$ is a sequence of (mutually) independently and identically distributed market microstructure noises with $\mathcal{E}[\mathbf{v}(t_i^n)] = \mathbf{0}$ and $\mathcal{E}[\mathbf{v}(t_i^n)\mathbf{v}(t_i^n)^{'}] = \mathbf{\Sigma}_v$ (> 0 a positive definite matrix).
We assume that these market microstructure noises are independent of the p-dimensional continuous-time stochastic process $\mathbf{X}(t)$, which is given by

$$\begin{aligned}\mathbf{X}(t) &= \mathbf{X}(0) + \int_0^t \boldsymbol{b}(s)ds + \int_0^t \boldsymbol{\sigma}(s)d\mathbf{W}(s) + \int_0^t \int_{\|\mathbf{x}\|<1} \mathbf{\Delta}(s,x)(\boldsymbol{\mu} - \boldsymbol{\nu})(ds, d\mathbf{x}) \\ &\quad + \int_0^t \int_{\|\mathbf{x}\|\geq 1} \mathbf{\Delta}(s,\mathbf{x})\boldsymbol{\mu}(ds, d\mathbf{x}) \; , \end{aligned} \tag{2.2}$$

where $\boldsymbol{b}(s)$ and $\boldsymbol{\sigma}(s)$ are the p-dimensional adapted drift process and the $p \times q_1$ ($q_1 \leq p$) instantaneous predictable volatility process, $\mathbf{W}(s) = (W_j(s))$ is the $q_1 \times 1$ standard Brownian motions, $\mathbf{\Delta}(\omega, s, \mathbf{x})$ is a $\mathbf{R}^p$-valued predictable function on $\Omega \times [0,\infty) \times \mathbf{R}^{q_2}$ ($q_2 \leq p$), $\boldsymbol{\mu}(\cdot)$ is a Poisson random measure on $[0,\infty) \times \mathbf{R}^{q_2}$ and $\boldsymbol{\nu}(ds, d\mathbf{x}) = ds \otimes \lambda(d\mathbf{x})$ is the predictable compensator or intensity measure of $\boldsymbol{\mu}$ with a σ-finite measure λ on $(\mathbf{R}^p, \mathcal{B}^p)$. The jump terms are denoted as $\Delta\mathbf{X}(s) = (\Delta X_j(s))$ ($\Delta X_j(s) = X_j(s) - X_j(s-)$, $X_j(s-) = \lim_{u\uparrow s} X_j(u)$ at any $s \in [0,1]$), and $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^p$. We use the notation $\boldsymbol{c}(s) = \boldsymbol{\sigma}(s)\boldsymbol{\sigma}^{'}(s) = (\boldsymbol{c}_{gh}(s))$ ($p \times p$ matrix) and for $p \times 1$ vectors $\mathbf{y}_i = \mathbf{Y}(t_i^n)$, $\mathbf{x}_i = \mathbf{X}(t_i^n)$, and $\mathbf{v}_i = \mathbf{v}(t_i^n)$ $(i = 1, \cdots, n)$.

We summarize the basic assumptions :

Assumption 2.1. *(a) The path $t \mapsto \boldsymbol{b}(t,\omega)$ is locally bounded.*

(b) The process $\boldsymbol{\sigma}$ is continuous.

(c) We have $\sup_{\omega,\mathbf{x}} \|\boldsymbol{\Delta}(\omega,t,\mathbf{x})\|/\boldsymbol{g}(\mathbf{x})$ is locally bounded for a deterministic non-negative function satisfying $\int_{\mathbf{R}^p}(\boldsymbol{g}(\mathbf{x})^h \wedge 1)\lambda(d\mathbf{x}) < \infty$. for some $h \in (0,2)$ ($\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^p$).

(d) We have $\int_t^{t+u} \|\boldsymbol{\sigma}(s)\|ds > 0$ a.s. for all $t, u > 0$.

(e) The noise terms $\mathbf{v}(t_i^n)$ $(= v_j(t_i^n))$ $(i = 1,\cdots,n; j = 1,\cdots,p)$ are a sequence of i.i.d. random variables with $\mathcal{E}[\mathbf{v}(t_i^n)] = 0$, $\mathcal{E}[\mathbf{v}(t_i^n)\mathbf{v}^{'}(t_i^n)] = \boldsymbol{\Sigma}_v$ (a positive definite matrix) and $\mathcal{E}[v_j^2(t_i^n)v_k^2(t_i^n)] < +\infty$ $(j,k = 1,\cdots,p)$. Furthermore, the stochastic processes of $\mathbf{v}$ and $\mathbf{X}$ are independent.

The conditions in Assumption 2.1 are standard in the literature of high-frequency data analysis. We assume the condition (b) for the simplicity of our analysis and the presentation of our results but our results also hold under more general assumption that $\boldsymbol{\sigma}(t)$ is cádlág (condition (b')). The condition (b') implies that $\boldsymbol{\sigma}(t)$ and $\mathbf{X}(t)$ could have common-jumps. There are some empirical evidence that $\boldsymbol{\sigma}(t)$ and the original process $\mathbf{X}(t)$ have common jumps (see Jacod and Todorov (2010), and Bibinger and Winkelmann (2018) for example). We also refer to Jacod and Todorov (2009) and Bibinger and Winkelmann (2015) which have investigated statistical testing for common jumps of $\mathbf{X}(t)$ (see Aït-Sahalia and Jacod (2014), Jacod and Protter (2012) and Kurisu (2018) for more discussions on this point). The condition (e) such as the independence of $\mathbf{v}$ and $\mathbf{X}$ can be relaxed to some extent, but we do not pursue the generalization for the sake of simplicity.

The fundamental quantity for the continuous-time Itô semimartingale with $p \geq 1$ is the quadratic variation (QV) matrix, which is given by

$$\boldsymbol{\Sigma}_x = \int_0^1 \boldsymbol{c}(s)ds + \sum_{0 \leq s \leq 1} (\Delta\mathbf{X}(s))(\Delta\mathbf{X}(s))^{'} = (\sigma_{gh}^{(x)}) \ . \tag{2.3}$$

When the stochastic process is the diffusion-type, $\boldsymbol{\Sigma}_x$ becomes the integrated volatility $\int_0^1 \boldsymbol{c}(s)ds$. The class of Itô semimartingales and the quadratic variation, which have been standard in stochastic analysis, are fully explained by Ikeda and Watanabe (1989), and Jacod and Protter (2012) as the standard literature.

2.2 On the SIML Estimation

Kunitomo and Sato (2013) have developed the separating information maximum likelihood (SIML) estimation for general $p \geq 1$, but there are no jump terms. The

SIML estimator of $\hat{\mathbf{\Sigma}}_x$ for the integrated volatility is defined by

$$\hat{\mathbf{\Sigma}}_x := \mathbf{G}_m = \frac{1}{m_n}\sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} = (\hat{\sigma}_{gh}^{(x)}) \ , \tag{2.4}$$

where $\mathbf{z}_k = (z_{gk})$ $(j = 1, \cdots, p; k = 1, \cdots, m_n)$, which are constructed by the transformation from $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ $(n \times p)$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = \mathbf{K}_n \left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) \tag{2.5}$$

where $\mathbf{K}_n = h_n^{-1/2}\mathbf{P}_n\mathbf{C}_n^{-1}$, $h_n = 1/n$,

$$\mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{n \times n} , \tag{2.6}$$

$$\mathbf{P}_n = (p_{jk}^{(n)}) \ , \ p_{jk}^{(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}}\cos\left[\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})\right] \tag{2.7}$$

and $\bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0^{'}$.
By using the spectral decomposition $\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1} = \mathbf{P}_n\mathbf{D}_n\mathbf{P}_n^{'}$ and $\mathbf{D}_n$ is a diagonal matrix with the k-th element $d_k = 2[1-\cos(\pi(\frac{2k-1}{2n+1}))]$ $(k = 1, \cdots, n)$ and a_{kn} $(= n \times d_k)$ $= 4n\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right]$.

To assure some desirable asymptotic properties of the SIML estimator, we need the condition that the number of terms m_n should be dependent on n and we need the order requirement that $m_n = O(n^{\alpha})$ $(0 < \alpha < 0.5)$ for the consistency and $m_n = O(n^{\alpha})$ $(0 < \alpha < 0.4)$ for the asymptotic normality.

When $\mathbf{X}$ is an Itô semipmartingale with possible jumps, the asymptotic properties of the SIML estimator were stated in Chapter 9 of Kunitomo et al. (2018) (Proposition 9.1 and Corollary 9.2) without the detailed exposition. Because they are the starting points of further developments, we state an extended version of their result and we give some supplementary derivations in the Appendix, for the sake of convenience.

In the following results, we freely use the stable convergence arguments and $\mathcal{F}^{(0)}$-conditionally Gaussianity, which have been developed and explained by Jacod (2008) and Jacod and Protter (2012), and use the notation $\xrightarrow{\mathcal{L}-s}$ as stable convergence in law. For the general reference on stable convergence, we refer to Häusler and Luschgy (2015). We use the notation $\xrightarrow{d}$ and $\xrightarrow{p}$ as convergence in distribution and in probability, respectively.

Theorem 2.1 : Suppose Assumption 2.1 is satisfied and $\mathcal{E}[v_j^4(t_i^n)] < +\infty$ in (2.1).
(i) For $m_n = [n^\alpha]$ ($[\cdot]$ is the floor function) and $0 < \alpha < 0.5$, as $n \to \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}\ . \tag{2.8}$$

(ii) Let

$$\hat{\sigma}_{gh}^{(*)} = \sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right], \tag{2.9}$$

where $\hat{\sigma}_{gh}^{(x)}$ is the (g,h)-th component of $\hat{\mathbf{\Sigma}}_x$ and $\sigma_{gh}^{(x)}(s)$ is the (g,h)-th component of $\mathbf{\Sigma}_x$. Then, as $n \to \infty$, for $m_n = [n^\alpha]$ and $0 < \alpha < 0.4$, we have that

$$\begin{bmatrix} \hat{\sigma}_{gh}^{(*)} \\ \hat{\sigma}_{kl}^{(*)} \end{bmatrix} \xrightarrow{\mathcal{L}-s} N\left[\mathbf{0}, \begin{pmatrix} V_{gh} & V_{gh,kl} \\ V_{gh,kl} & V_{kl} \end{pmatrix}\right] \tag{2.10}$$

where

$$\begin{aligned} V_{gh} &= \int_0^1 \left[\boldsymbol{c}_{gg}(s)\boldsymbol{c}_{hh}(s) + \boldsymbol{c}_{gh}^2(s)\right] ds \\ &+ \sum_{0<s\leq 1} \left[\boldsymbol{c}_{gg}(s)(\Delta X_h(s))^2 + \boldsymbol{c}_{hh}(s)(\Delta X_g(s))^2 + 2\boldsymbol{c}_{gh}(s)(\Delta X_g(s)\Delta X_h(s))\right] \end{aligned}$$

and

$$\begin{aligned} V_{gh,kl} &= \int_0^1 \left[\boldsymbol{c}_{gk}(s)\boldsymbol{c}_{hl}(s) + \boldsymbol{c}_{gl}(s)\boldsymbol{c}_{hk}(s)\right] ds \\ &+ \sum_{0<s\leq 1} \left[\boldsymbol{c}_{gk}(s)\Delta X_h(s)\Delta X_l(s) + \boldsymbol{c}_{gl}(s)\Delta X_h(s)\Delta X_k(s)\right. \\ &\qquad \left. +\boldsymbol{c}_{hk}(s)\Delta X_g(s)\Delta X_l(s) + \boldsymbol{c}_{hl}(s)\Delta X_g(s)\Delta X_k(s)\right]\ . \end{aligned}$$

Corollary 2.2 : When $p = 1$ in Theorem 2.1, the asymptotic variance V_{gg} is given by

$$V_{gg} = 2\left[\int_0^1 \boldsymbol{c}_{gg}^2(s)ds + 2\sum_{0<s\leq 1} \boldsymbol{c}_{gg}(s)(\Delta X_g(s))^2\right]\ . \tag{2.11}$$

The notable point is the fact that the asymptotic distribution and limiting variance-covariances of the SIML estimator have the same forms as the ones of the realized volatility and co-volatilities when there is no noise terms if we replace n by m_n, which is dependent on n. It is the key fact to obtain the results of asymptotic properties from the estimated QV in the next section.

3. Asymptotic Properties of Characteristic Roots and Vectors

One of important observations on the asset price movements has been the empirical observation that although there are many financial assets traded in markets, many of them move in similar ways with their trends, volatilities and jumps. Then there is a question how to cope with many asset prices when the number of factors of volatilities or quadratic variation of asset prices is less than p, which is the dimension of observed prices. In this section we consider the case when the underlying continuous time stochastic process is a p-dimensional Itô semimartingale and the number of factors of quadratic variation q_x is less than p. In particular, we assume that there exists a $p \times r_x$ $(1 \leq r_x < p)$ matrix $\mathbf{B}$ with rank r_x such that

$$\mathbf{B}^{'} \left[\mathbf{X}(t) - \mathbf{X}(0) - \int_0^t \boldsymbol{b}(s) ds \right] = \mathbf{O} \ \ (0 \leq t \leq 1) \ . \tag{3.1}$$

Then if we use the notation $q_x = p_x^r$, we have

$$\operatorname{rank}(\boldsymbol{\Sigma}_x) = \operatorname{rank} \left[\int_0^1 \boldsymbol{c}(s) ds + \sum_{0 \leq s \leq 1} (\Delta \mathbf{X}(s))(\Delta \mathbf{X}(s))^{'} \right] = q_x < p \tag{3.2}$$

and

$$\boldsymbol{\Sigma}_x \mathbf{B} = \mathbf{O} \ , \tag{3.3}$$

where $\boldsymbol{\Sigma}_x = (\sigma_{gh}^{(x)})$.

To avoid the complications in the following derivations, we assume that $\boldsymbol{b}(s) = \mathbf{0}$ $(0 \leq s \leq 1)$ in this section. It has been known that the effects of drift terms are negligible in the estimation of volatility and quadratic variation under some conditions such as Assumption 2.1. See Aït-Sahalia, Y. and J. Jacod (2014), or Chapter 5 of Kunitomo et al. (2018), for instance.
There are more general situations when we can relax the conditions given by (3.1)-(3.3), but then there would be substantial complications involved. Hence in the following analysis we shall use these conditions for the resulting simplicity.

We notice that the present problem has the similar aspect in the reduced rank regression problem, which has been well-known in statistical multivariate analysis (See Anderson (1984) and Anderson (2003) for instance). The new feature in our formulation is the fact that we are dealing with the continuous-time stochastic process as the hidden process while we have discrete observations with measurement errors.

In the present situation, if we take $m_n = [n^{\alpha}]$ and $0 < \alpha < 0.5$, then as $n \to \infty$

$$\hat{\boldsymbol{\Sigma}}_x - \boldsymbol{\Sigma}_m \xrightarrow{p} \mathbf{O} \ , \tag{3.4}$$

where $\mathbf{\Sigma}_m = (\sigma_{gh.m})$,

$$\mathbf{\Sigma}_m = \mathbf{\Sigma}_x + a_m \mathbf{\Sigma}_v = (\sigma_{gh}^{(x)} + a_m \sigma_{gh}^{(v)}) \tag{3.5}$$

and

$$a_m = \frac{1}{m_n} \sum_{k=1}^{m} a_{kn} \tag{3.6}$$

and $a_{kn} = 4n \sin^2[\frac{\pi}{2}(\frac{2k-1}{2n+1})]$ $(k = 1, \cdots, n)$. We have the second term although $a_m \to 0$ as $m_n = n^{\alpha} \to \infty$ and $0 < \alpha < 0.5$.

Lemma 3.1 : We set $a_m = (1/m)\sum_{k=1}^{m} a_{kn}$ and $a_m(2) = (1/m)\sum_{k=1}^{m}[a_{kn}]^2$. Then we can evaluate that as $n \to \infty$ and $m \to \infty$,

$$\frac{n}{m^2} a_m = (\frac{n}{m^2}) \frac{1}{m} \sum_{k=1}^{m} a_{kn} \sim \pi^2 \int_0^1 s^2 ds \tag{3.7}$$

by using $\sin x \sim x - (1/6)x^3 + o(x^3)$ when x is small. (The differences should be negligible because $(n/m^2)a_m \sim \pi^2 (1/m)\sum_{k=1}^{m}(k/m)^2 = O(1)$.) Also we find that

$$\frac{n^2}{m^4} a_m(2) = \frac{1}{m} \sum_{k=1}^{m} (\frac{n^2}{m^4})[a_{kn}]^2 \sim \pi^4 \int_0^1 s^4 ds \ . \tag{3.8}$$

In the following derivations, we investigate the case when $\mathbf{\Sigma}_v$ is known and $|\mathbf{\Sigma}_v| \neq 0$. However, the results do not depend on this assumption if we use a consistent estimator of $\mathbf{\Sigma}_v$. The variance-covariance matrix $\mathbf{\Sigma}_v$ can be consistently estimated by

$$\hat{\mathbf{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ , \tag{3.9}$$

where $l_n = [n^{\beta}]$ $(0 < \alpha < \beta < 1)$. We can take β being slightly less than 1 and $\hat{\mathbf{\Sigma}}_v = \mathbf{\Sigma}_v + O_p(\frac{1}{\sqrt{l_n}})$ such that the effects of estimating $\mathbf{\Sigma}_v$ are negligible. (See Chapter 5 of Kunitomo et al. (2018).)
Let the characteristic equation be

$$|\mathbf{G}_m - \lambda \mathbf{H}| = 0 \ , \tag{3.10}$$

and $\hat{\mathbf{B}}$, the estimator of $\mathbf{B}$ in (3.1) and (3.3), is given by

$$\left[\mathbf{G}_m \hat{\mathbf{B}} - \mathbf{H} \hat{\mathbf{B}} \mathbf{\Lambda}\right] = \mathbf{O} \ , \tag{3.11}$$

where $\mathbf{G}_m$ $(= \hat{\mathbf{\Sigma}}_x)$ in (2.4), λ_i $(i = 1, \cdots, p)$ are the characteristic roots, $\mathbf{\Lambda} = \text{diag}(\lambda_1, \cdots, \lambda_{r_x})$ with $0 \leq \lambda_1 \leq \cdots \leq \lambda_p$, and $\mathbf{H}$ is any positive (known) definite matrix. For the resulting convenience, we take a $p \times r_x$ $(p = r_x + q_x)$

$$\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{I}_{r_x} \\ -\hat{\mathbf{B}}_2 \end{bmatrix}$$

for a normalization. We take $\mathbf{H} = \hat{\mathbf{\Sigma}}_v$ in most part of the following analysis. Since $\hat{\mathbf{\Sigma}}_v - \mathbf{\Sigma}_v = O_p(1/\sqrt{l_n})$, we can treat $\mathbf{\Sigma}_v$ as if it were $\hat{\mathbf{\Sigma}}_v$ if we take a large β (< 1) with $l_n = O([n^\beta])$. We set the characteristic roots in an ascending order as $0 \leq \lambda_1 \leq \cdots \leq \lambda_{r_x} \leq \cdots \leq \lambda_p$.

We take the probability limit of the determinantal equation

$$|\text{plim}_{m\to\infty}(\mathbf{G}_m - a_m\mathbf{\Sigma}_v) - (\text{plim}_{m\to\infty}\lambda - a_m)\mathbf{\Sigma}_v| = 0 \ . \tag{3.12}$$

The rank of $\mathbf{\Sigma}_x$ is q_x, which is less than p, and $a_m = O(m^2/n)$. Then we find that $\mathbf{G}_m - \mathbf{\Sigma}_x = O_p(a_m)$, $\mathbf{G}_m - (\mathbf{\Sigma}_x + a_m\mathbf{\Sigma}_v) = O_p(\frac{a_m}{\sqrt{m}}) = O_p(\frac{\sqrt{m^3}}{n})$, and

$$\lambda_i - a_m \xrightarrow{p} 0 \ (i = 1, \cdots, r_x) \tag{3.13}$$

if $\frac{\sqrt{m^3}}{n} \longrightarrow 0$ as $n \to +\infty$ (see (3.22) and (3.32) below). Then

$$[\text{plim}_{n\to\infty}(\mathbf{G}_m - a_m\mathbf{\Sigma}_v)][\text{plim}_{n\to\infty}\hat{\mathbf{B}} - \mathbf{B}] = \mathbf{O} \ . \tag{3.14}$$

By multiplying $\mathbf{\Pi}_*^{'}(q_x \times p)$ from the left-hand to

$$\mathbf{\Pi}_*^{'}[\text{plim}_{n\to\infty}(\mathbf{G}_m - a_m\mathbf{\Sigma}_v)]\text{plim}_{n\to\infty}\left[\hat{\mathbf{B}} - \mathbf{B}\right] = \mathbf{O} \ ,$$

such that $\mathbf{\Pi}^{*'}\text{plim}_{n\to\infty}\mathbf{G}_m[\begin{smallmatrix} \mathbf{O} \\ \mathbf{I}_{q_x} \end{smallmatrix}]$ is non-singular. By using the facts that the rank is q_x and the normalization of $\mathbf{B}$, we find

$$\text{plim}_{n\to\infty}\hat{\mathbf{B}} = \mathbf{B} \ . \tag{3.15}$$

In order to proceed the further step to evaluate the limiting random variables, we use the $\mathbf{K}_n-$transformation and we decompose the resulting random variables $\mathbf{z}_k = \mathbf{x}_k^* + \sqrt{a_{kn}}\mathbf{v}_k^*$ $(k = 1, \cdots, m)$ with $\mathcal{E}(\mathbf{v}_k^*\mathbf{v}_k^{*'}) = \mathbf{\Sigma}_v$ and

$$\mathbf{G}_m = \frac{1}{m}\sum_{k=1}^{m}(\mathbf{x}_k^* + \sqrt{a_{kn}}\mathbf{v}_k^*)(\mathbf{x}_k^* + \sqrt{a_{kn}}\mathbf{v}_k^*)^{'} \ , \tag{3.16}$$

where the $p \times 1$ random vectors $\mathbf{x}_i^*$ and $\mathbf{v}_i^*$ are defined by $(\mathbf{x}_i^{*'}) = \mathbf{K}_n(\mathbf{x}_i^{'})$ and $(\mathbf{v}_i^{*'}) = \mathbf{K}_n(\mathbf{v}_i^{'})$, which are $n \times p$ matrices.

Under the null-hypothesis $\mathrm{H}_0 : \mathbf{\Sigma}_x\mathbf{B} = \mathbf{O}$, we have $\mathbf{B}^{'}\mathbf{\Sigma}_x\mathbf{B} = \mathbf{O}$ $(r_x \times r_x)$. Then we have the representation that

$$\mathbf{B}^{'}\mathbf{G}_m = \frac{1}{m}\sum_{k=1}^{m}\mathbf{B}^{'}(\sqrt{a_{kn}}\mathbf{v}_k^*)(\mathbf{x}_k^* + \sqrt{a_{kn}}\mathbf{v}_k^*)^{'} \tag{3.17}$$

and

$$\mathbf{B}^{'}\mathbf{G}_m\mathbf{B} = \frac{1}{m}\sum_{k=1}^{m} a_{kn}\mathbf{u}_k\mathbf{u}_k^{'} \ , \tag{3.18}$$

where we define $\mathbf{B}^{'}\mathbf{v}_k^* = \mathbf{u}_k$ $(k = 1, \cdots, m_n)$.
Let $\boldsymbol{\beta}_j$ $(= (\beta_{hj}))$ be the j-th column vector of $\mathbf{B}$ $(j = 1, \cdots, r_x)$ and

$$\sqrt{m}[\mathbf{G}_m - \mathbf{\Sigma}_m]\boldsymbol{\beta}_j = \sqrt{m}[\sum_{h=1}^{p}(\hat{\sigma}_{gh}^{(x)} - \sigma_{gh.m})\beta_{hj}]_g \ , \tag{3.19}$$

where $\mathbf{\Sigma}_m = \mathbf{\Sigma}_x + a_m\mathbf{\Sigma}_v$ $(= (\sigma_{gh.m}))$ and $a_m = (1/m)\sum_{k=1}^{m} a_{kn}$.
Since we have the relations $\sum_{h=1}^{p}\sigma_{gh}^{(x)}\beta_{hj} = 0$ for $g, j = 1, \cdots, p$ under the rank condition, we decompose

$$[\mathbf{G}_m - \mathbf{\Sigma}_m]\boldsymbol{\beta}_j = \frac{1}{m}\sum_{k=1}^{m}\sqrt{a_{kn}}\mathbf{x}_k^*(\mathbf{v}_k^{*'}\boldsymbol{\beta}_j) + \frac{1}{m}\sum_{k=1}^{m} a_{kn}(\mathbf{v}_k^*\mathbf{v}_k^{*'} - \mathbf{\Sigma}_v)\boldsymbol{\beta}_j \ . \tag{3.20}$$

We can evaluate that the first term of (3.20) is $O_p(a_m/m) = O_p(m/n)$ and the second order is $O_p(a_m(2)/m) = O_p((m/n)^2)$ by using Lemma 3.1. Hence we find that the dominant term is the first term as $n \to \infty$. Hence the asymptotic conditional covariances given $\mathcal{F}^{(0)}$ are

$$\mathrm{ACov}[\sqrt{m}\sum_{h=1}^{p}(\hat{\sigma}_{gh}^{(x)} - \sigma_{gh.m})\beta_{hj}, \sqrt{m}\sum_{l=1}^{p}(\hat{\sigma}_{kl}^{(x)} - \sigma_{kl.m})\beta_{lj}|\mathcal{F}^{(0)}] = a_m\sigma_{gk}^{(x)}[\sum_{h,l=1}^{p}\sigma_{hl}^{(v)}\beta_{hj}\beta_{lj}] \ . \tag{3.21}$$

By applying the central limit theorem (CLT) to the first term of the right-hand side of (3.20), we have the asymptotic normality for (3.19) and (3.20).

Let

$$\sqrt{m}\mathbf{B}^{'}(\mathbf{G}_m - \mathbf{\Sigma}_m)\mathbf{B} = \frac{1}{\sqrt{m}}\sum_{k=1}^{m} a_{kn}(\mathbf{u}_k\mathbf{u}_k^{'} - \mathbf{\Omega}_v) \tag{3.22}$$

and $\mathbf{\Omega}_v = \mathbf{B}^{'}\mathbf{\Sigma}_v\mathbf{B}$ $(= (\omega_{gh}))$.
Under the assumption of the existence of fourth order moments of noise terms, we

can evaluate the conditional covariances given $\mathcal{F}^{(0)}$ as

$$\text{(3.23)} \qquad \mathrm{Cov}[\frac{1}{\sqrt{m}}\sum_{k=1}^{m} a_{kn}(u_{gk}u_{hk}-\omega_{gh}), \frac{1}{\sqrt{m}}\sum_{k=1}^{m} a_{kn}(u_{g'k}u_{h'k}-\omega_{g'h'})|\mathcal{F}^{(0)}]$$
$$= \frac{1}{m}\sum_{k=1}^{m}[a_{kn}]^2\left[\omega_{gg'}\omega_{hh'}+\omega_{gh'}\omega_{g'h}+\kappa_{ghg'h'}\right] ,$$

where $\kappa_{ghg'h'}$ are the 4-th order cumulants of $u_{gk}u_{hk}u_{g',k}u_{h',k}$ and $\mathbf{u}_k = (u_{gk})$ $(g, h, g', h' = 1, \cdots, p)$.
We can further use the following Lemma on the fourth-order moments of the $\mathbf{K}_n-$transformed random variables. The derivation will be given in the Appendix.

Lemma 3.2 : We take $m = [n^{\alpha}]$ with $1/2 < \alpha < 1$. Under the assumption of the existence of fourth order moments of noise terms, the effects of fourth-order cumulants $\kappa_{ghg'h'}$ are of smaller order when $n \to \infty, m \to \infty$.

By applying CLT to (3.22), under the assumption of existence of 4-th order moments, we have the asymptotic normality under Assumption 2.1(e).

Let an $r_x \times r_x$ diagonal matrix be $\mathbf{\Lambda} = (\mathrm{diag}\ (\lambda_i))$ and

$$\text{(3.24)} \qquad \mathbf{G}_m\hat{\mathbf{B}} - \mathbf{\Sigma}_v\hat{\mathbf{B}}[a_m\mathbf{I}_{r_x} + (\mathbf{\Lambda} - a_m\mathbf{I}_{r_x})] = \mathbf{O} ,$$

which can be written as

$$\text{(3.25)} \qquad [\mathbf{G}_m - a_m\mathbf{\Sigma}_v]\hat{\mathbf{B}} = \mathbf{\Sigma}_v\hat{\mathbf{B}}[\mathbf{\Lambda} - a_m\mathbf{I}_{r_x}] .$$

By multiplying $\hat{\mathbf{B}}^{'}(r_x \times r_x)$ from the left-hand to (3.24), we find

$$\text{(3.26)} \qquad \hat{\mathbf{B}}^{'}\mathbf{\Sigma}_v\hat{\mathbf{B}}[\mathbf{\Lambda} - a_m\mathbf{I}_{r_x}] = \hat{\mathbf{B}}^{'}[\mathbf{G}_m - a_m\mathbf{\Sigma}_v]\hat{\mathbf{B}} .$$

By multiplying $[\mathbf{O}, \mathbf{I}_{q_x}](q_x \times p)$ from the left-hand to (3.23),

$$\text{(3.27)} \quad [\mathbf{O}, \mathbf{I}_{q_x}][\mathbf{G}_m - a_m\mathbf{\Sigma}_v][\mathbf{B} + (\hat{\mathbf{B}} - \mathbf{B})] = [\mathbf{O}, \mathbf{I}_{q_x}]\mathbf{\Sigma}_v[\mathbf{B} + (\hat{\mathbf{B}} - \mathbf{B})][\mathbf{\Lambda} - a_m\mathbf{I}_{r_x}] .$$

Then we evaluate the order of each terms of the above equation. If we have the condition $m/n \to 0$ as $n \to \infty$, it is asymptotically equivalent to

$$\text{(3.28)} \qquad [\mathbf{O}, \mathbf{I}_{q_x}]\sqrt{\frac{n}{m}}[\mathbf{G}_m - a_m\mathbf{\Sigma}_v]\mathbf{B} = [\mathbf{O}, \mathbf{I}_{q_x}]\mathbf{\Sigma}_x[\begin{array}{c}\mathbf{O}\\ \mathbf{I}_{q_x}\end{array}]\sqrt{\frac{n}{m}}(\hat{\mathbf{B}}_2 - \mathbf{B}_2) + o_p(1) .$$

Then we set the normalization factor c_m as

$$\text{(3.29)} \qquad c_m = \sqrt{\frac{m}{a_m}} = O(\sqrt{\frac{n}{m}}) ,$$

which goes to infinity as $n \to \infty$ when $m/n \to 0$ as $n \to \infty$.
We denote the j-th column vectors of $-\hat{\mathbf{B}}_2$ and $-\mathbf{B}_2$ as $-\hat{\boldsymbol{\beta}}_{2j}$ and $-\boldsymbol{\beta}_{2j}$ as the $(p-r_x)\times$ lower part vectors of $\hat{\boldsymbol{\beta}}_j$ and $\boldsymbol{\beta}_j$. Because $\boldsymbol{\Sigma}_m = \boldsymbol{\Sigma}_x + O(a_m)$, the limiting distribution of $c_m[\hat{\boldsymbol{\beta}}_{2j} - \boldsymbol{\beta}_{2j}]$ $(j = 1, \cdots, r_x)$ is $\mathcal{F}^{(0)}$-conditionally normal with zero means and the asymptotic conditional variance-covariance matrix given $\mathcal{F}^{(0)}$ is

$$\text{(3.30)} \qquad \text{AVar}(c_m\hat{\boldsymbol{\beta}}_{2j}|\mathcal{F}^{(0)}) = [\sum_{h,l=1}^{p} \sigma_{hl}^{(v)}\beta_{hj}\beta_{lj}][(\mathbf{O},\mathbf{I}_q)\boldsymbol{\Sigma}_x(\mathbf{O},\mathbf{I}_q)^{'}]^{-1} \; .$$

When we take $\mathbf{H} = \hat{\boldsymbol{\Sigma}}_v$ and $l_n = [n^{\beta}]$ with $0 < \alpha < \beta, 1/2 < \beta < 1$, the effects of estimating the variance and covariance matrix of the market microstructure noises are small. It is because we have asymptotically similar equations as (3.24)-(3.28). For instance, if we replace $\boldsymbol{\Sigma}_v$ by $\hat{\boldsymbol{\Sigma}}_v$ in (3.26), then it corresponds to $\boldsymbol{\Sigma}_v + O_p(1/\sqrt{l_n})$ and $a_m(\hat{\boldsymbol{\Sigma}}_v - \boldsymbol{\Sigma}_v) = O(m^2/n) \times (1/\sqrt{l_n})$, which is of smaller order than $\mathbf{G}_m - a_m\boldsymbol{\Sigma}_v$ (it is of $O_p(a_m/\sqrt{m})$ if $m/l \to 0$ as $n \to \infty$.
By calculating the variance and covariances of the limiting random variables and applying CLT to (3.28), we summarize the asymptotic distributions of $c_m(\hat{\mathbf{B}}_2 - \mathbf{B}_2)$ as the next proposition.

Theorem 3.1 : Suppose Assumption 2.1 is satisfied in (2.1) and (2.2). We take $m = m_n = [n^{\alpha}]$, $l_n = [n^{\beta}]$ with $0 < \alpha < \beta < 1, 0 < \alpha < 2/3$, and $\mathbf{H} = \hat{\boldsymbol{\Sigma}}_v$. Let $c_m \to \infty$ as $n \to \infty$. Then, as $n \to \infty$, we have that

$$\text{(3.31)} \qquad c_m\left[\hat{\mathbf{B}}_2 - \mathbf{B}_2\right] \stackrel{\mathcal{L}-s}{\longrightarrow} N_{r_x q_x}\left(\mathbf{0}, (\mathbf{B}^{'}\boldsymbol{\Sigma}_v\mathbf{B}) \otimes [(\mathbf{O},\mathbf{I}_{q_x})\boldsymbol{\Sigma}_x(\mathbf{O},\mathbf{I}_{q_x})^{'}]^{-1}\right) \; .$$

Next, we investigate the limiting distribution of the characteristic roots of $\boldsymbol{\Lambda}$ and the related statistics. Because of Lemma 3.1,

$$\text{(3.32)} \qquad \mathbf{B}^{'}(\mathbf{G}_m - \boldsymbol{\Sigma}_m)\mathbf{B} = O(\sqrt{\frac{a_m(2)}{m}}) = O(\frac{a_m}{\sqrt{m}})$$

and

$$\mathbf{B}^{'}\mathbf{G}_m\begin{pmatrix}\mathbf{O}\\ \mathbf{I}_q\end{pmatrix}(\hat{\mathbf{B}}_2 - \mathbf{B}_2) = O_p(\sqrt{\frac{a_m}{m}}) \times O_p(\sqrt{\frac{a_m}{m}}) = O_p(\frac{a_m}{m});.$$

By using the decomposition $\hat{\mathbf{B}} = \mathbf{B} + [\hat{\mathbf{B}} - \mathbf{B}]$, we can evaluate as

$$\begin{aligned}
\text{(3.33)} \qquad & \hat{\mathbf{B}}^{'}[\mathbf{G}_m - a_m\boldsymbol{\Sigma}_v]\hat{\mathbf{B}} \\
&= \mathbf{B}^{'}[\mathbf{G}_m - a_m\boldsymbol{\Sigma}_v]\mathbf{B} - \mathbf{B}^{'}[\mathbf{G}_m - a_m\boldsymbol{\Sigma}_v][\begin{matrix}\mathbf{O}\\ \mathbf{I}_{q_x}\end{matrix}]\boldsymbol{\Sigma}_*^{-1}[\mathbf{O},\mathbf{I}_{q_x}][\mathbf{G}_m - a_m\boldsymbol{\Sigma}_v]\mathbf{B} \\
&= O_p(\frac{a_m}{\sqrt{m}}) + O_p(\frac{m}{n}) \; ,
\end{aligned}$$

where

$$\mathbf{\Sigma}_* = [\mathbf{O}, \mathbf{I}_{q_x}]\mathbf{\Sigma}_x[\begin{array}{c}\mathbf{O}\\ \mathbf{I}_{q_x}\end{array}] .$$

Since the first term is $O_p(\frac{a_m}{\sqrt{m}}) = O_p(\frac{m\sqrt{m}}{n})$ and the second term is $O_p(\frac{a_m}{m}) = O_p(\frac{m}{n})$, the second term of the right-hand-side is asymptotically negligible. Hence the limiting distribution of

$$\sqrt{m}(\mathbf{B}^{'}\mathbf{\Sigma}_v\mathbf{B})[(\frac{1}{a_m})\mathbf{\Lambda} - \mathbf{I}_{r_x}] \sim \frac{\sqrt{m}}{a_m}\mathbf{B}^{'}(\mathbf{G}_m - \mathbf{\Sigma}_m)\mathbf{B} \tag{3.34}$$

is $\mathcal{F}^{(0)}$-conditionally normal by applying CLT to (3.22).
We define

$$\mathbf{E}_m = (e_{ij}) = \frac{\sqrt{m}}{a_m}\mathbf{B}^{'}(\mathbf{G}_m - \mathbf{\Sigma}_m)\mathbf{B} . \tag{3.35}$$

We also denote $\mathbf{\Omega}_v = \mathbf{B}^{'}\mathbf{\Sigma}_v\mathbf{B} = (\omega_{ij})$ and $\mathbf{D} = \mathbf{\Omega}_v^{-1}\mathbf{E}$ $(= (d_{ij}))$. Then by using Lemma 3.2,

$$\begin{aligned} d_{ij} &= \mathrm{Cov}[\sum_{k=1}^{r}\omega^{ik}e_{ki}, \sum_{k^{'}=1}^{r}\omega^{jk^{'}}e_{k^{'}j}|\mathcal{F}^{(0)}] \\ &= \frac{a_m(2)}{a_m^2}\sum_{k,k^{'}=1}^{r}\omega^{ik}\omega^{jk^{'}}[\omega_{kk^{'}}\omega_{ij} + \omega_{kj}\omega_{k^{'}i}] + o_p(1) \\ &= \frac{a_m(2)}{a_m^2}[\delta(i,j) + \omega_{ij}\omega^{ij}] + o_p(1) \ , \end{aligned} \tag{3.36}$$

for $e_{kj} = (\sqrt{m}/a_m)(\mathbf{B}^{'}(\mathbf{G}_m - \mathbf{\Sigma}_m)\mathbf{B})_{kj}$ and $\mathbf{\Omega}_v^{-1} = (\omega^{ij})$. (We shall ignore the last term $o_p(1)$ of (3.36) in the following expression for the resulting simplicity.)
When we use a consistent estimator of $\mathbf{H} = \hat{\mathbf{\Sigma}}_v$, the resulting expression of the limiting distribution becomes simple, which may be useful in practice. By using the fact that $\sqrt{a_m(2)}/a_m \sim 3/\sqrt{5}$ and Lemma 3.2, we obtain the next result on the asymptotic distributions of the smaller characteristic roots λ_i $(i = 1, \cdots, r_x)$.

Theorem 3.2 : Assume the conditions on the Itô semimartingale in (2.1) and (2.2) as Theorem 3.1. We take $m = m_n = [n^{\alpha}]$ $(1/2 < \alpha < 2/3)$, $l_n = [n^{\beta}]$ with $1/2 < \alpha < \beta < 1$ and $\mathbf{H} = \hat{\mathbf{\Sigma}}_v$. As $n \to \infty$,$m \to \infty$, and

$$\frac{\sqrt{m}}{a_m}[\lambda_i - a_m] \xrightarrow{\mathcal{L}-s} N(0, \frac{9}{5}d_{ii}) \tag{3.37}$$

for $i = 1, \cdots, r_x$ $(= p - q_x)$ and $[\sqrt{m}/a_m]\,[\lambda_i - a_m]$ are asymptotically normal jointly. The covariances of $[\sqrt{m}/a_m][\lambda_i - a_m]$ and $[\sqrt{m}/a_m][\lambda_j - a_m]$ $(i, j = 1, \cdots, r_x)$ are given by

$$d_{ij} = \frac{9}{5}[\delta(i,j) + \omega_{ij}\omega^{ij}] .$$

Let

$$\lambda_i^* = \frac{\sqrt{m}}{a_m}[\lambda_i - a_m] \quad (i = 1, \cdots, r_x). \tag{3.38}$$

Since the effect of estimating $\mathbf{\Sigma}_v$ is asymptotically negligible, we find the asymptotic variance as

$$\begin{aligned}
\text{AVar}[\sum_{i=1}^{r_x} \lambda_i^*] &= \mathcal{E}\left[\sum_{i,i'=1}^{r_x} \tilde{\lambda}_i^* \tilde{\lambda}_{i'}^*\right] \\
&= \mathcal{E}\left[\mathcal{E}\left[\sum_{i,i'=1}^{r_x} \tilde{\lambda}_i^* \tilde{\lambda}_{i'}^* \middle| \mathcal{F}^{(0)}\right]\right] \\
&= \mathcal{E}\left[\frac{a_m(2)}{a_m^2} \sum_{i,i'=1}^{r_x} \left[\delta(i,i') + \omega_{ii'}\omega^{ii'}\right]\right] \\
&= \frac{9}{5} 2r_x \ ,
\end{aligned}$$

where we have used the notation that $\tilde{\lambda}_i^*$ are characteristic roots of the characteristic equation $|\mathbf{G}_m - \lambda \mathbf{\Sigma}_v|$.

When $2/3 \leq \alpha < 1$, we do not have (3.13) and $\lambda_i - a_m$ $(i = 1, \cdots, r_x)$ diverge as $n \to +\infty$ because $\mathbf{G}_m - (\mathbf{\Sigma}_x + a_m \mathbf{\Sigma}_v) = O_p(\frac{a_m}{\sqrt{m}}) = O_p(\frac{\sqrt{m^3}}{n})$. By using the normalizing factor $\frac{m}{a_m(2)}$ to (3.10), we re-write

$$\left| \sqrt{\frac{m}{a_m(2)}}(\mathbf{G}_m - a_m \mathbf{\Sigma}_v) - \sqrt{\frac{m}{a_m(2)}}(\lambda - a_m)\mathbf{\Sigma}_v \right| = 0 \ . \tag{3.39}$$

Then it is asymptotically equivalent to

$$\left| \frac{1}{\sqrt{a_m(2)}} \frac{1}{\sqrt{m}} \sum_{k=1}^{m} a_{kn}(\mathbf{v}_k^* \mathbf{v}_k^{*'} - \mathbf{\Sigma}_v) - \sqrt{\frac{m}{a_m(2)}}(\lambda - a_m)\mathbf{\Sigma}_v \right| = 0 \ .$$

Hence we can show that the sum of smaller r_x characteristic roots of (3.10) with $\mathbf{H} = \mathbf{\Sigma}_v$ as $\sum_{i=1}^{r_x}(\lambda_i - a_m)$ has the asymptotic distribution of the smaller r_x characteristic roots of

$$\frac{1}{\sqrt{a_m(2)}} \frac{1}{\sqrt{m}} \sum_{k=1}^{m} a_{kn}(\mathbf{v}_k^* \mathbf{v}_k^{*'} - \mathbf{\Sigma}_v) \ , \tag{3.40}$$

which is the same as the normalized sum of characteristic roots of (3.22).

Hence we summarize our main result on the trace-statistic $\sum_{i=1}^{r_x} \lambda_i$, which will be used as the key statistic for the application.

Theorem 3.3 : Assume the conditions in Theorem 3.2. We take $m = m_n = [n^\alpha]$ $(1/2 < \alpha < 1)$, $l_n = [n^\beta]$ with $\alpha < \beta < 1$ and $\mathbf{H} = \hat{\boldsymbol{\Sigma}}_v$. As $n \to \infty$ and $m \to \infty$,

$$\text{(3.41)} \qquad \sqrt{\frac{m}{a_m(2)}}\left[\sum_{i=1}^{r_x}(\lambda_i - a_m)\right] \xrightarrow{d} N(0, 2r_x) \; .$$

and

$$\text{(3.42)} \qquad \frac{m}{a_m(2)}\frac{1}{2r_x}\left[\sum_{i=1}^{r_x}(\lambda_i - a_m)\right]^2 \xrightarrow{d} \chi^2(1) \; .$$

For the practical purposes, there are some remaining problems. The key parameter m can be small because we take $m = [n^\alpha]$, it may be important to find an improvement of the limiting mixed normal distribution. By using (3.33) and (3.34), the limiting distribution of $\frac{\sqrt{m}}{a_m}[\boldsymbol{\Lambda} - a_m\mathbf{I}_{r_x}]$ can be corrected by adding the term

$$\text{(3.43)} \qquad \mathbf{C}_{m,r_x,q_x} = [\frac{-1}{\sqrt{m}}]\sum_{j=1}^{q_x}\mathbf{c}_j\mathbf{c}_j^{'} \; ,$$

where $\mathbf{c}_j$ $(j = 1, \cdots, q_x)$ are independently distributed as $N_{r_x}(\mathbf{0}, \mathbf{I}_{r_x})$.
The additional term for the sum of diagonal elements of $\frac{\sqrt{m}}{a_m}[\boldsymbol{\Lambda} - a_m\mathbf{I}_r]$ can be approximated as a χ^2-distribution as

$$\text{(3.44)} \qquad \mathrm{tr}(\mathbf{C}_{m,r_x,q_x}) \sim \frac{1}{\sqrt{m}}\chi^2(r_x \times q_x) \; .$$

More generally, if we take a non-singular (known) matrix $\mathbf{H}$, then we need a sligtly restrictive condition. When we use any $\mathbf{H}$, let

$$\text{(3.45)} \qquad \boldsymbol{\Lambda}_0^{**} = (\mathbf{B}^{'}\mathbf{H}\mathbf{B})^{-1}\boldsymbol{\Omega}_v \; (= (\lambda_{ij}^{**})) \; ,$$

which corresponds to the probability limits of smaller characteristic roots.
Then we can replace $\mathbf{B}$ and $\boldsymbol{\Omega}_v$ $(= \mathbf{B}^{'}\boldsymbol{\Sigma}_v\mathbf{B})$ by their consistent estimators. By using Lemma A.1 in the Appendix, we have the following result.

Corollary 3.1 : In addition to the conditions in Theorem 3.2. assme that $\mathbf{v}(t_i^n)$

follows the Gaussian distribution. We take $m = m_n = [n^{\alpha}]$ $(0 < \alpha < 2/3)$ and a non-singular (constant) matrix $\mathbf{H}$. As $n \to \infty$, $m \to \infty$ and

(3.46)
$$\frac{m}{a_m(2)} \frac{1}{r_*} \left[\sum_{i=1}^{r_x} (\lambda_i - a_m \lambda_{ii}^{**}) \right]^2 \xrightarrow{d} \chi^2(1) \ ,$$

where $r_* = \sum_{i,i'=1}^{r_x} \mathbf{Cov}(U_{ii}^*, U_{i',i'}^*)$and the covariances are given as Lamma A.1 in the Appendix with the definition of U_{ii}.

It may be convenient to take $\mathbf{H} = \mathbf{I}_{r_x}$ for instance. Then there is a complication in the expression of asymptotic distribution as well as we need an additional condition on m to have the convergence.

A Test for Detecting the Number of Factors of Quadratic Variation

It is straight-forward to develop the testing procedure for the hypothesis $H_0 \ : \ r_x = r_0$ ($r_0 \geq 1$ is a specified number) against $H_A \ : \ r_x = r_0 + 1$, it may be reasonable to use the r_0-th smaller characteristic root and the rejection region can be constructed by the limiting normal or χ^2 distribution under H_0. (H_0 corresponds to the case of $q_x = p - r_0$ while H_A corresponds to $q_x = p - (r_0 + 1)$. Hence it may be natural to use the sum of smaller characteristic roots as

(3.47)
$$R_0 = \sum_{i=1}^{r_0} (\lambda_i - a_m)$$

where $0 \leq \lambda_1 \leq \cdots \leq \lambda_p$. From Corollary 3.3, we can use

(3.48)
$$T_n(r_0) = \frac{m}{a_m(2)} \frac{1}{2r_0} \left[\sum_{i=1}^{r_0} (\lambda_i - a_m) \right]^2$$

as the test statistics for detecting the number of factors of the underlying Itô semi-martingale. The rejection region with $1 - \alpha$ significance level should be

(3.49)
$$T_n(r_0) \geq \chi^2_{1-\alpha}(1),$$

where $\chi^2_{1-\alpha}(1)$ is the $(1-\alpha)$-quantile of $\chi^2(1)$. Under H_A, the r_0+1-th characteristic root $\lambda_{r_0+1} \xrightarrow{p} \infty$ and the test should be consistent.
Formally, we employ the following stopping rule for the proposed sequential test.

(1) Compute the test statistics $T_n(p-1)$. If $T_n(p-1) < \chi^2_{1-\alpha}(1)$, we finish the sequential test and conclude $r_x = p - 1$. If $T_n(p-1) \geq \chi^2_{1-\alpha}(1)$, we proceed to the next step to test $H_0 \ : \ r_x = p - 2$ against $H_A \ : \ r_x = p - 1$.

(2) We iterate the test $H_0 \ : \ r_x = r_0 - 1$ against $H_A \ : \ r_x = r_0$ ($r_0 = p-1, p-2, \ldots, 1$) sequentially until the null hypothesis is accepted.

(3) We finish the sequential test and conclude $r_x = r_0 - 1$ at the time when $H_0 \ : \ r_x = r_0 - 1$ is accepted for the first time.

Our method corresponds to an extension of the standard statistical method to the case when we have hidden continuous stochastic process and to detect the number of factors in the statistical multivariate analysis. See Anderson (1984), Anderson (2003), and Robin and Smith (2000).

4. Simulations

In this section, we give some simulation results on the characteristic roots and test statistics and we discuss the finite sample properties of the characteristic roots and test statistics we have developed in the previous sections.

4.1 Simulated Models

We simulate three dimensional Itô semimaritingales.
Let $\tilde{\mathbf{X}} = (\tilde{\mathbf{X}}(t))_{t\geq 0}$ $(= (\tilde{X}_1(t), \tilde{X}_2(t), \tilde{X}_3(t))')_{t\geq 0})$ be the vector of Itô semimartingale satisfying

$$d\tilde{X}_j(t) = \sigma_j(t) dW_j(t), \ j = 1, 2, \tag{4.1}$$

$$d\tilde{X}_3(t) = Z(t)dN(t) \ , \tag{4.2}$$

where $\mathbf{W} = (W_1, W_2)'$ is the two dimensional (standard) Brownian motion vector, N is the Poisson process with intensity 10. We assume that N is independent of $\mathbf{W}$ and $Z = (Z(t))_{t\geq 0}$ is the jump sizes with $Z(t) \sim \mathrm{N}(0, 5^{-2})$. (See Cont and Tankov (2004) for the generation of jump processes.)
For the volatility process σ of the diffusion part, we set

$$d(\sigma_j(t))^2 = a_j(\mu_j - (\sigma_j(t))^2)dt + \kappa_j \sigma_j(t) dW^{\sigma_j}(t), \quad j = 1, 2, \tag{4.3}$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are independent, $\mathbf{W}^{\sigma} = (W^{\sigma_1}, W^{\sigma_2})'$ is the two dimensional (standard) Brownian motion vector, and $a_1 = 2$, $a_2 = 3$, $\mu_1 = 0.8$, $\mu_2 = 0.7$, $\kappa_1 = \kappa_2 = 0.5$, $\mathcal{E}[dW^{\sigma_j}(t) dW_j(t)] = \rho_j dt$, $\rho_1 = \rho_2 = -0.5$. In our simulation, we consider the following two models :

$$\text{Model 1} \ : \ \mathbf{Y}(t) = \mathbf{\Gamma}_1 (\tilde{X}_1(t), \tilde{X}_2(t))' + \mathbf{v}(t) \tag{4.4}$$

and

$$\text{Model 2} \ : \ \mathbf{Y}(t) = \mathbf{\Gamma}_1 (\tilde{X}_1(t), \tilde{X}_2(t))' + \mathbf{\Gamma}_2 \tilde{X}_3(t) + \mathbf{v}(t). \tag{4.5}$$

Here we denote the coefficients matrices ($p \times 2$ and $p \times 1$, respectively) as $\mathbf{\Gamma}_1 = (\boldsymbol{\gamma}_1^{(1)}, \ldots, \boldsymbol{\gamma}_1^{(p)})'$, where $\boldsymbol{\gamma}_1^{(j)'} = (\boldsymbol{\gamma}_{1,1}^{(j)}, \boldsymbol{\gamma}_{1,2}^{(j)})$, $\mathbf{\Gamma}_2 = (\boldsymbol{\gamma}_2^{(1)}, \ldots, \boldsymbol{\gamma}_2^{(p)})'$and they are sampled as $\boldsymbol{\gamma}_{1,1}^{(j)} \sim U([0.25, 1.75])$, $\boldsymbol{\gamma}_{1,2}^{(j)} \sim U([0.1, 0.25])$ and $\boldsymbol{\gamma}_2^{(j)} \sim U([0.25, 1.75])$, $j = 1, \ldots, p$. The observation vectors are

$$\mathbf{Y}(t_i^n) = (Y_1(t_i^n), \ldots, Y_p(t_i^n))',\ i = 1, \ldots, n$$

and we set $t_i^n = \frac{i}{n}$ ($i = 1, \cdots, n$) and $\Delta = \Delta_n = 1/n$. As the market microstructure noise vectors, we set $\mathbf{v}(t_i^n) = (v_1(t_i^n), \ldots, v_p(t_i^n))'$, and use independent Gaussian noises for each component, that is,

$$(v_1(t_i^n), \ldots, v_p(t_i^n))' \sim i.i.d. N_p(\mathbf{0}, c\mathbf{I}_p) \ \ (i = 1, \ldots, n) \ ,$$

with a pre-specified value c. In all simulations, we set $p = 10$ and $n = 20000$. We note that Models 1 and 2 can be seen as special cases, which investigated in Li, Tauchen and Todorov (2017a, b).

4.2 Simulation Results

Let N be the number of Monte Carlo iterations. We plotted the mean value of the eigenvalues of the SIML estimator for the quadratic variation in Figures 4.1 and 4.2. To compute the SIML estimators, we set $m_n = 2 \times [n^{0.646}]$ ($= 2 \times 600$) and $l_n = 1.5 m_n$.

$$\hat{\mathbf{\Sigma}}_x = \frac{1}{m_n} \sum_{j=1}^{m_n} \mathbf{z}_j \mathbf{z}_j' \ , \ \ \hat{\mathbf{\Sigma}}_v = \frac{1}{l_n} \sum_{j=n-l_n+1}^{n} a_{jn}^{-1} \mathbf{z}_j \mathbf{z}_j', \tag{4.6}$$

where $a_{kn} = 4n \sin^2 \left[\frac{\pi}{2} \left(\frac{2k-1}{2n+1} \right) \right]$.

In the following Figures, we set $0 \leq \hat{\lambda}_1 \leq \ldots \leq \hat{\lambda}_p$ are eigenvalues of $\hat{\mathbf{\Sigma}}_v^{-1} \hat{\mathbf{\Sigma}}_x$.

In Model-1 and Model-2 we have 10 dimensions observation vectors ($p = 10$). Model-1 has two factor of diffusion type ($q_x = 2, r_x = 8$) while Model-2 has two diffusion type factors and one jumps factor ($q_x = 3, r_x = 7$). Figures 4.1-4.2 show that the estimated characteristic roots reflect the true rank of hidden stochastic process. Figures 4.3-4.4 show the distributions of the test statistic we are developed in this paper. In the first case (Figure 4.3) there is a clear indication that there are two non-zero characteristic roots, which are far from zero by any meaningful criterion while in the second case (Figure 4.4) there is a clear indication that there are three non-zero characteristic roots, which are far from zero with any meaningful criterion. It seems that our method of evaluating the rank condition of hidden volatility factors based on the characteristic roots and the SIML estimation of characteristic roots detects the number of factors properly in these two numerical simulations.

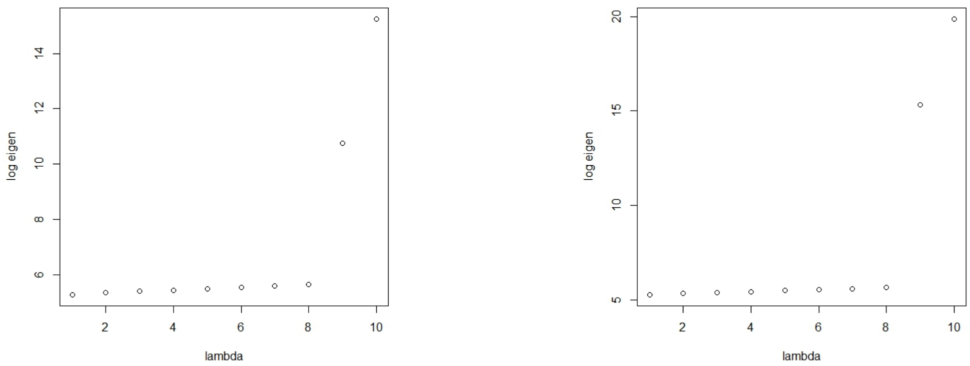

Figure 4.1 : Mean of estimated log characteristic roots (log eigenvalues) of Model 1 when $c = 10^{-6}$(left) and $c = 10^{-8}$(right). We set $\Delta = 1/20000$, $m_n = 2 \times [n^{0.646}]$ $(= 2 \times 600)$ and $l_n = 1.5m_n$. The number of Monte Carlo iteration is 300.

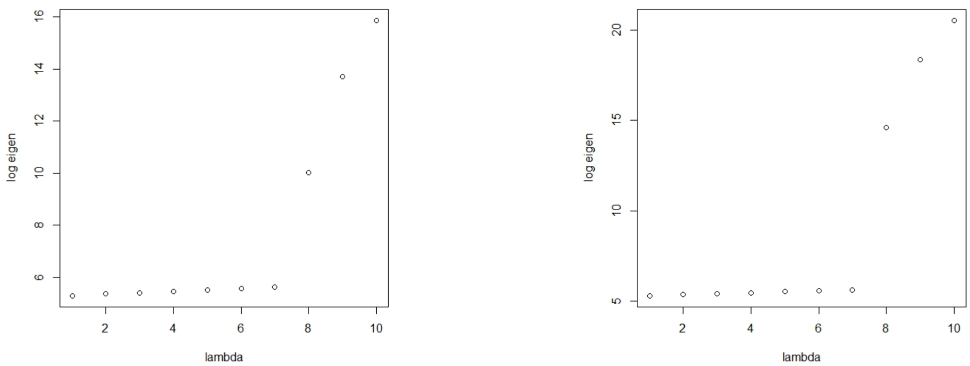

Figure 4.2 : Mean of estimated log characteristic roots (log eigenvalues) of Model 2 when $c = 10^{-6}$(left) and $c = 10^{-8}$(right). We set $\Delta = 1/20000$, $m_n = 2 \times [n^{0.646}]$ $(= 2 \times 600)$ and $l_n = 1.5m_n$. The number of Monte Carlo iteration is 300.

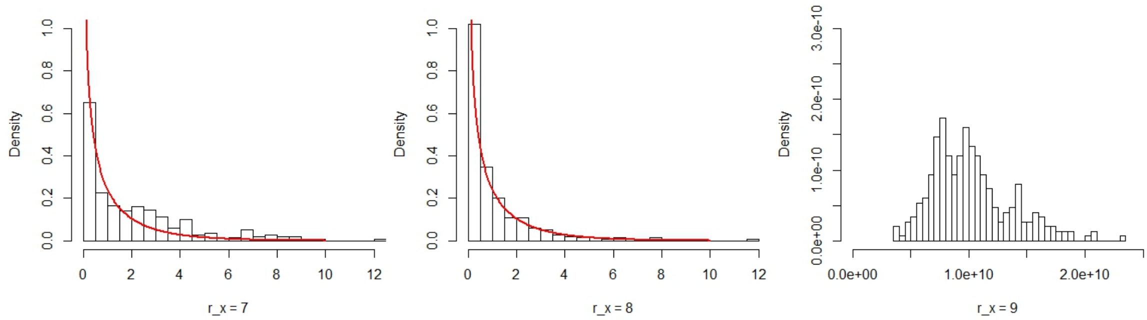

Figure 4.3 : Empirical distributions of test statistic $T_n(r_0)$ of Model 1 when $r_0 = 7$(left), $r_0 = 8$(center) and $r_0 = 9$(right) when $r_x = 8$. We set $\Delta = 1/20000$, $c = 10^{-8}$, $m_n = 2 \times [n^{0.646}]$ $(= 2 \times 600)$ and $l_n = 1.5m_n$. The number of Monte Carlo iteration is 300. The red line is the density of the chi square distribution with 1 degree of freedom.

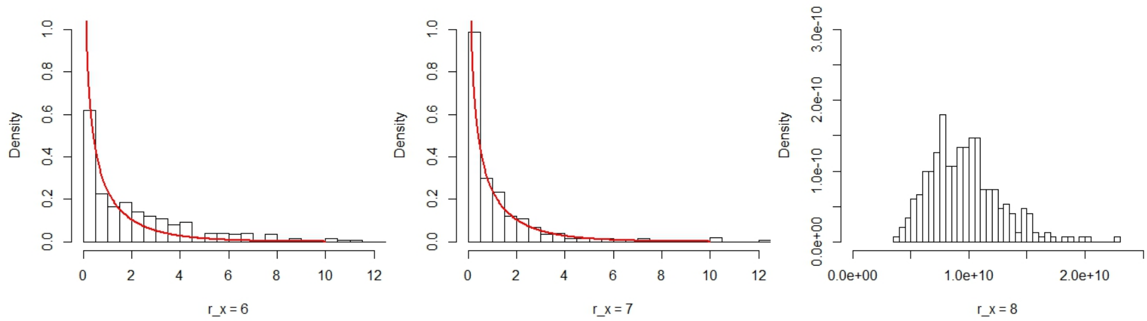

Figure 4.4 : Empirical distributions of test statistic $T_n(r_0)$ of Model 2 when $r_0 = 6$(left), $r_0 = 7$(center) and $r_0 = 8$(right) when $r_x = 7$. We set $\Delta = 1/20000$, $c = 10^{-8}$, $m_n = 2 \times [n^{0.646}]$ $(= 2 \times 600)$ and $l_n = 1.5m_n$. The number of Monte Carlo iteration is 300. The red line is the density of the chi square distribution with 1 degree of freedom.

5. An Empirical Example

In this section we report one empirical data analysis by using the proposed method developed in the previous sections. It is no more than an illustration on our proposed method in this paper. We have used the intra-day observations of top five financial stocks (Mitsubishi UFJ Financial Group, Inc., Mizuho Financial Group, Inc., Nomura Holdings, Inc., Resona Holdings, Inc., and Sumitomo Mitsui Financial

Group, Inc.) traded in the Tokyo Stock Exchange (TSE) on January 25 in 2016, which may be regarded as a typical one day. We have picked 5 major financial stocks listed at TSE because they are actively traded in each day with high liquidity. Hence we do not have serious disturbing effects due to actual non-synchronous trading process in TSE market. We sub-sampled returns of each asset every 1 second ($\Delta = n^{-1} = 1/18000$) and we have taken the nearest trading (past) prices at every unit of time.

For the SIML estimation we have set $m_n = 2 \times [n^{0.51}]$ $(= 294)$ and $l_n = 1.5m_n$. Since all companies belong to the same market division (First Section) of TSE, it would be reasonable to expect that the number of factor of these assets is smaller than 5 (i.e. $q_x < 5$). Figure 5.1 shows the estimated eigenvalues of the quadratic variation of these stocks by using (3.10) and (3.11). In this example, we have two large eigenvalues while there are three smaller eigenvalues and two roots are dominant. Then we have the statistics $T_n(5) = 91.37832$, $T_n(4) = 40.41634$, $T_n(3) = 5.479696$ and $T_n(2) = 10.60642$. Therefore, at a significance level of 0.01, the null hypotheses $H_0 : r_x = 5$ and $H_0 : r_x = 4$ are rejected, but the null hypothesis $H_0 : r_x = 3$ is not rejected. In this example, there is a large root and the second larger root is much smaller than the the largest root, but cannot be ignored while other roots are small. It implies that the quadratic variation has two factors in the particular day.

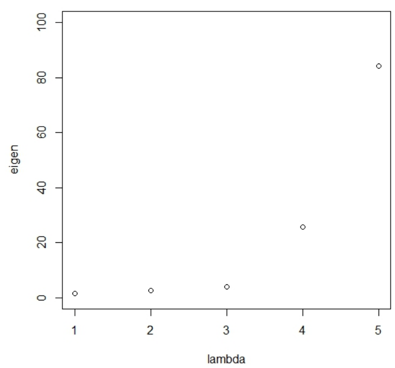

Figure 5.1 : Estimated eigenvalues. In this case, $\Delta = 1/18000$, $m_n = 2 \times [n^{0.51}]$ $(= 294)$ and $l_n = 1.5m_n$.

Figure 5.2 shows the intra-day movements of five stock prices in the TSE afternoon session of January 25, 2016. (There is a lunch break in Tokyo.) We set the same values for the starting prices because we want to focus on the volatility structure (or quadratic variation) of five asset prices. There is a strong evidence on two types of intra-day movements of stock prices, which is consistent with our data analysis reported.

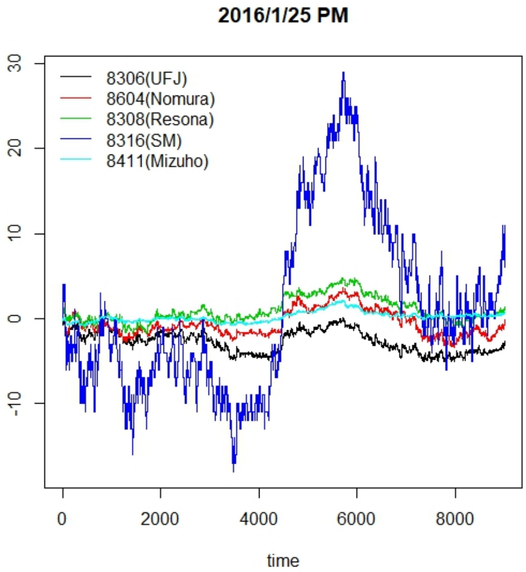

Figure 5.2 : Intra-day Movements of 5 Stock Prices at Tokyo (January 25, 2016)

6. Conclusions

In financial markets we usually have many assets traded and then it is important to find hidden (small number of) factors behind many assets. As we have shown in this paper, it is straight-forward to detect the number of hidden factors by using the SIML method when the true hidden stochastic process is the class of Itô semimartingales and there can be market microstructure noises. Our procedure is essentially the same as the standard statistical method in multivariate analysis except the fact that we have Itô semimartingales as the hidden state variables. We have derived the asymptotic distributions of characteristic roots and vectors, which are new. Then it is possible to develop the test statistics for the reduced rank condition, which has been developed in the standard statistical multivariate analysis. From our limited simulations and an empirical application, our approach works well in practical situations.

There can be possible extensions of our approach we have developed in this paper. Since the conditions in (3.1)-(3.3) are restrictive, it would be interesting to find the situations when we can lead to useful results. Also there are several unsolved problems remained. Since we need to choose m_n and l_n in the SIML testing in practice, which may be different from the problem of choosing m_n and l_n in the SIML estimation. The testing power of test procedure will be another unsolved problem although the trace statistic we used may be a natural choice of test procedure.

More importantly, the notable feature of our approach under (3.1)-(3.3) is that the method is simple and our result is useful. We have a promising experiment even in the case when the dimension of observations is not small, say 100. The number of asset prices in actual financial markets is large in practical financial risk management. There can be a number of empirical applications. These problems are currently under investigation.

References

[1] Aït-Sahalia, Y. and J. Jacod (2014), *High-Frequency Financial Econometrics*, Princeton University Press.

[2] Aït-Sahalia, Y. and D. Xiu (2017a), "Using Principal Component Analysis to Estimate a High Dimensional Factor Model with High-Frequency Data," *J. Econometrics*, 201, 384-399.

[3] Aït-Sahalia, Y. and D. Xiu (2017b), "Principal Component Analysis of High Frequency Data," To appear in *J. Amer. Statist. Assoc.* DOI: 10.1080/01621459.2017.1401542

[4] Anderson, T.W. (1984), "Estimating Linear Statistical Relationships," *Ann. Statist.*, 12, 1-45.

[5] Anderson, T.W. (2003), *An Introduction to Multivariate Statistical Analysis*, 3rd Edition, Wiley.

[6] Bibinger, M. and L. Winkelmann (2015), "Econometrics of Co-jumps in High-Frequency Data with Noise," *J. Econometrics*, 184, 361-378.

[7] Bibinger, M. and L. Winkelmann (2018), "Common Price and Volatility Jumps in Noisy High-Frequency Data," *Electron. J. Statist.*, 12, 2018-2073.

[8] Christensen, K., M. Podolskij and M. Vetter (2013), "On Covariation Estimation for Multivariate Continuous Itô Semimartingales with Noise in Non-Synchronous Observation Schemes," *J. Multivariate Anal.*, 120, 59-84.

[9] Cont, R. and P. Tankov (2004), *Financial Modeling with Jump Process*, Chapman & Hall.

[10] Fissler, T. and M. Podolskij (2017), "Testing the Maximal Rank of the Volatility Process for Continuous Diffusions Observed with Noise," *Bernoulli*, 23, 3021-3066.

[11] Häusler, E. and H. Luschgy (2015), *Stable Convergence and Stable Limit Theorems.* Springer.

[12] Ikeda, N. and S. Watanabe (1989), *Stochastic Differential Equations and Diffusion Processes*, 2nd Edition, North-Holland.

[13] Jacod, J. (2008). "Asymptotic Properties of Realized Power Variations and Related Functionals of Semimartingales," *Stochastic Process. Appl.*,118, 517-559.

[14] Jacod, J. and M. Podolskij (2013), "A Test for the Rank of the Volatility Process: the Random Perturbation Approach," *Ann. Statist.*, 41, 2391-2427.

[15] Jacod, J. and P. Protter (2012), *Discretization of Processes*, Springer.

[16] Jacod, J. and V. Todorov (2009), "Testing for Common Arrivals of Jumps for Discretely Observed Multidimensional Processes," *Ann. Statist.*, 37, 1792-1838.

[17] Jacod, J. and V. Todorov (2010), "Do Price and Volatility Jump Together?" *Ann. Appl. Probab.*, 20, 1425-1469.

[18] Kunitomo, N. and S. Sato (2013), "Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise," *North American J. Economics and Finance*, 26, 282-309.

[19] Kunitomo, N., Sato and D. Kurisu (2018), *Separating Information Maximum Likelihood Estimation for High Frequency Financial Data*, Springer.

[20] Kurisu, D. (2018), "Power Variations and Testing for Co-jumps : The Small Noise Approach," *Scand. J. Statist.*, 45, 2018, 482-512.

[21] Li, J., G. Tauchen and V. Todorov (2017a), "Jump Regressions," *Econometrica*, 85, 173-195.

[22] Li, J., G. Tauchen and V. Todorov (2017b), "Adaptive Estimation of Continuous-Time Regression Models using High-Frequency Data," *J. Econometrics*, 200, 36-47.

[23] Malliavin, P. and M. E. Mancino (2002), "Fourier Series Method for Measurement of Multivariate Volatilities," *Finance and Stochastics*, 6, 49-61.

[24] Malliavin, P. and M. E. Mancino (2009), "A Fourier Transform Method for Nonparametric Estimation of Multivariate Volatility," *Ann. Statist.*, 37, 1993-2010.

[25] Mancino, M. E. and S. Stanfelici (2008), "Robustness of Fourier Estimator of Integrated Volatility in the Presence of Microstructure Noise," *Comput. Statist. Data Anal.*, 52, 2966-2989.

[26] Mancino, Maria Elvira, Recchioni, Maria Cristina, and Sanfelici, Simona (2017), "Fourier-Malliavin Volatility Estimation Theory and Practice," Springer.

[27] Robin, J-M. and Smith, R.J. (2000), “Test of Rank,” *Econometric Theory*, 16, 151-175.

APPENDIX : On Mathematical Derivations

In this Appendix we give some mathematical details used in the previous sections. First, we give an outline of the Derivation of Theorem 2.1, which is supplementary to Chapter 5 of Kunitomo et al. (2018). Second, we give the derivation of Lemma 3.1 and Lemma A.1, which are elementary but useful for the derivation of Corollaries in Section 3.

(i) **An Outline of the Derivation of Theorem 2.1** :
(**Step 1**) : We give an intuitive argument for the above results. The basic method of proof is essentially the same to the one based on the notations, derivations and their extensions given in Chapter 5 of Kunitomo et al. (2018). We first consider the case when $\mathbf{X}$ has no jumps.
Let $\mathbf{x}_k^{*'} = (x_{kj}^*)$ and $v_k^{*'} = (v_{kj}^*)$ $(k = 1, \cdots, n)$ be the k-th row vector elements of $n \times p$ matrices

(A.1) $$\mathbf{X}_n^* = \mathbf{K}_n(\mathbf{X}_n - \bar{\mathbf{Y}}_0)\ ,\ \mathbf{V}_n^* = \mathbf{K}_n\mathbf{V}_n ,$$

respectively, where we denote $\mathbf{X}_n = (\mathbf{x}_k^{'}) = (x_{kg})$, $\mathbf{V}_n = (\mathbf{v}_k^{'}) = (v_{kg})$, $\mathbf{Z}_n = (\mathbf{z}_k^{'})$ $(= (z_{kg}))$ are $n \times p$ matrices with the notations $z_{kg} = x_{kg}^* + v_{kg}^*$ in Section 3. We write z_{kg} as the g-th component of $\mathbf{z}_k$ $(k = 1, \cdots, n; g = 1, \cdots, p)$. We use the decomposition of $\mathbf{z}_k^{'} = (z_{kg})$ for investigating the asymptotic distribution of $\sqrt{m_n}[\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x] = (\sqrt{m_n}(\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)})_{gh})$ for $g, h = 1, \cdots, p$. We decompose

(A.2) $$\begin{aligned} & \sqrt{m_n}\left[\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x\right] \\ = & \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k\mathbf{z}_k^{'} - \mathbf{\Sigma}_x\right] \\ = & \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{x}_k^*\mathbf{x}_k^{*'} - \mathbf{\Sigma}_x\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}[\mathbf{v}_k^*\mathbf{v}_k^{*'}] \\ & + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{v}_k^*\mathbf{z}_k^{*'} - \mathcal{E}[\mathbf{v}_k^*\mathbf{v}_k^{*'}]\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{x}_k^*\mathbf{v}_k^{*'} + \mathbf{v}_k^*\mathbf{x}_k^{*'}\right]\ . \end{aligned}$$

Then we can investigate the conditions that three terms except the first one are $o_p(1)$. When these conditions are satisfied, we could estimate the variance and covariance of the underlying processes consistently as if there were no noise terms because other terms can be ignored asymptotically as $n \to \infty$.
Let $\mathbf{b}_k = (b_{kj}) = h_n^{-1/2}\mathbf{e}_k^{'}\mathbf{P}_n\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0, \cdots, 1, 0, \cdots)$ be an $n \times 1$ vector. We write $v_{kg}^* = \sum_{j=1}^n b_{kj}v_{jg}$ for the noise part and use the relation

(A.3) $$h_n^{-1}(\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'})_{k,k'} = \delta(k, k')4n\sin^2[\frac{\pi}{2n+1}(k - \frac{1}{2})]\ .$$

Then because we have $\sum_{j=1}^{n} b_{kj} b_{k'j} = \delta(k,k^{'}) a_{kn}$ and $\mathbf{\Sigma}_v$ is bounded, it is straightforward to find K_1 (a constant) such that

$$\text{(A.4)} \qquad \mathcal{E}[(v_{kg}^*)]^2 = \mathcal{E}[\sum_{i=1}^{n} b_{ki} v_{ig} \sum_{j=1}^{n} b_{kj} v_{jg}] \leq K_1 \times a_{kn} \ .$$

As Chapter 5 of Kunitomo at al. (2018), we have

$$\frac{1}{m_n} \sum_{k=1}^{m_n} a_{kn} = \frac{1}{m_n} 2n \sum_{k=1}^{m_n} \left[1 - \cos(\pi \frac{2k-1}{2n+1}) \right] = O(\frac{m_n^2}{n})$$

and the second term becomes

$$\text{(A.5)} \qquad \frac{1}{\sqrt{m_n}} \sum_{k=1}^{m_n} \mathcal{E}[v_{kn}^*]^2 \leq K_1 \frac{1}{\sqrt{m_n}} \sum_{k=1}^{m_n} a_{kn} = O(\frac{m_n^{5/2}}{n}) \ ,$$

which is $o(1)$ if we set α such that $0 < \alpha < 0.4$.
For the fourth term of (A.2),

$$\begin{aligned} \mathcal{E}\left[\frac{1}{\sqrt{m_n}} \sum_{k=1}^{m_n} x_{kg}^* v_{kg}^* \right]^2 &= \frac{1}{m_n} \sum_{k,k'=1}^{m_n} \mathcal{E}\left[x_{kg}^* x_{k',g}^* v_{kg}^* v_{k',g}^* \right] \\ &= O(\frac{m_n^2}{n}) \ . \end{aligned}$$

We set

$$s_{jk} = \cos[\frac{2\pi}{2n+1}(j - \frac{1}{2})(k - \frac{1}{2})]$$

for $j, k = 1, 2, \cdots, n$ and then we have the relation

$$|\sum_{j=1}^{n} s_{jk} s_{j,k'}| \leq [\sum_{j=1}^{n} s_{jk}^2] = \frac{n}{2} + \frac{1}{4} \ \text{ for any } k \geq 1 \ .$$

For the third term of (A.2), we need to consider the variance of

$$(v_{kg}^*)^2 - \mathcal{E}[(v_{kg}^*)^2] = \sum_{j,j'=1}^{n} b_{kj} b_{k,j'} \left[v_{jg} v_{j',g} - \mathcal{E}(v_{jg} v_{j',g}) \right] \ .$$

Then by using the assumption on the existence of the fourth order moments of market microstructure noise terms, we can find a positive constant K_2 such that

$$\begin{aligned}
&\mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}((v_{kg}^*)^2-\mathcal{E}[(v_{kg}^*)^2])\right]^2 \\
&= \frac{1}{m_n}\sum_{k_1,k_2=1}^{m_n}\mathcal{E}\left[\sum_{j_1,j_2,j_3,j_4=1}^{n} b_{k_1,j_1}b_{k_1,j_2}(v_{j_1,g}v_{j_2,g}-\mathcal{E}(v_{j_1,g}v_{j_2,g}))\right. \\
&\qquad\qquad \times b_{k_3,j_3}b_{k_4,j_4}(v_{j_3,g}v_{j_4,g}-\mathcal{E}(v_{j_3,g}v_{j_4,g}))] \\
&\leq K_2\frac{1}{m_n}[\sum_{k=1}^{m_n}a_{kn}]^2 \\
&= O(\frac{1}{m_n}\times(\frac{m_n^3}{n})^2)\ ,
\end{aligned}$$

which is $O(m_n^5/n^2)$. Thus the third term of (A.2) is negligible if we set α such that $0<\alpha<0.4$.

The second step is to give the asymptotic variance-covariance of the first term, that is,

$$\text{(A.6)}\qquad \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{x}_k^*\mathbf{x}_k^{*'}-\mathbf{\Sigma}_x\right]$$

because it is of the order $O_p(1)$. We can write

$$\begin{aligned}
&\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{x}_k^*\mathbf{x}_k^{*'} \\
&= \frac{1}{m_n}(\frac{2}{n+\frac{1}{2}})\sum_{k=1}^{m_n}[\sum_{i=1}^{n}\mathbf{r}_i\cos[\pi(\frac{2k-1}{2n+1})(i-\frac{1}{2})]\sum_{j=1}^{n}\mathbf{r}_j^{'}\cos[\pi(\frac{2k-1}{2n+1})(j-\frac{1}{2})]] \\
&= \sum_{i=1}^{n}c_{ii}\mathbf{r}_i\mathbf{r}_i^{'}+\sum_{i\neq j}c_{ij}\mathbf{r}_i\mathbf{r}_j\ ,
\end{aligned}$$

where $\mathbf{r}_i=\mathbf{x}_i-\mathbf{x}_{i-1}=(r_{ig})$ and

$$\begin{aligned}
c_{ii} &= (\frac{2n}{2n+1})\left[1+\frac{1}{m}\frac{\sin 2\pi m(\frac{i-1/2}{2n+1})}{\sin(\pi\frac{i-1/2}{2n+1})}\right]\ , \\
c_{ij} &= \frac{1}{2m}(\frac{2n}{2n+1})\left[\frac{\sin 2\pi m(\frac{i+j-1}{2n+1})}{\sin(\pi\frac{i+j-1}{2n+1})}+\frac{\sin 2\pi m(\frac{j-i}{2n+1})}{\sin(\pi\frac{j-i}{2n+1})}\right]\quad (i\neq j)\ .
\end{aligned}$$

Then Kunitomo et al. (2018) have shown that when $\mathbf{r}_i$ are i.i.d. random variables

$$\text{(A.7)}\qquad \frac{\sqrt{m_n}}{n}\sum_{i=1}^{n}\left[\mathbf{r}_i\mathbf{r}_i^{'}-\mathbf{\Sigma}_x+(c_{ii}-1)\mathbf{r}_i\mathbf{r}_i^{'}\right]=o_p(1)$$

and by re-writing (A.2) as

$$\text{(A.8)} \qquad \frac{\sqrt{m_n}}{n}\sum_{i=1}^{n}\left[c_{ii}\ \mathbf{r}_i\mathbf{r}_i^{'} - \mathbf{\Sigma}_x\right] + \frac{\sqrt{m_n}}{n}\sum_{i\neq j}^{n}\left[c_{ij}\ \mathbf{r}_i\mathbf{r}_j^{'}\right]$$

we need to evaluate the asymptotic variance of its second term. Kunitomo and Sato (2013) have also shown that if there is no jump terms, then the variance of the limiting distribution of the (g,g)-th element of the limiting variance-covariance matrix is the limit of

$$V_n(g,g) = 2\sum_{i,j=1}^{n} m_n c_{ij}^2 \int_{t_{i-1}^n}^{t_i^n} \boldsymbol{c}_{gg}(s)ds \int_{t_{j-1}^n}^{t_j^n} \boldsymbol{c}_{gg}(s)ds.$$

The resulting arguments of the derivations are the result of straightforward calculations and lengthy, but the final form becomes simple. From the Lemma 3 in Kunitomo and Sato (2013), we have that

$$\text{(A.9)} \qquad \sum_{i,j=1}^{n} c_{ij}^2 = \frac{4}{m_n}\left[\frac{n}{2}+\frac{1}{4}\right]^2 .$$

Then following a similar argument of Lemma 7 in Kunitomo and Sato (2013), we have that

$$\text{(A.10)} \qquad V_n(g,g) \xrightarrow{p} V(g,g) = 2\int_0^1 \boldsymbol{c}_{gg}^2(s)ds$$

as $n \to \infty$. For the covariances of the hidden terms when there is no jump part, we have the similar arguments and obtain the corresponding asymptotic variance as

$$\text{(A.11)} \qquad V(g,h) = \int_0^1 \left[\boldsymbol{c}_{gg}(s)\boldsymbol{c}_{hh}(s) + \boldsymbol{c}_{gh}^2(s)\right] ds\ .$$

For the more general case when $\mathbf{X}$ have jumps, we can also show that the effect of noise terms is asymptotically negligible by using the same argument in this step (See also Step 3). Then, as with the proof of the stable convergence of the SIML estimator in this step, it suffice to follow the proof of the stable convergence of the realized volatility (See Chapter 5 of Jacod and Protter (2012) for details of the proof.) to show our result for the more general case.

(**Step 2**) : In this step, we illustrate the basic arguments on jump-diffusion processes when there is no market microstructure noise. (It may make the underlying arguments in a clear manner.) Let the (true) return vector process $\mathbf{r}_i = \mathbf{X}(t_i^n) - \mathbf{X}(t_{i-1}^n)$ $(= (r_{gi})$ $(i = 1, \cdots, n; g = 1, \cdots, p)$ in the decomposition of $\mathbf{Z}_n^{(1)}$.

Let $p \times p$ matrices $\mathbf{A}_n = (A_n(gh))$ and $\mathbf{A} = (A(gh))$ be

$$\text{(A.12)} \qquad A_n(gh) = \sum_{i=1}^{n} (X_g(t_i^n) - X_g(t_{i-1}^n))(X_h(t_i^n) - X_h(t_{i-1}^n))$$

and

$$\text{(A.13)} \qquad A(gh) = \sum_{i=1}^{n} \left(\int_{t_{i-1}^n}^{t_i^n} \boldsymbol{c}_{gh}(s)ds + \sum_{t_{i-1}^n < s \leq t_i^n} \Delta X_g(s) \Delta X_h(s) \right) ,$$

where we set $\mathbf{X}(0) = (X_g(0))$, $\mathbf{X}_i = (X_g(t_i^n))$, $\Delta \mathbf{X}_i = (\Delta X_g(t_i^n))$, and $\boldsymbol{c}(s) = (\boldsymbol{c}_{gh}(s))$. Let $q = 1$ also be the number of Brownian motions for the resulting notational simplicity. By using the basic arguments of stochastic processes, we can approximate

$$\begin{aligned} & \sqrt{n}\left[A_n(gh) - A(gh)\right] \\ \sim \ & \sqrt{n} \left\{ \sum_{i=1}^{n} \left[\boldsymbol{\sigma}_{g1}(t_{i-1}^n)(W_i - W_{i-1}) + \sum_{t_{i-1}^n < s \leq t_i^n} \Delta X_g(s) \right] \right. \\ & \left. \times \left[\boldsymbol{\sigma}_{h1}(t_{i-1}^n)(W_i - W_{i-1}) + \sum_{t_{i-1}^n < s \leq t_i^n} \Delta X_h(s) \right] \right\} - A(gh) \ , \end{aligned}$$

where $\boldsymbol{\sigma}(s) = (\boldsymbol{\sigma}_{gh}(s))$ and $W_i = W(t_i^n)$. Then the above quantity can be asymptotically decomposed into

$$\begin{aligned} \text{(A.14)} \qquad & \sqrt{n} \ \left[\sum_{i=1}^{n} \boldsymbol{\sigma}_{g1}(t_{i-1}^n)\boldsymbol{\sigma}_{h1}(t_{i-1}^n)(W_i - W_{i-1})^2 - \int_0^1 \boldsymbol{c}_{gh}(s)ds \right] \\ & + \sqrt{n} \left[\sum_{i=1}^{n} \boldsymbol{\sigma}_{g1}(t_{i-1}^n)(W_i - W_{i-1}) \sum_{t_{i-1}^n < s \leq t_i^n} \Delta X_h(s) \right] \\ & + \sqrt{n} \left[\sum_{i=1}^{n} \boldsymbol{\sigma}_{h1}(t_{i-1}^n)(W_i - W_{i-1}) \sum_{t_{i-1}^n < s \leq t_i^n} \Delta X_g(s) \right] . \\ = \ & (I) + (II) + (III) \ \ \text{(say)}. \end{aligned}$$

We denote $(I), (II)$ and (III) in each terms of the last equality and we can evaluate their asymptotic distributions. The resulting asymptotic variance is V_{gh} when $p = 1$. When we have market microstructure noise terms, we apply the similar arguments as (A.6)-(A.11) and then we have some additional jump terms in the limiting distribution.

(**Step 3**) : As the 3rd step, we apply the martingale convergence theorems as the method of evaluating the asymptotic distributions of the SIML estimator used in Chapter 5 when there are market microstructure noises. For $g, h = 1, \cdots, p$, let

$$U_n = \sum_{j=2}^{n} [2 \sum_{i=1}^{j-1} \sqrt{m_n} c_{ij} r_{ig}] r_{jh}, \tag{A.15}$$

where $c_{ij} = (2/m) \sum_{k=1}^{m} s_{ik} s_{jk} \ \ (i, j = 1, \cdots, n)$ and

$$s_{jk} = \cos\left[\frac{2\pi}{2n+1}(j - \frac{1}{2})(k - \frac{1}{2})\right] . \tag{A.16}$$

Then the variance of the limiting random variables can be calculated as the variance of three terms in (A.15) except the factor $\sqrt{m_n}$ instead of $\sqrt{n}$ as stated. In fact, we need to evaluate the effects of c_{ij} $(i, j = 1, \cdots, n)$ as in Chapter 5 of Kunitomo et al. (2018), which are omitted. The resulting calculations in the general case with $p \geq 1, q \geq 1$ become tedious, but they are straightforward as we have given the details for the diffusion cases in Chapter 5 of Kunitomo et al. (2018).

(**Step 4**) : Finally, we use the stable-convergence in law explained by Jacod and Protter (2012), for instance, when the elements of $\mathbf{\Sigma}_x$ are random. The essential arguments are the same and thus we have omitted their details.
(**Q.E.D**)

(ii) **Derivation of Lemma 3.2** : Let $p \times 1$ vectors $\mathbf{w}_k = (w_{gk}) = \mathbf{V}_n^{'} \mathbf{P}_n^{'} \mathbf{C}_n^{'-1} \mathbf{e}_k$ where $\mathbf{V}_n = (\mathbf{v}_k^{'}) = (v_{kg})$ and $\mathbf{e}_k = (0, \cdots, 0, 1, 0, \cdots, 0)^{'}$ $(k = 1, \cdots, n)$ is the $n \times 1$ unit vectors.
Then by using (2.7) and tedious but straight-forward calculations as illustrated in Chapter 5 of Kunitomo et al. (2018), $\mathcal{E}[w_{gk}] = 0$,

$$\mathcal{E}[w_{gk} w_{hk}] = \sigma_{gh}^{(v)} a_{kn} , \tag{A.17}$$

and

$$\mathcal{E}[w_{gk} w_{hk} w_{g',k} w_{h',k}] = \left[\sigma_{gh}^{(v)} \sigma_{g',h'}^{(v)} + \sigma_{gg'}^{(v)} \sigma_{h,h'}^{(v)} + \sigma_{g,h'}^{(v)} \sigma_{h,g'}^{(v)}\right] [a_{kn}]^2 + o(a_{kn}^2) . \tag{A.18}$$

(It can be shown that the effects of diagonal elements in the quadtratic forms can be negligible if $1 > \alpha > 1/2$.)

(iii) Finally, for the sake of convenience, we summarize an important, but simple calculation of variances used in Corollaries 3.3 and 3.4 as a lemma.

Lemma A.1 : Assume the normality of market microstructure noises. Let a $p \times p$ matrix

$$\mathbf{U}^* = (U_{ij}^*) = (\mathbf{B}^{'} \mathbf{H} \mathbf{B})^{-1} (\sqrt{\frac{m}{a_m(2)}}) \mathbf{B}^{'} (\mathbf{G}_m - \mathbf{\Sigma}_m) \mathbf{B} , \tag{A.19}$$

and $\mathbf{A}^{-1} = (a^{ij}) = (\mathbf{B}^{'}\mathbf{H}\mathbf{B})^{-1}$ and $\mathbf{\Omega}_v = (\omega_{ij}) = (\mathbf{B}^{'}\mathbf{\Sigma}_v\mathbf{B})$. We denote X^* as the limiting random matrix of $\sqrt{\frac{m}{a_m(2)}}\mathbf{B}^{'}(\mathbf{G}_m - \mathbf{\Sigma}_m)\mathbf{B}$. Then the asymptotic variance is given by

$$
\begin{aligned}
\text{(A.20)} \qquad \mathbf{AV}(U_{ii}^*) &= \mathcal{E}[\sum_{j=1}^{p} a^{ij} X_{ji}^* \sum_{j^{'}=1}^{p} a^{ij^{'}} X_{j^{'}i}^*] \\
&= \sum_{j,j^{'}=1}^{p} a^{ij} a^{ij^{'}} Cov(X_{ji}^*, X_{j^{'},i}^*) \\
&= \sum_{j,j^{'}=1}^{p} a^{ij} a^{ij^{'}} [\omega_{jj^{'}}\omega_{ii} + \omega_{ji}\omega_{j^{'}i}] \ .
\end{aligned}
$$

When $\mathbf{H} = \mathbf{\Omega}_v$, it becomes

$$
\omega_{ii} \sum_{j=j^{'}=1}^{p} \omega^{ij}\omega^{ij^{'}}\omega_{jj^{'}} + \sum_{j,j^{'}=1}^{p} \omega^{ij}\omega_{ji}\omega^{ij^{'}}\omega_{j^{'}i} = \omega_{ii}\omega^{ii} + 1 \ .
$$

Also we find the asymptotic covariances as

$$
\begin{aligned}
\text{(A.21)} \qquad \mathbf{ACov}(U_{ii}^*, U_{kk}^*) &= \mathcal{E}[\sum_{j=1}^{p} a^{ij} X_{ji} \sum_{j^{'}=1}^{p} a^{kj^{'}} X_{j^{'}k}^*] \\
&= \sum_{j,j^{'}=1}^{p} a^{ij} a^{kj^{'}} \mathbf{Cov}(X_{ji}^*, X_{j^{'},k}^*) \\
&= \sum_{j,j^{'}=1}^{p} a^{ij} a^{kj^{'}} [\omega_{jj^{'}}\omega_{ik} + \omega_{ji}\omega_{j^{'}k}] \\
&= \sum_{j^{'}=1}^{p} [\sum_{j=1}^{p} a^{ij}\omega_{jj^{'}}] a^{kj^{'}}\omega_{ik} + \sum_{j,j^{'}=1}^{p} a^{ij}\omega_{ji} a^{kj^{'}}\omega_{j^{'}k} \ .
\end{aligned}
$$

When $\mathbf{H} = \mathbf{\Omega}_v$, it becomes

$$
\sum_{j^{'}=1}^{p} \omega^{kj^{'}}\omega_{ik} + 1 = \omega^{ki}\omega_{ik} + 1 \ .
$$