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Local SIML Estimation of Some Brownian and Jump Functionals under Market Microstructure Noise *

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Abstract

To estimate Brownian and jump functionals from high-frequency financial data under market microstructure noise, we introduce a new local estimation method of the integrated volatility and higher-order variation of Ito's semi-martingale processes. Although it is straight-forward to extend the realized volatility (RV) estimation to more general cases without micro-market noise, it may not be straight-forward to estimate Brownian and Jump functionals in the presence of micro-market noise. We develop the local SIML (LSIML) method, which is an extension of the separating information maximum likelihood (SIML) method proposed by Kunitomo, Sato and Kurisu (2018) and Kunitomo and Kurisu (2021). The new method is simple and the LSIML estimator has some desirable asymptotic properties as well as reasonable finite sample properties.

Key Words

High-Frequency Data Analysis, Integrated Volatility, Separating Information Maximum Likelihood (SIML), Higher-Order Brownian and Jump Functionals, Stable Convergence, Local SIML Estimation

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1. Introduction

In financial econometrics, several statistical methods have been proposed to estimate the integrated volatility and co-volatility from high-frequency data. The integrated volatility is one type of Brownian functionals and the realized volatility (RV) estimate has been often used when there does not exist any micro-market noise and the underlying diffusion process is directly observed. The asymptotic distribution of the RV estimator depends on the fourth-order integrated Brownian functional and then we need to estimate the fourth-order integrated moments to make statistical inference on the integrated volatility when the number of observations increases in a fixed interval. However, it has been known that the RV estimator is quite sensitive to the presence of micro-market noise in high-frequency financial data. Then several statistical methods have been proposed to estimate the integrated volatility and co-volatility. See Zhang, L., Per A. Mykland, and Y. Ait-Sahalia (2005), Bandorff-Nielsen, Hansen, Lunde and Shepard (2008), Jacod, J., Y. L., Per A. Mykland, M. Podolskijc, and M. Vetter (2009), Ait-Sahalia and Jacod (2014) for the detail of recent developments of financial econometrics. In particular, Malliavin and Mancino (2009) have developed the Fourier series method while independently Kunitomo, Sato and Kurisu (2018), referred as KSK (2018), developed the SIML (separating information maximum likelihood) estimation. We shall use the latter formulation in this paper, which is closely related to the former method. (See Mancino, Recchioni and Sanfelici (2017).)

When the market micro-structure noise cannot be ignored in high-frequency financial data, KSK (2018) have developed the SIML method for estimating the volatility and co-volatilities of security prices when the underlying processes are the class of diffusion processes. In this paper we extend the SIML method and develop the local SIML (LSIML) estimation method for estimating higher-order Brownian and Jump functionals such as the fourth-order integrated moments and the jump part of quadratic variation. The LSIML method was originally suggested in Chapter 8 of KSK (2018), but they did not give its detailed exposition. (To avoid the possible duplication of explanations on the SIML method, we will sometimes refer to the corresponding parts of KSK (2018), and Kunitomo and Kurisu (2021, referred as KK (2021).) The main motivation for developing the LSIML method is to improve the SIML method and to estimate some Brownian and jump functionals, which are general than the volatility and co-volatility. For instance, the fourth order integrated moments appear as the asymptotic variance of the limiting distribution of several estimation methods including the SIML estimation. Since the main purpose here is to propose the use of the LSIML method, we shall try to make our formulation not in the most general case, but concentrate on the simple cases, which make the results easy to be understood.

In this paper, we show that the LSIML method has some desirable asymptotic properties such as the consistency and asymptotic normality, and more importantly, there could be some application to the jump-diffusion cases. It also has reasonable finite sample properties, which are illustrated by several simulations. Since the LSIML method is a straightforward extension of the SIML estimation and it is quite simple, it will be useful for practical applications. Although there could be other methods for estimating higher-order Brownian functionals and jump functionals, the LSIML method has some merits such as its simplicity and desirable asymptotic properties.

In Section 2, we discuss the framework of estimation problem of some Brownian and jump functionals when we have market micro-structure noise in high-frequency financial data. In Section 3, we generalize the estimation problem of realized volatility and explain the method of local estimation in our study. Then, in Section 4, we propose the LSIML method under market micro-structure noise, which is a generalization of the SIML method originally developed by KSK (2018). In Section ,5 we investigate the asymptotic properties of the local SIML method such as consistency as well as the asymptotic normality. Then, in Section 6 we discuss the problem of selecting key parameters needed in the LSIML estimation method. In Section 7 we discuss the possible generalizations of our results in more general settings including the jump-diffusion and the multivariate models. In Section 8, we give some finite sample properties of the LSIML estimation based on a set of Monte Carlo simulation, and as an illustration we give an empirical result of high frequency data analysis. In Section 9, we give some concluding remarks. Mathematical details are given in the Appendix.

2. Estimation of Brownian and Jump Functionals

To see the essential feature of the local estimation method in this paper, we first consider the basic and simple time-varying cases when $p = 1$ (where p is the dimension). Let

$$Y(t_i^{(n)}) = X(t_i^{(n)}) + \epsilon_n v(t_i^{(n)}) \ \ (i = 1, \cdots, n) \tag{2.1}$$

be the (one dimensional) observed (log-)price at $t_i^{(n)}$ $(0 = t_0^{(n)} \leq t_1^{(n)} \leq \cdots \leq t_n^{(n)} = 1)$ and $v(t_i^{(n)})$ $(= v_i)$ be a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$ and $\mathbf{E}[v_i^2] = \sigma_v^2$ (> 0). We consider the case when

$$\epsilon_n = \frac{1}{n^{\delta}} \ , \tag{2.2}$$

where δ (≥ 0) is a constant. When $\delta = 0$, it is the market micro-structure noise model, while it is the high-frequency financial model without micro-market noise when $\delta = +\infty$. When $0 < \delta < +\infty$, it corresponds to the small-noise high-frequency model.

The underlying continuous-time Brownian martingale is given by

$$X(t) = X(0) + \int_0^t \sigma_s dB_s \quad (0 \leq s \leq t \leq 1) \ , \tag{2.3}$$

which is independent of $v(t_i^{(n)})$, σ_s is the (instantaneous) volatility function, which is bounded and Lipschitz-continuous and B_s is the standard Brownian motion.

Although it may be possible to apply the LSIML method to more general Itô semi-martingales, we first consider this situation because it gives the essential feature of the LSIML method in a simple way. (See Section 7 for its possible extensions.) We assume that when the volatility process is stochastic, it has a representation of Ito's

Brownian semi-martingale as

$$\sigma_t^2 = \sigma_0 + \int_0^t \mu_s^\sigma ds + \int_0^t \omega_s^\sigma dB_s^\sigma \ (0 \le s \le t \le 1) \ , \tag{2.4}$$

where B_s^σ is another Brownian motion, which may be correlated with B_s, and μ_s^σ and ω_s^σ are the drift and diffusion coefficients which are assumed to be deterministic, bounded and Lipschitz-continuous. They can be relaxed to some extent, but this paper will not pursue the generalization of the underlying process except some in Section 7.

The first problem of our interest is how to estimate Brownian functionals of the form

$$V(g, 2r) = \int_0^1 g(s)\sigma_s^{2r} ds \tag{2.5}$$

for any positive integer r and a known function $g(s)$ from a set of observations of $Y(t_i^n)$ $(i = 1, \cdots, n)$. We denote $V(2r) = V(g, 2r)$ when $g(s) = 1$ $(0 \le s \le 1)$ for convenience.

There are important examples of this type of Brownian functionals. An obvious example is the integrated volatility that corresponds to the case when $r = 1$.

Example 1 : When r=1, we have the integrated volatility, which is given by

$$V(2) = \int_0^1 \sigma_s^2 ds \ . \tag{2.6}$$

Example 2 : The asymptotic variance of the SIML estimator of integrated volatility $V(2)$ is given by

$$2V(4) = 2\int_0^1 \sigma_s^4 ds \ . \tag{2.7}$$

It should be noted that the estimation of $V(4)$ with $r = 2$ under market micro-structure noise is a non-trivial task. Zhang, L., Per A. Mykland, and Y. Ait-Sahalia (2005), Bandorff-Nielsen, Hansen, Lunde and Shepard (2008), Jacod, J., Y. L., Per A. Mykland, M. Podolskijc, and M. Vetter (2009), and Ait-Sahalia and Jacod (2014) discussed different estimation methods of the integrated quarticity ($\int_0^1 \sigma_u^4 du$), a higher-order Brownian functional with different $g(s)$ functions, but it seems that they are more complicated than the method developed herein.

One important class of continuous time processes is Ito's jump-diffusion process. A simple process may be expressed as

$$X(t) = X(0) + \int_0^t \sigma_s dB_s + \sum_{0 \le s \le t} \Delta X_s \quad (0 \le s \le t \le 1) \ , \tag{2.8}$$

where the jump term with $X_t - X_{t-} \neq 0$ and $\Delta X = X_t - X_{t-}$, which is independent of B_s (the Brownian motion). The term $\sum_{0 \le s \le t} \Delta X_s$ is formally defined as $\int_0^{t+} \int_X f_1(s, x, \cdot) N_p(dsdx) + \int_0^{+t} \int_X f_2(2, x.\cdot) \hat{N}_p(ds, dx)$ with measurable functions f_i $(i = 1, 2)$, Poisson random measure $N_p(dtdx)$ and the compensator $\hat{N}_p(dtdx)$. (See Chapter II of Ikeda and Watanabe (1989).)
In this study, we shall use the simple cases when the number of jumps is finite in $[0, 1]$, the sizes of jumps f_i $(i = 1, 2)$ are bounded, and we often treat as if they were fixed in the following analysis for the resulting simplicity.
Since we have the market micro-structure noise, which could be regarded as jump component at each observation, there is a difficulty to distinguish the jump term in the underlying the Ito's semi-martingales from the market micro-structure noise, or measurement error in the statistical terminology. In the general theory of stochastic processes, there can be small jumps as well as large jumps. (See Ikeda and Watanabe (1989), Jacod and Protter (2012) for the detail.) Our interpretation of jumps in the present study would be to detect large jumps of Ito's semi-martingales from noisy high-frequency observations.
In this situation, the fundamental quantity of the stochastic process is Quadratic Variation (QV), which is an extension of the integrated volatility, given by

$$V(2) = \int_0^1 \sigma_s^2 ds + \sum_{0 \le s \le 1} (\Delta X_s)^2 \ . \tag{2.9}$$

Example 3 : When we have jumps under market micro-structure noise, we may be interested in the continuous part of QV by

$$V_C(2) = \int_0^1 \sigma_s^2 ds \tag{2.10}$$

and the jump part of QV by

$$V_J(2) = \sum_{0\leq s\leq 1} (\Delta X_s)^2 \ , \tag{2.11}$$

respectively.

When there exists market micro-structure noise, it may be difficult to distinguish the random jump process from it. For many applications, however, the roles of stochastic jumps and market micro-structure noise (or measurement error) are different, and it is important to estimate them in high-frequency finanical data.

3. Local Estimation for the No-Market Microstructure-Noise Case

For simplicity, we take $t_j^{(n)} - t_{j-1}^{(n)} = 1/n$ $(j = 1, \cdots, n)$ and $t_0^n = 0$. We divide $(0, 1]$ into $b(n)$ sub-intervals and in every interval we allocate $c(n)^*$ observations. We consider the sequence $c^*(n)$ such that $c^*(n) \to \infty$ and we can take $b(n) \to \infty$ and $b(n) \sim n/c^*(n)$ as $n \to \infty$. A typical choice of observations in each interval would be $c^*(n) = [n^\gamma]$ $(0 < \gamma < 1)$, whereupon $b(n) \sim n^{1-\gamma}$. Because there are some extra observations (n may not be equal to $b(n)c^*(n)$) and $b(n)$ is a positive integer, we need to adjust the number of terms in each interval $c(n) = c^*(n) +$ (several terms). Although there can be finite sample effects, we will ignore the effects of extra terms in the following development because they are asymptotically negligible and hence we take $b(n)c(n) = n$.

When there exsts market micro-structure noise, we simply use the log-return process $r_j = y(t_j^{(n)}) - y(t_{j-1}^{(n)})$ from the log-price process $y(t_j^{(n)})$. We order the data r_j in each sub-intervals and denote $r_{k,(i)}$ $(k = 1, \cdots, c(n); i = 1, \cdots, b(n))$.
When $p = 1$, let the 2nd moment of $r_{k,(i)}$ in the i-th interval be

$$M_{2,(i)} = \sum_{k=1}^{c(n)} [r_{k,(i)}]^2 \ . \tag{3.1}$$

Then we define the local realized moment (LRM) estimator of $V^*(2r)$ by

$$\hat{V}^*(2r) = n^{r-1} \sum_{i=1}^{b(n)} M_{2,(i)}^r \quad . \tag{3.2}$$

When $r=1$, it is the realized volatility (RV).
In this construction of the local realized moment (LRM) estimation, we need to normalize the sample moment due to the scale factor n^{r-1} and to use the local Gaussianity of underlying continuous martingales.

For the LRM estimator, we have the next result on the asymptotic properties, which could be obtained straight-forwardly by extending the standard arguments developed in the existing literature to the present case. (See Section 3.4 of Ait-Sahalia and Jacod (2014) on the standard arguments, for example.)

Proposition 1 : Assume that there is no market micro-structure noise, i.e. $\epsilon_n = 0$ with $p = 1$ and $r \geq 1$ in (2.1), (2.3) and (2.4). Also assume that $Y(t_i^{(n)}) = X(t_i^{(n)})$ and σ_s $(0 \leq s \leq 1)$ is bounded and Lipschitz-continuous.
(i) As $n \longrightarrow \infty$

$$\hat{V}^*(2r) - V(2r) \xrightarrow{p} 0 \ . \tag{3.3}$$

(ii) As $n \longrightarrow \infty$

$$\sqrt{n}\left[\hat{V}^*(2r) - V(2r)\right] \stackrel{\mathcal{L}-s}{\rightarrow} N\left[0, W\right] \ , \tag{3.4}$$

where $\mathcal{L} - s$ means the stable convergence and

$$W = 2r^2 \int_0^1 \sigma_s^{4r} ds \ . \tag{3.5}$$

We notice that we have used the stable-convergence in Proposition 1 because W is a random variable when the volatility function is stochastic in general. We shall use the stable-convergence in the following analysis and we give a brief discussion on the CLT(central limit theorem) and stable-convergence at the end of the Appendix.

4. Local SIML Estimation

We consider the estimation problem of some Brownian and jump functionals when we have the market micro-structure noise as (2.1), (2.2) with $\delta \geq 0$, and (2.3) or (2.8). We utilize the same localization of the estimation method in Section

3, and divide $(0,1]$ into $b(n)$ sub-intervals and at every interval we allocate $c^*(n)$ observations. We consider the sequence $c^*(n)$ such that $c^*(n) \to \infty$ and we take $b(n) \to \infty$ and $b(n) \sim n/c^*(n)$ as $n \to \infty$. We choose that the observations in each interval would be $c^*(n) = [n^\gamma]$ $(0 < \gamma < 1)$, whereupon $b(n) \sim n^{1-\gamma}$ and we assume $n = b(n)c(n)$.

Then we apply the SIML method developed by KSK (2018) to each sub-intervals. To use the SIML transformation in each local interval, we set $m_c = [c(n)^\alpha]$ $(0 < \alpha < 0.5)$ in the i-th interval $(i = 1, \cdots, b(n))$ and the transformed data are denoted as $z_{k,(i)}$ as the k-th data in the i-th interval $I_c(i)$ $(k = 1, \cdots, c(n); i = 1, \cdots, b(n))$. Here we explain the procedure for the general case when $p \geq 1$ by following the notations in Chapter 3 of KSK (2018) for the $p-$dimensional stochastic process $\mathbf{y}(t_i^{(n)})$. In each sub-intervals, we transform $c(n) \times p$ observation matrix $\mathbf{Y}_{c(n),(i)}$ to $c(n) \times p$ matrix $\mathbf{Z}_{n,(i)}$ $(= (\mathbf{z}'_{k,(i)}))$ $(i = 1, \cdots, b(n))$ by

$$(4.1) \qquad \mathbf{Z}_{c(n),(i)} = h_{c(n)}^{-1/2} \mathbf{P}_{c(n)} \mathbf{C}_{c(n)}^{-1} \left(\mathbf{Y}_{c(n),(i)} - \bar{\mathbf{Y}}_{0,(i)} \right)$$

where $h_{c(n)} = 1/c(n)$, and $c(n) \times c(n)$ matrices

$$(4.2) \qquad \mathbf{C}_{c(n)}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} ,$$

$$(4.3) \qquad \mathbf{P}_{c(n)} = (p_{jk}) \ , \ p_{jk} = \sqrt{\frac{2}{c(n) + \frac{1}{2}}} \cos \left[\frac{2\pi}{2c(n)+1} (k - \frac{1}{2})(j - \frac{1}{2}) \right] .$$

The initial conditions are given by the $p \times 1$ vector $\mathbf{y}_{0,(i)}$ and

$$(4.4) \qquad \bar{\mathbf{Y}}_{0,(i)} = \mathbf{1}_{c(n)} \cdot \mathbf{y}'_{0,(i)} .$$

Then we have the spectral decomposition

$$(4.5) \qquad \mathbf{C}_{c(n)}^{-1} \mathbf{C}_{c(n)}^{\prime -1} = \mathbf{P}_{c(n)} \mathbf{D}_{c(n)} \mathbf{P}'_{c(n)} ,$$

where $\mathbf{D}_{c(n)}$ is a diagonal matrix with the k-th element $d_k = 2\left[1-\cos(\pi(\frac{2k-1}{2c(n)+1}))\right]$ $(k = 1,\cdots,c(n))$. We define

$$a_{k,c(n)} = c(n)d_k = 4c(n)\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2c(n)+1}\right)\right] \ (k=1,\cdots,n)\ . \tag{4.6}$$

When $p = 1$ and for any positive integer r, let the 2nd moment in the i-th sub-interval be

$$M_{2,(i)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}]^2\ . \tag{4.7}$$

Then, we define the LSIML estimator of $V(2r)$ by

$$\hat{V}(2r) = b(n)^{r-1}\sum_{i=1}^{b(n)}[M_{2,(i)}]^r\ . \tag{4.8}$$

If we take $c(n) = n, b(n) = 1$ and $r = 1$, then we have the SIML estimator for integrated volatility as a special case. In this construction of the LSIML estimator, we have $c(n)$ observations in each interval and then we need to normalize (4.8) because the scale factor is $c(n)/n = b(n)^{-1}$.

5. Asymptotic Properties of Local SIML

We consider the case when σ_s is a time-varying continuous and bounded function when $p = 1$. First, we consider the asymptotic properties of the LSIML estimation for the case of $r = 1$. Then we shall discuss the case when $r \geq 2$. The SIML estimation method was originally developed for the case of constant volatility, but it has some desirable asymptotic properties when the instantaneous volatility is time-dependent and also stochastic in the form of (2.4). The LSIML estimation shares these asymptotic properties of the SIML method. Since we need some arguments based on the stable convergence (SC) and the martingale central limit theorem (MCLT) in the stochastic case, we explain the asymptotic properties in this section as if the time-varying volatility was deterministic function. Some discussion on SC will be given at the end of Appendix.

(i) The case when $r = 1$

First, we consider the asymptotic behavior of the quantity $\mathbf{M}_{2,(i)} = (1/m_c) \sum_{k=1}^{m_c} z_{k,(i)}^2$ in the i-th interval $I_c(i) = ((i-1)\frac{c(n)}{n}, i\frac{c(n)}{n}]$ (we use the notation $t_{i-1}^{(n)} = (i-1)\frac{c(n)}{n}$, and $t_i^{(n)} = i\frac{c(n)}{n}$ $(i = 1, \cdots, b(n))$), when we take $n = b(n)c(n)$, $m_c = [c(n)^{\alpha}]$ $(0 < \alpha < 0.5)$ and $m_c \to \infty$ as $n \to \infty$. We summarize the result for the case of $r = 1$, which corresponds to Proposition 1 without any market micro-structure noise. This presentation may be useful to understand the results in more general cases with market micro-structure noise. The derivation is given in the Appendix.

Theorem 2 : When $r = 1$ and $p = 1$ in (2.1), (2.2), (2.3) and (2.4) with $\delta \geq 0$. Also assume that $v(t_i^{(n)})$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$, $\mathbf{E}[v_i^4] < +\infty$, σ_s $(0 \leq s \leq 1)$ is bounded and Lipschitz-continuous. We set $\alpha_1^* = 1 + [4\delta - 1]/[3\gamma]$, $\alpha_2^* = 1 + [4\delta - 3]/[5\gamma]$, and $0 < \gamma < 1$.
Then we have the following asymptotic properties of the LSIML estimator with $0 < \gamma < 1$.
(i) For $m_c = [c(n)^{\alpha}]$ and $0 < \alpha < \min\{0.5, \alpha_1^*\}$, as $n \longrightarrow \infty$

$$\hat{V}(2) - V(2) \xrightarrow{p} 0 \; . \tag{5.1}$$

(ii) For $m_c = [c(n)^{\alpha}]$ and $0 < \alpha < \min\{0.4, \alpha_2^*\}$, as $n \longrightarrow \infty$

$$\sqrt{m_c b(n)} \left[\hat{V}(2) - V(2)\right] \overset{\mathcal{L}-s}{\to} N\left[0, W\right] \tag{5.2}$$

in the stable convergence sense, where

$$W = 2\int_0^1 \sigma_s^4 ds \; . \tag{5.3}$$

If we take $\delta = 0.0$ and $\gamma = 3/4$ $(4/5)$, then the first condition for consistency implies $0 < \alpha < 1/4$ $(3/8)$ while the second condition for asymptotic normality implies $0 < \alpha < 1/5$ $(1/4)$.

(ii) The case when $r \geq 2$

We investigate the asymptotic properties of the Local-SIML estimator when $p = 1$ and $r \geq 2$. As a generalizeation of Theorem 2 when $r \geq 2$ and $p = 1$ as follows, which is the summary of the asymptotic properties of the LSIML estimation. The derivation is given in the Appendix.

Theorem 3 : When $p = 1$ and $r \geq 2$ in (2.1), (2.2), (2.3) and (2.4) with $\delta \geq 0$, assume that $v(t_i^{(n)})$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$, $\mathbf{E}[v_i^{4r}] < +\infty$ and σ_s $(0 \leq s \leq 1)$ is bounded and Lipschitz-continuous. We define $\alpha_1^* = 1 + [4\delta - 1]/[3\gamma]$, $\alpha_2^* = 1 + [4\delta - 3]/[5\gamma]$, and $0 < \gamma < 1$. Then, we have the following asymptotic properties of the LSIML estimator.

(i) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.5, \alpha_{1r}^*\}$ $((\alpha_1^* > 0)$, as $n \longrightarrow \infty$

$$\hat{V}(2r) - V(2r) \xrightarrow{p} 0 \ . \tag{5.4}$$

(ii) We assume the additional condition $\gamma\alpha > 1 - \gamma$ and $0 < \gamma < 1$. For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.4, \alpha_{2r}^*\}$ $(\alpha_2^* > 0)$, as $n \longrightarrow \infty$

$$\sqrt{m_c b(n)}\left[(\hat{V}(2r) - V(2r)) + (V(2r) - V^*(2r))\right] \stackrel{\mathcal{L}-s}{\rightarrow} N\left[0, W\right] \tag{5.5}$$

in the stable convergence sense, where

$$W = 2r^2 \int_0^1 \sigma_s^{4r} ds \ , \tag{5.6}$$

and

$$V^*(2r) = [b(n)]^{r-1} \sum_{i=1}^{b(n)} \left(\int_{t_{i-1}^{(n)}}^{t_i^{(n)}} \sigma_s^2 ds\right)^r \ . \tag{5.7}$$

When $r \geq 2$, an asymptotic bias term as (5.5) and (5.7) in the limiting distribution appears. As we will show in the Appendix, we have some complications in the evaluation of stochastic orders in this case. When $r = 1$, however, there does not exist any bias term and we have the result in Theorem 2.

When $\gamma = 3/4$ (or4/5), the condition $\gamma\alpha > 1 - \gamma$ in Part (ii) implies $\alpha > 1/3$ (or1/4). It may be interesting to find that the form of the asymptotic variance for the LSIML estimation is the same as the one for RV as in Proposition 1 when there is no market micro-structure noise except that n $(= b(n)c(n))$ is replaced by $b(n)m_c$.

6. An Optimal Choice of α and γ

Because the properties of the LSIML estimation method depends crucially on the choice of $c(n)$ and m_c, which are dependent on n, we need to investigate the asymptotic effects as well as the small-sample effects of their choice.

As we will explaine in the derivation of Theorem 2 in the Appendix ((A.17), (A.26), and (A.27)), the asymptotic bias of the LSIML estimator is proportional to

$$\text{AB}_n \sim [b(n) \times \frac{m_c^2}{c(n)}](\epsilon_n)^2 \tag{6.1}$$

and the asymptotic is proportional to

$$\text{AV}_n \sim \frac{1}{m_c b(n)} = \frac{1}{n}[c(n)]^{1-\alpha} \quad . \tag{6.2}$$

Hence when n is large, we may approximate the mean squared error of the LSIML estimator as

$$g_n = c_{1g}\frac{1}{n}[c(n)]^{1-\alpha} + c_{2g}[b(n) \times \frac{m_c^2}{c(n)}]^2(\epsilon_n)^4 \ , \tag{6.3}$$

where c_{1g} and c_{2g} are some constants.
By setting $c(n) = n^{\gamma}$ and $b(n) = n^{1-\gamma}]; (0 < \gamma < 1)$, we can rewrite

$$g_n^* = c_{1g}\frac{1}{n}[c(n)]^{1-\alpha} + c_{2g}\left[n^{2(1-\gamma)-2\gamma+4\alpha\gamma-4\delta}\right] \quad . \tag{6.4}$$

Then, by ignoring the difference of $c(n) = [n]^{\gamma}$ and n^{γ} and similar terms and differentiating MSE with respect to α we have the condition such that $n^{-1}c(n)^{1-\alpha}$ $(= n^{-1+\gamma(1-\alpha)})$ is proportional to $n^{-[2(1-\gamma)-2\gamma+4\alpha\gamma-4\delta]}$. By rearranging the related terms, we have the next result.

Theorem 4 : When $p = 1$ and $r = 1$ in (2.1), (2.2), (2.3) and (2.4) with $\delta \geq 0$, assume that $v(t_i^{(n)})$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$ and $\mathbf{E}[v_i^{4r}] < +\infty$, and σ_s $(0 \leq s \leq 1)$ is bounded and Lipschitz-continuous. An optimal choice of $m_c = [c(n)^{\alpha}]$ and $c(n) = [n^{\gamma}]$ (with $\epsilon_n = n^{-\delta}$ $(0 < \gamma < 1$ and $\delta \geq 0)$ to minimize MSE when n is large, is approximately given by

$$1 - \gamma(1-\alpha) = 2(1-\gamma) - 2\gamma + 4\alpha\gamma + 4\delta \ , \tag{6.5}$$

which means the choice as

$$\alpha^* = \frac{1-\gamma-2(1-\gamma)+4\delta}{3\gamma} = 1 + \frac{4\delta-1}{3\gamma} \ . \tag{6.6}$$

For example, when $\delta = 0$, $\alpha^* = 1 - 1/[3\gamma]$. When $\delta = 0$ and we take α^*, then the MSE is proportional to $n^{-[1-\gamma+\gamma\alpha^*]}$, which is

$$\text{MSE} \sim n^{-2/3} \ . \tag{6.7}$$

Because the MSE in Proposition 1 is proportional to n^{-1}, we have some loss of efficiency when we have market micro-structure noise in high-frequency data

When $r \geq 2$, the result of Theorem 4 holds if the volatility function is constant in $[0,1]$. In the general case, however, we need slightly different conditions and there may be a further complication. It is because we have an additional bias term due to $V(2r) - V^*(2r)$ of (5.5) in Part (ii) of Theorem 3 in the general case.

7. Possible Extensions

There are possible generalizations of our results in the previous sections. We will discuss two cases of the jump-diffusion process and the multivariate diffusion models.

7.1 Continuous-Part and Jump-Part of Quadratic Variation

We consider the estimation problem in Example 3 in Section 2. When there is no market micro-structure noise in the continuous-time Ito-process as $X(t) = X(0) + \int_0^t \sigma_s dB_s + \sum_{0 \leq s \leq t} \Delta X_s$ $(0 \leq s \leq t \leq 1)$, the method of estimating the continuous and jump parts of quadratic variation has been known. For instance, Chapters 9 and 13 of Jacod and Protter (2012) have developed the truncation functionals and reported many theoretical results in the high-frequency asymptotics. When there is some market microstructure noise, however, it seems that there has not been an unified method available. The LSIML method gives a useful solution for this purpose. In this subsection, we investigate the simple case of diffusion-jump model and assume that $p = 1$, the jump-size is bounded, and there can be a finite number of jumps in $[0, 1]$. The discussion is based on Section 2 of KK (2021), and the general discussion of diffusion-jump processes has been given in Jacod and Protter (2012).

We consider the truncated functionals of the LSIML estimation. From Section 3, when $p = 1$ and $r = 1$, let the 2nd moment in the i-th sub-interval be $M_{2,(i)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}]^2$. We define the truncated LSIML functionals $V_J(2)$ and $V_C(2)$ by

$$\hat{V}_J(2) = \sum_{i=1}^{b(n)} M_{2,(i)}\mathrm{I}(M_{2,(i)} > u_n) \tag{7.1}$$

and

$$\hat{V}_C(2) = \sum_{i=1}^{b(n)} M_{2,(i)}\mathrm{I}(M_{2,(i)} \le u_n)\ , \tag{7.2}$$

respectively, where $\mathrm{I}(\cdot)$ is the indicator function.

Here we take the truncation parameter u_n (a sequence of positive constants) such that

$$A_n = \frac{1}{u^2}[\frac{1}{b(n)} + \frac{b(n)}{c(n)^{2-4\alpha}}] \xrightarrow{p} 0\ . \tag{7.3}$$

(See Lemma A-3 in the Appendix.) Then, we can estimate the continuous-part and jump-part of the quadratic variation in a simple way.

We summarize the asymptotic properties of the truncated LSIML estimator as the next result. The proof is given in the Appendix.

Theorem 5 : When $r = 1$ and $p = 1$ in (2.1), (2.2), (2.4) with $\delta \ge 0$, and (2.8) instead of (2.3), assume that $v(t_i^{(n)})$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$, $\mathbf{E}[v_i^4] < +\infty$, and σ_s $(0 \le s \le 1)$ is Lipschitz-continuous, and jumps are bounded. We also assume that $\alpha_1^* > 0$, $\alpha_2^* > 0$ in Theorem 2, and the condition (7.3) on A_n. Then we have the following asymptotic properties of the trancated LSIML estimator with $0 < \gamma < 1$.

(i) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.5, \alpha_1^*\}$, as $n \longrightarrow \infty$

$$\hat{V}_C(2) - V_C(2) \xrightarrow{p} 0 \tag{7.4}$$

and

$$\hat{V}_J(2) - V_J(2) \xrightarrow{p} 0 \ . \tag{7.5}$$

(ii) For $m_c = [c(n)^{\alpha}]$ and $0 < \alpha < \min\{0.4, \alpha_1^*\}$, as $n \longrightarrow \infty$

$$\sqrt{m_c b(n)} \left[\hat{V}_J(2) - V_J(2)\right] \overset{\mathcal{L}-s}{\to} N\left[0, W_J\right] \tag{7.6}$$

and

$$\sqrt{m_c b(n)} \left[\hat{V}_C(2) - V_C(2)\right] \overset{\mathcal{L}-s}{\to} N\left[0, W_J\right] \tag{7.7}$$

in the stable convergence sense, where

$$W_J = 4 \sum_{0<s\leq 1} \sigma_s^2 (\Delta X(s))^2 \tag{7.8}$$

and

$$W_C = 2 \int_0^1 \sigma_s^4 ds \ , \tag{7.9}$$

respectively.

For example, if we take $c(n) = n^{\gamma}$, $b(n) = n^{1-\gamma}$, and $m_c = [c(n)]^{\alpha}$, $[b(n)]^{-1} u_n^2 [c(n)]^{2-4\alpha} = u_n^2 n^{-1+\gamma(3-4\alpha)}$. It is possitive and can converges to zero if we set $\alpha = .39$ and $\gamma = .75$ because of $-1 + \gamma(3 - 4\alpha) < 0$.
KK (2021) have derived the central limit theorem (CLT) for the SIML estimation when the underlying process is the class of Ito's jump-diffusion process in the multivariate case. When $p = 1$, in their Corollary 2.1, the asymptotic variance of the limiting distribution is given by

$$W = 2 \left[\int_0^1 \sigma^4(s) ds + 2 \sum_{0<s\leq 1} \sigma^2(s)(\Delta X(s))^2\right] \ . \tag{7.10}$$

Since $W = W_J + W_C$, it can be regarded as a decomposition of the variance and Theorem 5 is an extension of Theorem 2.1 of KK (2021).

7.2 Multivariate Processes

There are possible generalizations to multivariate processes when $p \geq 1$. Let

$$\mathbf{Y}(t_i^{(n)}) = \mathbf{X}(t_i^{(n)}) + \epsilon_n \mathbf{v}(t_i^{(n)}) \ \ (i = 1, \cdots, n) \tag{7.11}$$

be the (p-dimensional) observed (log-)prices $\mathbf{Y}(t_i^{(n)}) = (Y_j(t_i^{(n)}))$ at t_i^n $(0 = t_0^n \leq t_1^{(n)} \leq \cdots \leq t_n^{(n)} = 1)$ and $\mathbf{v}(t_i^{(n)})$ $(= (v_j(t_i^{(n)}))$ be a sequence of $(p \times 1)$ i.i.d. random vectors with $\mathbf{E}[\mathbf{v}(t_i^{(n)})] = 0$ and $\mathbf{E}[\mathbf{v}(t_i^{(n)})\mathbf{v}(t_i^{(n)})'] = \boldsymbol{\Sigma}_v$ (> 0).

As the underlying continuous-time process, we consider the class of multi-dimensional diffusion processes. As the theory of continuous-time stochastic processes $\mathbf{X}(t_i^{(n)})$ $(= (X_j(t_i^{(n)}))$, a general form of the SDE for the p-dimensional continuous-time stochastic processes is given by

$$(7.12) \qquad d\mathbf{X} = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{B}_t \ ,$$

which has been called the diffusion-type continuous process, where $\boldsymbol{\mu}(s)$ is the $p \times 1$ drift vector, $\boldsymbol{\sigma}(s)$ is the $p \times q_1$ diffusion matrix, and $\mathbf{B}_t$ is the $q_1 \times 1$ Brownian motions. It also has the representation as

$$(7.13) \qquad \mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \boldsymbol{\mu}(s)ds + \int_0^t \boldsymbol{\sigma}(s)d\mathbf{B}_s \ ,$$

where the first term is an integration in the sense of Riemann while the second term is an Itô's stochastic integration with respect to the Brownian motion B_t ($q_1 \times 1$ vector).

We need some regularity conditions on $\boldsymbol{\mu}(\cdot)$ and $\boldsymbol{\sigma}(\dot{})$. A detailed theory of stochastic differential equation (SDE) and stochastic integration has been explained by Ikeda and Watanabe (1989). When the volatility process $\boldsymbol{\sigma}(t) = (\sigma_{ij}(t))$ is stochastic, we take a diffusion type process as

$$(7.14) \qquad \sigma_{ij}(t) = \sigma_{ij}(0) + \int_0^t \mu_{ij}^{\sigma}(s)ds + \int_0^t \boldsymbol{\omega}_{ij}^{\sigma}(s)d\mathbf{B}_s^{\sigma} \ (0 \leq s \leq t \leq 1) \ ,$$

where $\mu_{ij}(s)$ is the drift coefficient, $\boldsymbol{\omega}_{ij}^{\omega}(s)$ is $1 \times q_2$ diffusion coefficients and $\mathbf{B}_s^{\sigma}$ is another $q_2 \times 1$ Brownian motion vector, which may be correlated with $\mathbf{B}_s$.

An example of the estimation problem, we may assume $p \times p$ variance-covariance (or the integrated volatility) matrix $\boldsymbol{\Sigma}_x = \int_0^1 \boldsymbol{\sigma}_s \boldsymbol{\sigma}_s' ds$, which is the same as $\mathbf{V}(2) = (V_{gh}(2))$ in our notation. In this case, the terms $(1/m_c)\sum_{k=1}^{m_c}[z_{k,(i)}]^2$ and the asymptotic variance $2\int_0^1 [\sigma_x(s)]^4 \, ds$ in Section 5 are replaced by

$$(7.15) \qquad \hat{V}(g,h;2) = \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{g,k,(i)} z_{h,k,(i)}]$$

and

$$(7.16)\qquad \int_0^1 \left[\sigma_{gg}^{(x)}(s)\sigma_{hh}^{(x)}(s)+(\sigma_{gh}^{(x)}(s))^2\right] ds\ ,$$

where we set $p=2$ and

$$\mathbf{\Sigma}_x=\int_0^1 \mathbf{\Sigma}_x(s)ds=\begin{pmatrix} \sigma_{gg}^{(x)} & \sigma_{gh}^{(x)} \\ \sigma_{gh}^{(x)} & \sigma_{hh}^{(x)} \end{pmatrix} .$$

The most important fact is that both the SIML method and the LSIML method are simple and it is straightforward to use them when the dimension p of underlying processes is large. This aspect is quite different from other methods proposed in the past. Recently, KK (2021) have considered a statistical procedure to detect factors of the hidden covariation r_x when it is substantially less than the dimension p, for instance. We expect that under a set of regularity conditions, we have the similar results on the asymptotic properties of the local SIML estimator in more general settings.

8. Simulations and An Empirical Data Analysis

8.1 Simulations

We have done some simulation when $r=1$ and $r=2$ on the estimation of the true parameters of $V(2)$, $V(4)$, $V_C(2)$ and $V_J(2)$. We note that the estimated variance of the SIML estimator of integrated volatility corresponds to $2\hat{V}(4)$ in the univariate case. In our simulations we set $b(n)=[n^{1-\gamma}]$, $c(n)=[n^{\gamma}]$ such that $b(n)c(n)=n$ and the number of replications is 3,000. Also we have investigated several cases in which the instantaneous volatility function σ_s^2 is given by

$$(8.1)\qquad \sigma_s^2=\sigma_0^2\left[a_0+a_1 s+a_2 s^2\right],$$

where a_i $(i=0,1,2)$ are constants and we have some restrictions such that $\sigma_s>0$ for $s\in[0,1]$. This is a typical time-varying (but deterministic) case and the integrated volatility $V(2)$ is given by

$$(8.2)\qquad V(2)=\int_0^1 \sigma_s^2 ds=\sigma_x(0)^2\left[a_0+\frac{a_1}{2}+\frac{a_2}{3}\right] .$$

In this case we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

(i) As the first exercise, we take (2.1), (2.2), and (2.3) with $\delta = 0.0$. In Tables 8.1-8.5, the true parameter values of V(2) and V(4) are $\int_0^1 \sigma_s^2 ds$ and $\int_0^1 \sigma_s^4 ds$, respectively. In Tables *mean* and *Var* are the mean and variance of simulated variables for

$$\hat{V}(2r) = b(n)^{r-1} \sum_{i=1}^{b(n)} (M_{2,(i)})^r \tag{8.3}$$

and

$$\hat{V}^{**}(2r) = \frac{b(n)^{r-1}}{a_r} \sum_{i=1}^{b(n)} M_{2r,(i)} \tag{8.4}$$

where

$$M_{2r,(i)} = \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}]^{2r} \tag{8.5}$$

and $a_r = \frac{2r!}{r!\,2^r}$.

In Tables, AV stands for the limiting variances calculated from (5.2)-(5.3) and (5.5)-(5.6). (Kunitomo and Sato (2018) discussed $\hat{V}^{**}(2r)$ in some detail.)

If we take $c(n) = n, b(n) = 1$ and $r = 1$, then we have the SIML estimator for integrated volatility as a special case.

In this construction of the LSIML estimator, we have $c(n)$ observations in each interval and then we need to normalize (8.3) and (8.4) because the scale factor is $c(n)/n = b(n)^{-1}$ while we need the local Gaussianity for underlying continuous martingales. Tables 8.1 and 8.2 correspond to the case of flat volatility while other tables correspond to the case of time-varying, but non-stochastic volatility.

Table 8.1 : Estimation of integrated fourth-order functional

$(a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005,\ b(n) = 5, c(n) = 521;\ \alpha = 0.4, \gamma = 0.795)$

n=2,605	$V(2) = 2.0$	$V(4) = 4.0$	n=2,605	$V(2) = 2.0$	$V(4) = 4.0$
mean	2.001	4.671	mean	2.009	4.053
Var	0.133	2.837	Var	0.134	2.837
AV	0.133	3.10	AV	0.133	2.843

Table 8.2 : Estimation of integrated fourth-order functional

$(a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005,\ b(n) = 10, c(n) = 1,000;\ \alpha = 0.33, \gamma = 0.75)$

n=10,000	$V(2) = 2.0$	$V(4) = 4.0$	n=10,000	$V(2) = 2.0$	$V(4) = 4.0$
mean	2.012	4.950	mean	2.013	4.056
Var	0.092	2.400	Var	0.092	1.973
AV	0.090	2.400	AV	0.089	1.895

Table 8.3 : Estimation of integrated fourth-order functional

$(a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_v^2 = 0.0005,\ b(n) = 10, c(n) = 1,000;\ \alpha = 0.33, \gamma = 0.75)$

n=10,000	$V(2) = 2.0$	$V(4) = 7.2$	n=10,000	$V(2) = 2.0$	$V(4) = 7.2$
mean	2.012	8.700	mean	2.023	7.167
Var	0.160	19.400	Var	0.160	15.093
AV	0.160	17.056	AV	0.160	17.056

Table 8.4 : Estimation of integrated fourth-order functional

$(a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_v^2 = 0.0005,\ b(n) = 40, c(n) = 1,261;\ \alpha = 0.45, \gamma = 0.66)$

n=50,440	$V(2) = 2.0$	$V(4) = 7.2$	n=50,440	$V(2) = 2.0$	$V(4) = 7.2$
mean	2.069	8.100	mean	2.070	7.457
Var	0.015	1.510	Var	0.016	1.650
AV	0.015	1.599	AV	0.015	1.599

Table 8.5 : Estimation of integrated fourth-order functional

$(a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_v^2 = 0.0005,\ b(n) = 18, c(n) = 5,622;\ \alpha = 0.33, \gamma = 0.75)$

n=101,196	$V(2) = 2.0$	$V(4) = 7.2$	n=101,196	$V(2) = 2.0$	$V(4) = 7.2$
mean	2.021	8.124	mean	2.022	7.273
Var	0.048	5.057	Var	0.049	5.128
AV	0.047	5.016	AV	0.047	5.016

In Tables 8.1-8.5 we first confirm that the LSIML method work well for the estimation of the integrated volatility. Although there may be some loss of estimation accuracy when the underlying true stochastic process is known, the LSIML method gives desirable finite and asymptotic properties. The most important result in our simulation is the estimation of 2V(4), which is the asymptotic variance of the SIML estimator of integrated volatility. As we see in Tables, the mean and SD (standard deviation) have reasonable values.

To investigate the asymptotic distribution of the LSIML estimator, we give some typical empirical distribution of a set of simulated data in Figure 8.1 ($r = 1, b(n) = 14, c(n) = 3371,\ \alpha = 0.4, a_0 = 6.0, a_1 = -24.0, a_2 = 24.0$) and Figure 8.2 ($r = 2, b(n) = 76, c(n) = 677,\ \alpha = 0.4, a_0 = 6.0, a_1 = -24.0, a_2 = 24.0$). We confirm that we have the asymptotic normality of the SIML estimator and the limiting normal distribution gives a reasonable approximation of the finite sample distribution. Also we found that when $r = 2$, we have a small bias with the limiting normal distribution, which is consistent to Theorem 3.

(ii) As the second example, we give Tables 8.6-8.10 for the jump-diffusion case under market micro-structure noise. We set the true parameter values of $V_C(2) = 2.0$ and $\lambda = 3/n$ for the diffusion-Poisson-jump model with the intensity λ/n. In Tables *mean* and *Var* are the mean and variance of simulated.
In Tables 8.6-8.10 we confirm that the LSIML estimation method of the continuous and jump parts work well. In our experiment, after some trials we have set the threshhold value as

$$u_n = (\text{mean of } M_{2,(i)}) + Q995 \times SD(M_{2,(i)} < \text{mean of } M_{2,(i)}) \ , \tag{8.6}$$

where $SD(\cdot)$ is the standard deviation and $Q995$ is the .995 quantile.
We also show the emipirical distribution of the continupus-part and jump-part of the LSIML estimator under market micro-structure noise in Figure 8.3. We confirm that the limiting normal distribution in Theorem 5 gives reasonable approximation to the finite sample distributions of estimator of continuous part and the jump part of quadratic variation.

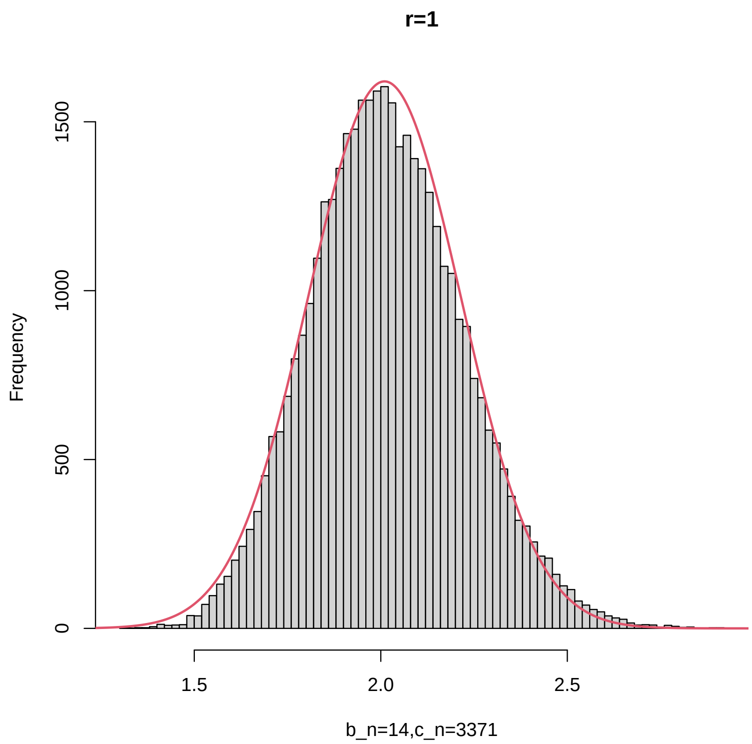

Figure 8.1: Normalized Histogram and Normalized Distribution ($r = 1$)

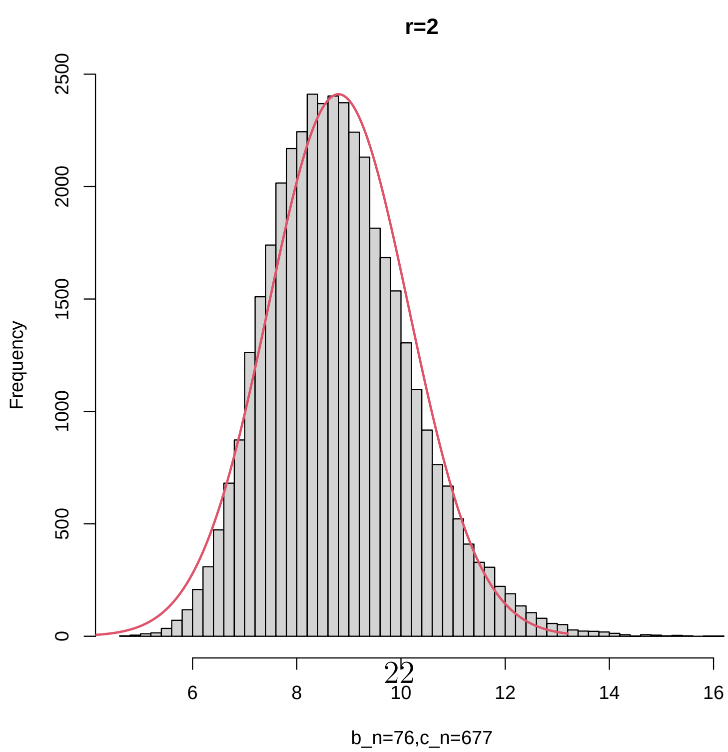

Figure 8.2: Normalized Histogram and Normalized Distribution ($r = 2$)

Table 8.6 : Estimation of V(2)l

($a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005$, $b(n) = 100, c(n) = 1000$;

$\alpha = 0.4, \lambda = 3/n$, size =0.7)

n=100,000	$V_C(2) = 2.0$	$V_J(2)$
mean	2.107	0.030
Var	0.007	0.001

Table 8.7 : Estimation of V(2)

($a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005$, $b(n) = 10, c(n) = 10,000$;

$\alpha = 0.4, \lambda = 3/n$, size =0.7)

n=10,000	$V_C(2) = 2.0$	$V_J(2)$
mean	1.991	0.0209
Var	0.026	0.006

Table 8.8 : Estimation of V(2)

($a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005$, $b(n) = 100, c(n) = 1,000$;

$\alpha = 0.4, \lambda = 3/n$, size =0.7)

n=100,000	$V_C(2) = 2.0$	$V_J(2)$
mean	2.073	1.52
Var	0.007	0.803

Table 8.9 : Estimation of V(2)

($a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005$, $b(n) = 10, c(n) = 10,000$;

$\alpha = 0.4, \lambda = 3/n$, size =0.7)

n=100,000	$V_C(2) = 2.0$	$V_J(2)$
mean	2.875	0.623
Var	1.214	0.372

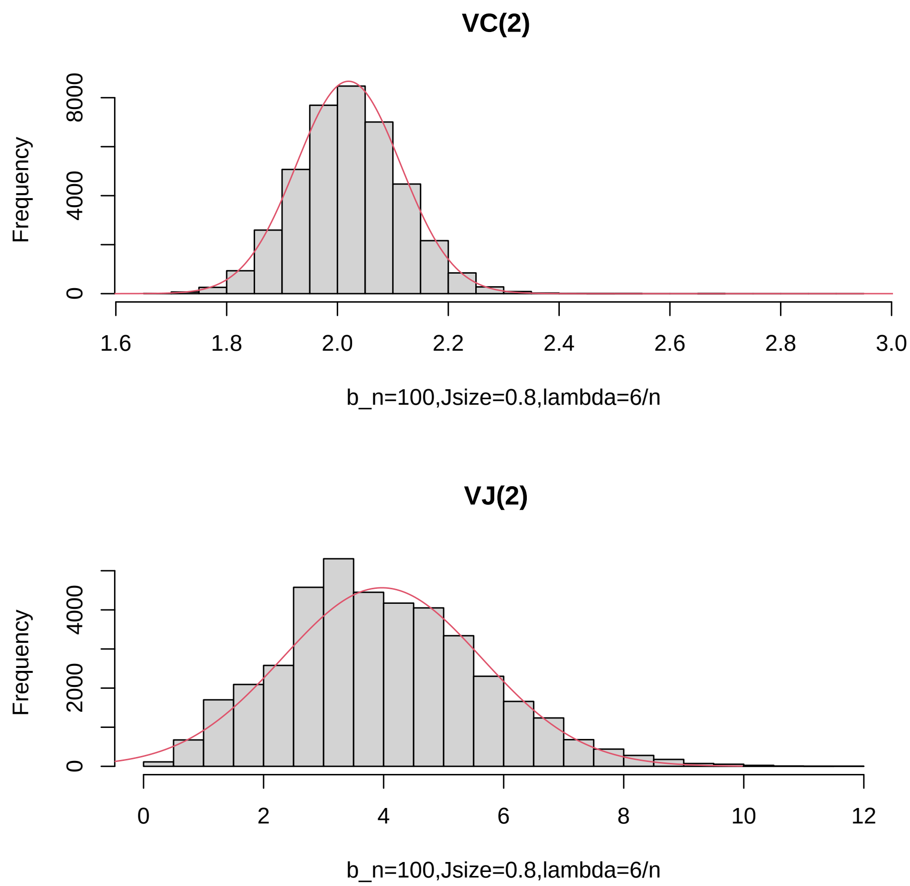

Figure 8.3: Normalized Histogram of LSIML estimator of $V_C(2)$ and $V_J(2)$.

Table 8.10 : Estimation of V(2)

$(a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_v^2 = 0.0005,\ \ b(n) = 200, c(n) = 500;$

$\alpha = 0.4, \lambda = 3/n,$ size $=0.7)$

n=100,000	$V_C(2) = 2.0$	$V_J(2)$
mean	2.257	1.502
Var	0.005	0.808

(iii) From our simulations we found that the LSIML estimator of integrated volatility V(2) and V(4) perform quite well as we expected. We also have confirmed that the estimation of continuous part and jump-part of the quadratic variation in the presence of market micro-structure noise is possible. The behaviors of the LSIML estimator for higher Brownian and jump functionals as $r = 1$ and $r = 2$ are reasonable given the difficulties of the problem involved because of the presence of market micro-structure noise.

8.2 An Empirical Data Analysis

As an illustration for the use of the method we have discussed in previous sections, we give an empirical result of high-frequency data analysis in Table 8.11. For the comparative purpose, we have used the same data set in Section 4 of KSK (2018), which is the high-frequency tick-data of Nikkei-225 Futures at April 16, 2007 traded at OSE (Osaka Stock Exchange). The data are 1s, 5s and 10s (see Kunitomo et al. (2018) for more details) and we have taken $\alpha = 0.4, 1.0$ and several values of γ.

There are several interesting findings. The estimated values of the LSIML estimator are fairly stable and they do not depend on the choice of observation lengths (1s, 5s and 10s) except the case when $\alpha = 1.0$. (The last case does not satisfy the conditions in Section 5.) The estimated standard deviation of $V(2)$ is $\sqrt{2\hat{V}(4)/m_n}$, where $\hat{V}(4)$ is an estimated value of V(4)), and its values are highly significant in all cases. The estimated values of V(2) are quite similar to the estimated values of the SIML estimator reported in Section 4 of KSK (2018).

The estimated values of RV correspond to the case when $\alpha = 1.0$, and the estimated values of RV on V(2) and V(4) are significantly different from the LSIML estimates. (Note that $\hat{V}(4)$ is asymptotically the same as $\hat{V}^*(4)$ although there are small differences in finite samples.) This finding suggests that the estimated values of RV for V(4) as well as V(2) have significant biases due to market micro-structure noise, and the use of RV may cause some problems in applications such as the risk managements.

Table 8.11 : Estimatation Result of Local SIML for Nikkei-225 Future

1s	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=1
	γ=0.66	γ=0.7	γ=0.75	γ=0.8	γ=0.85	γ=1
V(2)	5.64E-05	5.02E-05	5.21E-05	5.27E-05	5.28E-05	4.95E-04
V(4)	4.07E-09	2.88E-09	2.54E-09	3.32E-09	3.35E-09	2.74E-07
5s	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=1
	γ=0.66	γ=0.7	γ=0.75	γ=0.8	γ =0.85	γ=1
V(2)	4.86E-05	4.50E-05	4.80E-05	5.33E-05	4.17E-05	2.60E-04
V(4)	3.76E-09	2.14E-09	2.68E-09	2.86E-09	2.55E-09	8.19E-08
10s	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=0.4	alpha=1
	γ=0.66	γ=0.7	γ=0.75	γ=0.8	γ=0.85	γ=1
V(2)	5.11E-05	4.98E-05	4.78E-05	4.21E-05	3.79E-05	1.76E-04
V(4)	4.20E-09	3.80E-09	2.56E-09	2.60E-09	1.70E-09	3.82E-08

9. Concluding Remarks

In this paper, we have developed the Local SIML (LSIML) method for estimating higher-order Brownian functionals and second-order jump functionals, which is a new statistical method. We extend the separating information maximum likelihood (SIML) method, which was proposed by KSK (2018). The main motivation of the LSIML method is to estimate higher order Brownian and jump functionals including the integrated volatility and co-volatility when we have market micro-structure noise in high-frequency financial data. We have shown that the LSIML method has desirable asymptotic properties such as the consistency and asymptotic normality in the stable-convergence sense, and it also has reasonable finite sample properties, which are illustrated by several simulations and an empirical data analysis. Although there could be other methods for estimating higher-order Brownian and jump functionals, the LSIML method is simple and it has desirable asymptotic properties. Hence it should be useful for practical application including the measurement of financial β with possible jumps under market micro-structure noise. Some empirical applications are currently under investigation.

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APPENDIX : Mathematical Derivations

In this Appendix, we give some details of the derivations of the results in Section 5 and Section 7. Since we have used the stable convergence in Theorem 2, Theorem 3, and Theorem 5, we will give some discussion how we can apply the basic arguments of the CLT and stable-convergence to our situation at the end of this Appendix. We will use some notation of KSK (2018) and KK (2021).

1. **Some Lemmas**

Lemma A-1 : Let r be any positive integer and $a_{k,c(n)}$ is given by (4.6). Then

(A.1)
$$\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)}^r \sim (\frac{\pi^{2r}}{2r+1})\frac{m_c^{2r}}{c(n)^r}$$

as $c(n), m_c \to \infty$ and $m_c/c(n) \to 0$.

Proof of Lemma A-1 : Since $m_c/c(n) \to 0$ as $n \to \infty$ and $\sin x \sim x$ when x is small, we can evaluate

$$\begin{aligned}\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)}^r &= [\pi]^{2r}\frac{m_c^{2r}}{c(n)^r}\left[\frac{1}{m_c}\sum_{k=1}^{m_c}(\frac{k}{m_c})^{2r} + o(1)\right] \\ &= \frac{\pi^{2r}}{2r+1}\left[\frac{m_c^{2r}}{c(n)^r} + o(1)\right]\end{aligned}$$

because

$$\frac{1}{m_c}\sum_{k=1}^{m_c}(\frac{k}{m_c})^{2r} - \int_0^1 x^{2r}dx = o(1) \ .$$

(**QED**)

Lemma A-2 : Let

(A.2) $$b_{kj} = \sqrt{c(n)}[p_{kj} - p_{k,j+1}] = \frac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{kj} - \frac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{k,j+1}$$

for $k = 1, \cdots, c(n); j = 1, \cdots, c(n)-1$ and

$$b_{k,c(n)} = \frac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{k,c(n)} \ , \theta_{kj} = \frac{2\pi}{2c(n)+1}(k-\frac{1}{2})(j-\frac{1}{2}) \ .$$

Then

$$\sum_{j=1}^{c(n)}[b_{kj}]^2 = [1 + O(\frac{1}{c(n)})]a_{k,c(n)} \ , \ \sum_{j=1}^{c(n)}[b_{kj}]^4 = [\frac{3}{2c(n)} + o(\frac{1}{c(n)})][a_{k,c(n)}]^2 \ ,$$

and for any positive integers k_1, k_2 there exists a constant K_1 such that

$$\sum_{j=1}^{c(n)}[b_{k_1j}]^2[b_{k_2j}]^2 \le K_1\frac{a_{k_1,c(n)}a_{k_2,c(n)}}{c(n)} \ .$$

Proof of Lemma A-2 : We use the decomposition as

$$\begin{aligned}\frac{2c(n)+1}{c(n)}\sum_{j=1}^{c(n)-1}[b_{kj}]^2 &= \sum_{j=1}^{c(n)-1}[(1-e^{i\theta_k})e^{i\theta_{kj}}]^2 + \sum_{j=1}^{c(n)-1}[(1-e^{-i\theta_k})e^{-i\theta_{kj}}]^2 \\ &\quad +2(c(n)-1)(1-e^{i\theta_k})(1-e^{-i\theta_k}) \ ,\end{aligned}$$

where $\theta_k = [2\pi/(2c(n)+1)](k-1/2)$ $(k = 1, \cdots, c(n))$.
and $\theta_{kj} = [2\pi/(2c(n)+1](k-1/2)(j-1/2)$. Then we use the relation

(A.3) $$\begin{aligned}\sum_{j=1}^{c(n)}[e^{i\theta_{kj}}]^2 &= e^{i\theta_k}\frac{1-e^{i(4\pi/(2c(n)+1))(k-1/2)c(n)}}{1-e^{2i\theta_k}} \\ &= e^{i\theta_{kj}}\frac{1-e^{i\pi(2k-1)}e^{-i\theta_k}}{1-e^{2i\theta_k}}\end{aligned}$$

because we have

$$e^{i(4\pi/(2c(n)+1)(k-1/2)c(n)} = e^{i(\pi/(2c(n)+1)(2k-1)(2c(n)+1-1)} = e^{i\pi(2k-1)}e^{-i\theta_k} \ .$$

We use the relation $e^{i\theta_{kc(n)}}+e^{-i\theta_{kc(n)}} = e^{i\pi(k-1/2)}[e^{-i\theta_k}-e^{i\theta_k}] = 2\sin\theta_k = 4\sin(\theta_k/2)\cos(\theta_k/2)$ for the last term $b_{k,c(n)}$ of (A.3). Then, by arranging each terms and use the relation

$$(1-e^{i\theta_k})(1-e^{-i\theta_k}) = (e^{-i\frac{\theta_k}{2}} - e^{i\frac{\theta_k}{2}})(e^{i\frac{\theta_k}{2}} - e^{-i\frac{\theta_k}{2}}) \ ,$$

we have the result.

By using the similar but tedious arguments for the fourth-powers, after some calculations (we only need to evaluate the dominant terms), we find that

$$\begin{aligned} \frac{[2c(n)+1]^2}{c(n)^2}\sum_{j=1}^{c(n)-1}[b_{kj}]^4 &= \sum_{j=1}^{c(n)-1}[e^{i\theta_{kj}}(1-e^{i\theta_k}) + e^{-i\theta_{kj}}(1-e^{-i\theta_k})]^4 \\ &= [6c(n)+O(1)]\times 4^2\sin^4\frac{\theta_k}{2} \ . \end{aligned}$$

The last evaluation follows by applying the Chaucy-Schwartz inequality to

$$\sum_{j=1}^{c(n)}[b_{k_1j}]^2[b_{k_2j}]^2 \le [\sum_{j=1}^{c(n)}[b_{k_1j}]^4]^{1/2}[\sum_{j=1}^{c(n)}[b_{k_1j}]^4]^{1/2} \ ,$$

and by using the above relation.

(**Q.E.D.**)

Lemma A-3 : Assume the conditions in Theorem 2 for the diffusion-plus-noise model. Let $M_{2,(i)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}]^2$ $(k=1,\cdots,c(n); i=1,\cdots,b(n))$.

Then

(A.4) $$\mathbf{E}[(M_{2,(i)})^2] = O(\frac{1}{(b(n))^2} + \frac{m_c^4}{(c(n))^2}) \ .$$

Also

(A.5) $$\sum_{i=1}^{b(n)} P(\|M_{2,(i)}\| > u_n) \xrightarrow{p} 0$$

if

(A.6) $$A_n = \frac{1}{u_n^2}[\frac{1}{b(n)} + \frac{b(n)}{c(n)^{2-4\alpha}}] \xrightarrow{p} 0 \ .$$

Proof of Lemma A-3 : (i) Let $M_{2,(i)}^{(1)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}]^2$ for $z_{k,(i)}^{(1)} = z_{k,(i)}$ when $\epsilon_n = 0$. Then by using (A.10) and (A.11) below, we decompose

$$\text{(A.7)} \quad M_{2,(i)}^{(1)} = \sum_{k,l=1}^{c(n)}[\delta(k,l)\int_{t_{k-1}^n(i)}^{t_k^n(i)}\sigma_s^2 ds + (c_{kl}r_{k,(i)}r_{l,(i)} - \delta(k,l)\int_{t_{k-1}^n(i)}^{t_k^n(i)}\sigma_s^2 ds)] \ .$$

Since the first term is of the order $O_p(c(n)/n)$ and the second term is of the order $O_p(1/[b(n)\sqrt{b(n)m_c}])$.
Let $M_{2,(i)}^{(2)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}]^2$ for $z_{k,(i)}^{(2)} = z_{k,(i)}$ when $\sigma_s = 0$ $(0 \le s \le 1)$. We need to evaluate the expected vales of $(\frac{1}{m_c})^2\sum_{k,k^{‘}=1}^{m_c}(\sum_{j=1}^{c(n)} b_{k,j}v_j)^2(\sum_{j^{‘}=1}^{c(n)} b_{k^{‘},j}v_{j^{‘}})^2$. Then, by using Lemma A-2, we can find a constant K_1 such that

$$\text{(A.8)} \qquad \mathbf{E}[M_{2,(i)}^{(2)}]^2 \le K_1[\frac{1}{m_c}\sum_{k=1}^{c(n)} a_{k,c(n)}]^2 \ .$$

By using Lemma A-1, we have the first result under the conditions in Theorem 2.
(ii) By using the Markov inequality,

$$\text{(A.9)} \qquad \begin{aligned} \sum_{i=1}^{b(n)} P(\|M_{2,(i)}\| > u_n) &= \sum_{i=1}^{b(n)} P(\|M_{2,(i)}\|^2 > u_n^2) \\ &\le \sum_{i=1}^{b(n)} \frac{\mathbf{E}[(M_{2,(i)})^2]}{u_n^2} \\ &= O(\frac{1}{u_n^2}b(n)[\frac{1}{b(n)^2} + \frac{(m_c)^4}{c(n)^2}]) \ . \end{aligned}$$

and $b(n) = n^{1-\gamma}$, we have the result.
(**Q.E.D.**)

2. **Derivation of Theorem 2** :
With the transformation (4.1) when $p = 1$ in the set $I_c(i) = (t_{i-1}^{(n)}, t_i^{(n)}]$, we write $z_{k,(i)} = z_{k,(i)}^{(1)} + z_{k,(i)}^{(2)}$, where $z_{k,(i)}^{(1)}$ and $z_{k,(i)}^{(2)}$ correspond to the $k-$th elements of $\mathbf{Z}_{c(n),(i)}^{(1)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}(\mathbf{X}_{c(n),(i)} - \bar{\mathbf{y}}_{0,(i)})$ and $\mathbf{Z}_{c(n),(i)}^{(2)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}\mathbf{V}_{c(n),(i)}$, respectively, where $\mathbf{X}_{c(n),(i)}$ and $\mathbf{V}_{c(n),(i)}$ are the $c(n) \times 1$ state vector and the noise vector in $I_c(i)$ $(i = 1, \cdots, b(n))$. Then, we have $\mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(1)}] = \mathbf{0}$ and $\mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(2)}] = \mathbf{0}$ and

$$\text{(A.10)} \qquad \mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(2)}\mathbf{Z}_{c(n),(i)}^{(2)'}] = \sigma_v^2 h_{c(n)}^{-1}\mathbf{D}_{c(n)}^{(2)} \ ,$$

where $\mathbf{D}_{c(n)}^{(2)} = \text{diag}(d_k)$ $(k \in I_c(i))$. (We take the interval $(t_i^{(n)}(k-1), t_i^{(n)}(k)] \in I_c(i)$ $(i = 1, \cdots, c(n); k = 1, \cdots, c(n)$, and we have $t_i^{(n)}(k) - t_i^{(n)}(k-1) = 1/n$.)
When σ_s is a time-varying and deterministic function,

$$\text{(A.11)} \qquad \mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(1)} \mathbf{Z}_{c(n),(i)}^{(1)'}] = h_{c(n)}^{-1} \mathbf{P}_{c(n)} \mathbf{D}_{c(n)}^{(1)} \mathbf{P}_{c(n)} \ ,$$

where $\mathbf{D}_{c(n)}^{(1)} = \text{diag}(\sigma^2(t_k^{(n)}(i))$ (for $k \in I_c(i)$) and $\sigma^2(t_k^{(n)}(i)) = \int_{t_{k-1}^{(n)}(i)}^{t_k^{(n)}(i)} \sigma_s^2 ds$, which is σ_s^2 at $s = t_{k-1}^{(n)}(i)$.
When σ_s is stochatic under the assumption that it is bounded and Lipschitz-continuous, we can use the similar argument based of the standard approximation in stochastic calculus for $\int_{t_{k-1}^{(n)}(i)}^{t_k^{(n)}(i)} \sigma_s^2 ds$ by $\sigma^2(t_{k-1}^{(n)}(i))[t_k^{(n)}(i) - t_{k-1}^{(n)}(i)]$. (See (A.30) below for some details.)
We use the decomposition

$$\begin{aligned}
\text{(A.12)} \qquad M_{2,(i)} &= \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)} + z_{k,(i)}^{(2)}]^2 \\
&= \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)}]^2 + \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(2)}]^2 + 2 \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)} z_{k,(i)}^{(2)}]^2 \\
&= M_{2,(i)}^{(1)} + M_{2,(i)}^{(2)} + 2 M_{2,(i)}^{(12)} \ (\text{, say}) \ .
\end{aligned}$$

For the third term, we utilize the well-known inequality that $[M_{2,(i)}^{(12)}]^2 \leq M_{2,(i)}^{(1)} M_{2,(i)}^{(2)}$, and we find that the effects $M_{2,(i)}^{(12)}$ $(i = 1, \cdots, b(n))$ are stochastically negligible.

In $I_c(i)$ we write $z_{k,(i)}^{(2)} = \sum_{j=1}^{c(n)} b_{kj} v_j(i)$, $v_j(i)$ are noise terms in $I_c(i)$ $(i = 1, \cdots, b(n))$ and b_{kj} are the corresponding coefficients of $h_{c(n)}^{-1/2} \mathbf{P}_{c(n)} \mathbf{C}_{c(n)}^{-1}$. We re-write

$$\text{(A.13)} \qquad z_{k,(i)}^{(1)} = \sqrt{\frac{4c(n)}{2c(n)+1}} \sum_{j=1}^{c(n)} s_{kj} r_{j,(i)} \ ,$$

$s_{kj} = \cos \theta_{kj}$ and $\theta_{kj} = (2\pi/(2m_c + 1))(k - 1/2)(j - 1/2)$ $(j, k = 1, \cdots, c(n))$.

We use the relations of trigonometric functions for $z_{k,(i)}^{(1)}$ and rewrite

$$\text{(A.14)} \qquad M_{2,(i)}^{(1)} = \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)}]^2 = \sum_{k,l=1}^{c(n)} c_{kl} r_{k,(i)} r_{l,(i)} \ ,$$

where $r_{k,(i)}$ $(= X(t_i^{(n)}(k)) - X(t_i^{(n)}(k-1)))$ are returns in the interval $(t_i^{(n)}(k-1), t_i^{(n)}(k)] \in I_c(i)$ $(i = 1, \cdots, c(n); k = 1, \cdots, c(n))$. and $c_{kl} = (2/m_c)\sum_{j=1}^{m_c} s_{kj}s_{lj}$. Due to the basic properties of c_{kl} $(k, l = 1, \cdots, c(n))$ (see Lemmas 5.1 and 5.2 of KSK (2018)), (A.12) is a consistent estimator of the hidden volatility $\int_{I_c(i)} \sigma_s^2 ds[c(n)/n]$ in a sense although we need the last term $c(n)/n$ due to the fact that the total sample size is n while we use its fraction in the intervals $I_c(i)$ $(i = 1, \cdots, b(n))$ in the local estimation. When there exist the market micro-structure noise, however, we further need to evaluate the effects of noise in $M_{2,(i)}$. (We used the notations and derivations of the asymptotic properties of the SIML method in Chapter 5 of KSK (2018) when $p = 1$, which corresponds to the case when $\epsilon_n = 1$.)

By using the analogous arguments as Chapter 5 of KSK (2018) to the local interval $I_c(i)$ $(i = 1, \cdots, b(n))$, we evaluate the conditions that the effects of noises are negligible and the proper order of the stochastic part around the volatility

$$\text{(A.15)} \qquad \sqrt{m_c}\sum_{k,l=1}^{c(n)}[c_{kl}r_{k,(i)}r_{l,(i)} - \delta(k,l)\int_{t_i^n(k-1)}^{t_i^n(k)} \sigma_s^2 ds]$$

which is of the order $O_p(c(n)/n)$. Since each term with i $(i = 1, \cdots, b(n))$ are asymptotically uncorelated, the normalized stochastic part around the volatility should be in the form of

$$\text{(A.16)} \qquad \sqrt{m_c b(n)}\sum_{i=1}^{b(n)}\sum_{k,l=1}^{c(n)}[c_{kl}r_{k,(i)}r_{l,(i)} - \delta(k,l)\int_{t_i^n(k-1)}^{t_i^n(k)} \sigma_s^2 ds] \ ,$$

Next, we use Lemma A-1 in the Appendix (see the proof of Lemma 5.3 of KSK (2018)) to evaluate the asymptotic bias. If $\sigma_s = 0$ $(0 \leq s \leq 1)$ and ϵ_n $(= \epsilon)$ is a fixed constant, by using (4.5) and (4.6), the bias term is proportional to $\mathbf{E}[m_c^{-1}\sum_{k=1}^{m_c}[z_{k,(i)}^{(2)}]^2]$ in all intervals and it is given as $\epsilon_n^2\ \sigma_v^2$ times

$$\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)} = O(\frac{1}{m_c} \times \frac{m_c^3}{c(n)}) = O(\frac{m_c^2}{c(n)}) \ .$$

In the general case with micro-market noises, we use the transformation (4.1) in the local SIML estimation and then the bias term of $\sum_{i=1}^{b(n)}[1/m_c]\sum_{k=1}^{m_c} z_{k,(i)}^2$ in each interval is asymptotically equivalent to a constant $((\pi^2/3)\sigma_v^2)$ times

$$\text{(A.17)} \qquad \mathrm{AB}_{1n} = b(n)\frac{(m_c)^2}{c(n)}[\epsilon_n]^2 \ .$$

Because σ_s is Lipschitz-continuous, in $I_c(i)$ $(i = 1, \cdots, b(n))$, we can take a positive constant K_2 such that

$$\begin{aligned} \left| \int_{s \in (t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds - \sigma^2(t_{i-1}^{(n)})[\frac{c(n)}{n}] \right| &\leq K_2 \left| \int_{s \in (t_{i-1}^{(n)}, t_i^{(n)}} [s - (i-1)\frac{c(n)}{n}] ds \right| \\ &= O_p\left((\frac{c(n)}{n})^2 \right) , \end{aligned}$$

which is $O_p(1/b(n)^2)$. Then we have the relation that

$$\left[\sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} z_{k,(i)}^2 \right] - \int_0^1 \sigma_s^2 ds \xrightarrow{p} 0 , \tag{A.18}$$

provided that the bias can be negligible, that is $\max\{\frac{1}{b(n)}, \frac{1}{m_c}\} \longrightarrow 0$ and $\mathrm{AB}_{1n} \to 0$ as $n \to \infty$.

For the asymptotic normality of $\hat{V}(2)$ without any asymptotic bias term, we use the fact that the dominant factor of (A.14) is a martingale part. A sufficient condition for the asymptotic normality (see Theorem 3.3 of KSK (2018)) would be

$$\mathrm{AB}_{2n} = \sqrt{m_c b(n)} b(n) \frac{(m_c)^2}{c(n)} [\epsilon_n]^2 \longrightarrow 0 \tag{A.19}$$

as $n \to \infty$.

If we set $c(n) = n^\gamma$, $b(n) = n^{1-\gamma}$ and $m_c = [c(n)]^\alpha$, then

$$\mathrm{AB}_{1n} = n^{1-2\gamma+2\gamma\alpha-2\delta} , \tag{A.20}$$

and

$$\mathrm{AB}_{2n} = n^{\frac{1-\gamma}{2} + \frac{\alpha\gamma}{2} + 1 - 2\gamma + 2\gamma\alpha - 2\delta} = n^{1-2\delta+\frac{5}{2}\alpha\gamma+\frac{1}{2}(1-5\gamma)} . \tag{A.21}$$

By setting $\alpha_1^* = 1 + [2\delta - 1]/[2\gamma]$, and $\alpha_2^* = 1 + [4\delta - 3]/[5\gamma]$, we have the result on the asymptotic distribution of the local SIML estimation in the simplest case.

The CLT in the stable-convergence sense will be discussed at the end of the Appendix.

3. **Derivation of Theorem 3** :

(i) For $r \geq 2$, we decompose

$$
\begin{aligned}
\text{(A.22)} \quad & \hat{V}(2r) - V(2r) \\
= \; & [b(n)]^{r-1} \sum_{i=1}^{b(n)} [M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds + \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds]^r - V(2r) \\
= \; & [b(n)]^{r-1} \sum_{i=1}^{b(n)} [\int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds]^r - V(2r) \\
& + [b(n)]^{r-1} \sum_{i=1}^{b(n)} {}_r\mathrm{C}_1 [\int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds]^{r-1} [M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds] \\
& + [b(n)]^{r-1} \sum_{i=1}^{b(n)} \sum_{j=2}^{r} {}_r\mathrm{C}_j [\int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds]^{r-j} [M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds]^j \; .
\end{aligned}
$$

There are three terms in (A.22) and we evaluate each term from the representation

$$
M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds = [M_{2,(i)}^{(1)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds] + [M_{2,(i)}^{(2)} + M_{2,(i)}^{(12)}] \; .
$$

First, we consider the effects of $M_{2,(i)}^{(1)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds$ $(i = 1, \cdots, b(n))$ as if it were $M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds$ in the above expression.

Under the assumption of Lipschitz-condition on σ_s^2, we can use the evaluation $[\int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds - \sigma^2(t_{i-1}^{(n)})[t_i^{(n)} - t_{i-1}^{(n)}]| = O_p([\frac{1}{b(n)}]^2)$. Then, the first tem of (A.22) converges to zero in probability because σ_s^2 is bounded and

$$
|\int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds| = O_p([t_i^{(n)} - t_{i-1}^{(n)}]) = O_p(\frac{c(n)}{n}) = O_p(\frac{1}{b(n)}) \; .
$$

For the second and third terms of (A.22), we use the fact that $b(n)[M_{2,(i)} - \int_{(t_{i-1}^{(n)}, t_i^{(n)}]} \sigma_s^2 ds] = O_p(1/\sqrt{m_c})$ from the proof of Theorem 2. Then, by evaluating the orders of other terms and using Theorem 2, we have the consistency.

(ii) For the asymptotic normality of the LSIML estimator, we want to evaluate the stochastic behavior of $\sqrt{b(n)m_c}[\hat{V}(2r) - V(2r)]$ $(r \geq 2)$. When $r \geq 2$, we have the term

$$
V(2r)^* = [b(n)]^{r-1} \sum_{i=1}^{b(n)} [\int_{(t_{i-1}^{(n)}, (t_i^{(n)}]} \sigma_s^2 ds]^r \; .
$$

Then, we find that $V^*(2r) - V(2r) = O_p(1/b(n))$, and $\sqrt{b(n)m_c}[V^*(2r) - V(2r)] = O_p(\sqrt{\frac{m_c}{b(n)}})$ is not necessarily negligible in the general case. (It is zero when the volatility function is constant.) Hence we need to normaliz $\sqrt{b(n)m_c}[\hat{V}(2r) - V(2r) +$

$(V(2r) - V^*(2r))]$.

Then we can evaluate the limiting distribution of the normalized LSIML estimator from the second term of (A.22). It is approximately equivalent to the random variable

$$\text{(A.23)} \qquad \mathrm{AM}_n(2r) = \sqrt{b(m)m_c}\sum_{i=1}^{b(n)} r[\sigma(t_{i-1}^{(n)})]^{2(r-1)}[M_{2,(i)}^{(1)} - \int_{(t_{i-1}^{(n)},(t_i^{(n)}]} \sigma_s^2 ds] \ ,$$

and its asymptotic variance is given by

$$\mathrm{AV}(2r) = r^2 \times 2\int_0^1 \sigma_s^{4(r-1)+4} ds \ .$$

The order of the third term of (A.22) multiplied by $\sqrt{b(m)m_c}$ is $O_p(\sqrt{b(m)m_c}) \times O_p((1/\sqrt{m_c})^2) = O_p(\sqrt{b(n)}/\sqrt{m_c})$. Then, it goes to zero if $b(n)/m_c \to 0$ as $n \to \infty$. A sufficent condition is $1 - \gamma - \gamma\alpha < 0$.

(iii) It remains to show that the effects of the bias due to the presence of market micro-structure on $M_{2,(i)}^{(2)}$ $M_{2,(i)}^{(12)}$ $(i = 1, \cdots, b(n))$ in the third term of (A.22) are stochastically negligible for the asymptotic normality of (A.23). We use the relation that for any positive integer $r \geq 2$,

$$[M_{2,(i)} - \int_{(t_{i-1}^{(n)},t_i^{(n)}]} \sigma_s^2 ds]^r = \sum_{j=0}^{r} {}_r\mathrm{C}_j [M_{2,(i)}^{(1)} - \int_{(t_{i-1}^{(n)},t_i^{(n)}]} \sigma_s^2 ds]^j [M_{2,(i)}^{(2)} + M_{2,(i)}^{(12)}]^{r-j} \ .$$

and evaluate the order of each term.

We need straight-forward, but tedious calculations for each each term involving $M_{2,(i)}^{(2)}$ $(i = 1, \cdots, b(n))$. We prepare the following lemma.

Lemma A-4 : Under the assumptions in Theorem 3, for any positive integer $r \geq 2$, there exists a constant K_r^* such that

$$\mathbf{E}[(M_{2,(i)})^r] \leq K_r^* [\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,(i)}]^r \ \ (i = 1, \cdots, b(n)) \ .$$

Then, we have

$$\text{(A.24)} \qquad [b(n)]^{r-1}\sum_{i=1}^{b(n)} \mathbf{E}[(M_{2,(i)})^r] \leq K_r^* [\mathrm{AB}_{1n}]^r \ ,$$

where AB_{1n} is given by (A.17).

We state Lemma A-4 for the general case, but we only illustrate a typical evaluation for the case $r = 2$ with the terms involving $[M_{2,(i)}^{(2)}]^2$. Because $\mathbf{E}[M_{2,(i)}] = (1/m_c)\sum_{k=1}^{m_c} a_{k,(i)}$ and we have fourth-order moments of market noise, by using Lemma A-2 and lengthy (but elementary) calculations, we can evaluate the result as $\mathbf{E}[(M_{2,(i)})^2] = [\mathbf{E}(M_{2,(i)}))]^2 + \mathbf{E}[M_{2,(i)} - \mathbf{E}(M_{2,(i)}))]^2$. Then, we can find a constant K_{21}^* $(= K_2^* - 1)$ such that

$$
\text{(A.25)} \qquad \begin{aligned}
& \mathbf{E}[m_c(M_{2,(i)} - \mathbf{E}(M_{2,(i)}))]^2 \\
= \; & [\sum_{k=1}^{m_c} \mathbf{E}[(z_{k,(i)}^{(2)})^2 - \mathbf{E}((z_{k,(i)}^{(2)})]^2 \\
= \; & \sum_{k_1,k_2=1}^{m_c} \mathbf{E}([\sum_{j_1=1}^{c(n)} b_{k_1,j_1}^2 (v_{j_1}(i)^2 - \sigma_v^2) + \sum_{j_1 \neq j_2} b_{k_1,j_1} b_{k_1,j_2} v_{j_1}(i) v_{j_2}(i)] \\
& \quad \times [\sum_{j_3=1}^{c(n)} b_{k_2,j_3}^2 (v_{j_3}(i)^2 - \sigma_v^2) + \sum_{j_3 \neq j_2} b_{k_2,,j_3} b_{k_2,,j_4} v_{j_3}(i) v_{j_4}(i)]) \\
\leq \; & K_{21}^* [\sum_{k=1}^{m_c} a_{k,(i)}]^2 \;,
\end{aligned}
$$

where we denote $v_j(i)$ is the j-th market noise in the set $I_c(i)$ $(i = 1, \cdots, b(n))$.

The effects of $M_{2,(i)}^{(12)}$ $(i = 1, \cdots, b(n))$ are evaluated in a similar way. After the results of many evaluations, we find that the third term of (A.22) is asymptotically negligible for the asymptotic normality of (A.23).

From (A.22) and (A.23), we find that the dominant factor of the normalized LSIML estimator is a linear combination of $M_{2,(i)}^{(1)}$ $(i = 1, \cdots, n)$, which is essentially the same as (A.14) and (A.15) in the derivation of Theorem 2. By using (A.17) and (A.19), the condition for consistency of the LSIML estimator of $V(2r)$ becomes

$$
\text{(A.26)} \qquad \mathrm{AB}_{1n} \longrightarrow 0 \;.
$$

Since the asymptotic bias from the third term of (A.22) is $\sqrt{b(n)m_c} \times [\mathrm{AB}_{1n}]^2 = \mathrm{AB}_{1n} \times \mathrm{AB}_{2n}$ at most and other terms are of higher orders, the condition for the asymptotic normality without bias becomes

$$
\text{(A.27)} \qquad \mathrm{AB}_{2n} \longrightarrow 0 \;.
$$

(iv) In the general case of stochastic volatility, we need the stable-convergence because the limiting terms as $\int_0^1 \sigma_s^{2r} ds$ are stochastic. Since we are considering higher order Brownian motions of the form (2.5) under (2.4) and the dominant terms are martingale differences, it is possible to show the stable-convergence. Some dscussion on the CLT and the stable-convergence will be given at the end of the Appendix

4. **Proof of Theorem 5** :

We use the decomposition method for the diffusion part and jump part, which was used in the derivation of Theorem 2 in Section 5 and Theorem 2.1 of KK (2021).

In the set $I_c(i) = (t_{i-1}^{(n)}, t_i^{(n)}]$, $(i = 1, \cdots, b(n))$ we set $z_{k,(i)} = z_{k,(i)}^{(1)} + z_{k,(i)}^{(2)}$, where $z_{k,(i)}^{(1)}$ and $z_{k,(i)}^{(2)}$ $(k = 1, \cdots, c(n))$ correspond to the $k-$th elements of $\mathbf{Z}_{c(n),(i)}^{(1)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}(\mathbf{X}_{c(n),(i)} - \bar{\mathbf{y}}_{0,(i)})$ and $\mathbf{Z}_{c(n),(i)}^{(2)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}\mathbf{V}_{c(n),(i)}$ ($\mathbf{X}_{c(n),(i)}$ and $\mathbf{V}_{c(n),(i)}$ are the $c(n) \times 1$ state vector and the noise vector, respectively, in $I_c(i)$ $(i = 1, \cdots, b(n))$). We use the decomposition $M_{2,(i)} = M_{2,(i)}^{(1)} + [M_{2,(i)}^{(2)} + M_{2,(i)}^{(12)}$ as in the derivation of Theorem 2.

Because $M_{2,(i)}^{(2)}\mathrm{I}(M_{2,(i)} \leq u) \leq M_{2,(i)}^{(2)}$, we can apply Conditions in Theorem 2 to $M_{2,(i)}^{(2)} = (1/m_c)\sum_{k=1}^{m_c}[z_{k,(i)}^{(2)}]^2$ and then the asymptotic bias AB_n in (A.17) is asymptotically negligible. Also we can use $|M_{2,(i)}^{(12)}\mathrm{I}(M_{2,(i)} \leq u)| \leq |M_{2,(i)}^{(12)}|$, and we find that the effects $M_{2,(i)}^{(12)}$ $(i = 1, \cdots, b(n))$ are stochastically negligible.

Next, we use the representation

$$M_{2,(i)}^{(1)} = \frac{1}{m_c}\sum_{k=1}^{m_c}[z_{k,(i)}^{(1)}]^2 = \sum_{k,l=1}^{c(n)} c_{kl} r_{k,(i)} r_{l,(i)} \ , \tag{A.28}$$

where $r_{k,(i)}$ $(= X(t_i^n(k)) - X(t_i^n(k-1)))$ are returns in the interval $(t_i^n(k-1), t_i^n(k)] \in I_c(i)$ $(i = 1, \cdots, c(n); k = 1, \cdots, c(n))$.

Here we take $t_i^n(k) - t_i^n(k-1) = 1/n$, $c_{kl} = (2/m_c)\sum_{j=1}^{m_c} s_{kj}s_{lj}$, and then approximate $r_{k,(i)}$ $(= X(t_k^{(n)}(i))) - X(t_{k-1}^{(n)}(i)))$ by

$$r_{k,(i)}^{(n)} = \sigma(t_k^{(n)}(i))\Delta B(t_k^{(n)}(i) + \sum_{s \in I_k(i)} \Delta X(s) \ , \tag{A.29}$$

where $\Delta B(t_k^{(n)}(i)) = B(t_k^{(n)}(i)) - B(t_{k-1}^{(n)}(i))$ and $I_k(i) = (t_{k-1}^{(n)}(i), t_k^{(n)}(i)]$.

In this approximation, by using Lipschitz condition on σ_s in $(t_{k-1}^{(n)}(i), t_k^{(n)}(i)](k =$

$1, \cdots, c(n); i = 1, \cdots, b(n))$, we can evaluate

$$\text{(A.30)} \qquad |\int_{t_{k-1}^{(n)}(i)}^{t_k^{(n)}(i)} \sigma_s dB - \sigma(t_{k-1}^{(n)}(i)) \Delta B(t_k^{(n)}(i)| = O_p(\frac{1}{n\sqrt{n}})$$

and hence we will ignore the differences of approximations, which are of the higher orders.

Let

$$\text{(A.31)} \qquad QV_i = \int_{t_{k-1}^{(n)}(i)}^{t_k^{(n)}(i)} \sigma_s^2 ds + \sum_{s \in I_k(i)} (\Delta X(s))^2$$

and we decomose

$$\begin{aligned}
M_{2,(i)}^{(1)} - QV_i &= \sum_{k=l=1}^{c(n)} \left[(r_{k,(i)})^2 - QV_i \right] \\
&= \sum_{k=l=1}^{n} (c_{kk} - 1) \left[(r_{k,(i)})^2 - QV_i \right] \\
&+ \sum_{k \neq l1}^{n} c_{kl} \left[r_{k,(i)} r_{l,(i)} - \sum_{s \in I_k(i)} \Delta X(s) \sum_{s \in l_k(i)} \Delta X(s) \right] \\
&+ \sum_{k \neq l1}^{n} c_{kl} \left[\sum_{s \in I_k(i)} \Delta X(s) \sum_{s \in l_k(i)} \Delta X(s) \right] \\
&= (I) + (II) + (III) + (IV) \ \ (\text{, say}) \ .
\end{aligned}$$

By using the similar evaluations (which are straight-forward, but tedious) as the proof of Theorem 2.1 of Kunitomo and Kurisu (2021), we have

$$\text{(A.32)} \qquad [(I) + (II) + (III) + (IV)] \xrightarrow{p} 0$$

under Condition (i) of Theorem 2. For (IV), because we have a finite number of jumps, we can take the jump times $0 < i_1(n) < \cdots < i_M < n$ and we can take $0 < s_1 < \cdots < s_M < 1$ such that $i_j(n)/n \to s_j$ $(j = 1, \cdots, M)$. Then

$$\text{(A.33)} \qquad \sqrt{m_c} \sum_{i=1}^{b(n)} \sum_{k \neq l=1}^{c(n)} c_{lk} \left[\sum_{s \in I_k(i)} \Delta X(s) \sum_{s \in I_l(i)} \Delta X(s) \right] = o_p(\frac{1}{\sqrt{m_c}})$$

as $m_c \to \infty$ because

$$c_{kl} = \frac{1}{2m} (\frac{2c(n)}{2c(n)+1}) \left[\frac{\sin 2\pi m_c (\frac{k+l-1}{2c(n)+1})}{\sin(\pi \frac{k+l-1}{2c(n)+1})} + \frac{\sin 2\pi m_c (\frac{l-k}{2c(n)+1})}{\sin(\pi \frac{l-k}{2c(n)+1})} \right] \quad (k \neq l) \ .$$

Next, for QV $(i = 1, \cdots, b(n))$ the first term is of the order $c(n) \times (1/n) = 1/b(n)$ and the second term is of the order $c(n)$. Then we find that for any positive constant $QV_i \mathrm{I}(M_{2(i)} < u)$ is of the order $1/b(n)$ when n is large. Hence, as $n \to \infty$ for any positive fixed constant u,

$$\text{(A.34)} \qquad \sum_{i=1}^{b(n)} \left[M_{2,(i)}^{(1)} \mathrm{I}(M_{2,(i)} < u) \right] - \int_0^1 \sigma_s^2 ds \xrightarrow{p} 0$$

and

$$\text{(A.35)} \qquad \sum_{i=1}^{b(n)} \left[M_{2,(i)}^{(1)} \mathrm{I}(M_{2,(i)} > u) \right] - \sum_{0 \le s \le 1} (\Delta X_s)^2 \xrightarrow{p} 0 \ .$$

By using Lema A-3 in the Appendix, we have the consistency of the continuous part and jump part of the quadratic variation.

(ii) For the asymptotic distribution, we consider

$$\begin{aligned}
\sum_{i=1}^{b(n)} \sum_{k \neq l = 1}^{c(n)} c_{kl} \left[r_{k,(i)} r_{l,(i)} \right] \ = \ & \sum_{i=1}^{b(n)} \sum_{k \neq l}^{c(n)} c_{kl} \Big[\sigma(t_{k-1}^{(n)}(i)) \sigma(t_{l-1}^{(n)}(i)) \Delta B(t_k^{(n)}(i)) \Delta B(t_l^{(n)}(i)) \\
& + \sigma(t_{k-1}^{(n)}(i)) \Delta B(t_k^{(n)}(i)) \sum_{s \in I_l(i)} \Delta X(s) \\
& + \sigma(t_{l-1}^{(n)}(i))) \Delta B(t_l^{(n)}(i)) \sum_{s \in I_k(i)} \Delta X(s) \Big] \ ,
\end{aligned}$$

Then we need to evaluate the asymptotic behaviors of the continuous part of the limiting disrtribution of

$$\text{(A.36)} \qquad \sqrt{b(n) m_c} \sum_{i=1}^{b(n)} \sum_{k \neq l}^{c(n)} c_{kl} \left[\sigma(t_{k-1}^{(n)}(i)) \sigma(t_{l-1}^{(n)}(i)) \Delta B(t_k^{(n)}(i)) \Delta B(t_l^{(n)}(i)) \right],$$

and the jump part of the limiting disrtribution of

$$\begin{aligned}
\text{(A.37)} \qquad \sqrt{b(n) m_c} \sum_{i=1}^{b(n)} \sum_{k \neq l}^{c(n)} c_{kl} \Big[& \sigma(t_{k-1}^{(n)}(i)) \Delta B(t_k^{(n)}(i)) \sum_{s \in I_l(i)} \Delta X(s) \\
& + \sigma(t_{l-1}^{(n)}(i))) \Delta B(t_l^{(n)}(i)) \sum_{s \in I_k(i)} \Delta X(s) \Big] .
\end{aligned}$$

Two parts of the limiting random variables in (A.36) and (A.37) are asymptotically independent. By using Lemma 5.6 of KSK (2018), we can derive the variances of the limiting distributions as (7.9) and (7.8). (See Lemma 8.3.3 of Anderson (1971) for the Feje-kernel operation.) Finally, we apply lengthy arguments for CLT, which are similar to the derivation of Theorem 2 in Section 2 and the proof of Theorem 2.1 in KK (2021). By using Lemma A-3 in the Appendix. We have the asymptotic normality of (A.36) and (A.37) in the stable-convergence sense. (We have omitted some details, but we give a discussion on the CLT and the stable convergence in the next subsection.)

(**Q.E.D.**)

5. **On Stable Convergence and MCLT** :

We give an outline of the underlying arguments of the CLT and stable-convergence in Theorem 2, Theorem 3, and Theorem 5. We consider the simple diffusion model of (2.1)-(2.4) when μ_s^σ and ω_s^σ in (2.3) and (2.4) are bounded and Lipschitz-continuous with $p = r = 1$ and $b(n) = 1$. Then we denote $c(n) = n$ and $m_c = m_n$ as in KSK (2018) under the conditions as $m_n \to \infty$ and $m_n = O(n^\alpha)$ $(0 < \alpha < .4)$. (For jump terms, we need some additional arguments on the validity of asymptotic normality in the sense of stable convergence.)

By using Itô's formula, we can represent

$$\sigma_t^4 = \sigma_0^4 + \int_0^t \mu_s^{\sigma *} ds + \int_0^t \omega_s^{\sigma *} dB_s^\sigma \ (0 \le s \le t \le 1) \ , \tag{A.38}$$

where $\mu_s^{\sigma *}$ and $\omega_s^{\sigma *}$ are the drift and diffusion coefficients and B_s^σ is Brownian motion, which may be correlated with B_s.

For $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ we write

$$V(4) = \sigma_0^4 + \sum_{j=1}^n [\int_{t_{j-1}^n}^{t_j^n} (\int_0^t \mu_s^{\sigma *} ds) dt + \int_{t_{j-1}^n}^{t_j^n} (\int_0^t \omega_s^{\sigma *} dB_s^\sigma) dt] \ . \tag{A.39}$$

Then, $V(4)$ is a diffusion process and the last term of $V(4)$ becomes the sum of

$$V_i^n = \int_{t_{i-1}^n}^{t_i^n} (\int_s^1 dt) \omega_s^{\sigma *} dB_s^\sigma \quad (i = 1, \cdots, n). \tag{A.40}$$

By using the standard arguments, we can show that the effects of drift terms are negligible as $n \to \infty$. By using the similar arguments in Chapter 5 of KSK (2018), the leading martingale term of the SIML estimator is

$$U_n = \sum_{j=2}^{n} U_j^n \ , \tag{A.41}$$

where $U_j^n = [\sum_{i=1}^{j-1} 2\sqrt{m_n} c_{ij} r_i] r_j$, $r_j = X(t_j^n) - X(t_{j-1}^n)$, $c_{ij} = (2/m_n)\sum_{k=1}^{m_n} s_{ki}s_{kj}$ and $s_{ij} = \cos\left[\frac{2\pi}{2c(n)+1}(i-\frac{1}{2})(j-\frac{1}{2})\right]$ $(i,j = 1,\cdots,n)$.
Then, we can evaluate the conditional expectations as

$$W_j^n = \mathbf{E}[U_j^n V_j^n | \mathcal{F}_{j-1,n}] = [\sum_{i=1}^{j-1} 2\sqrt{m} c_{ij} r_i] \int_{t_{j-1}^n}^{t_j^n} \sigma_s (1-s)\omega_s^{\sigma *} ds \ , \tag{A.42}$$

where $\mathcal{F}_{j-1,n}$ is the $\sigma-$field generated at t_{j-1}^n $(j = 1,\cdots,n)$. We notice that for any j $(j = 1,\cdots,n)$ $\int_{t_{j-1}^n}^{t_j^n} \sigma_s(1-s)\omega_s^{\sigma *} ds = O_p(n^{-1})$, which can be approximated as $[\sigma(t_{j-1}^n)(1-t_{j-1}^n)\omega^{\sigma *}(t_{j-1}^n)][B(t_j^n) - B(t_{j-1}^n)]$ with the error order being $O(n^{-2})$. By using (2.4) with $t = t_{j-1}^n$ for each j, $\sigma(t_{j-1}^n)$ can be further represented as the sum of drift terms and Brownian motion parts given $\mathcal{F}_{i-1,n}$ for $t_{j-1}^n > t_{i-1}^n$ $(j = 1,\cdots,n)$. By re-writing the sum of conditional expectations as

$$\sum_{j=2}^{n} W_j^n = \sum_{i=1}^{n-1}[\sum_{j=i+1}^{n} \sqrt{m_n} c_{ij} \int_{t_{j-1}^n}^{t_j^n} \sigma_s(1-s)\omega_s^{\sigma *} ds] r_i \ , \tag{A.43}$$

it is possible to show that as $n \longrightarrow \infty$

$$\sum_{j=2}^{n} W_j^n \xrightarrow{p} 0 \ . \tag{A.44}$$

In order to show this convergence, we use several facts that the function $\sigma_s(1-s)\omega_s^{\sigma *}$ is bounded and Lipschitz-continuous, σ_s is a Brownian semi-martingale with (2.4) for any s, and $r_j^n = \int_{t_{j-1}^n}^{t_j^n} \sigma_s dB_s$ can be approximated by $r_j^{*n} = \sigma(t_{j-1}^n)(B(t_j^n) - B(t_{j-1}^n))$ with errors order being $O(n^{-2})$. We also have the representation for $i \neq j$

$$c_{ij} = \frac{1}{2m_n}[\frac{\sin \frac{\pi}{2n+1}(i+j-1)m_n}{\sin \frac{\pi}{2n+1}(i+j-1)} + \frac{\sin \frac{\pi}{2n+1}(i-j)m_n}{\sin \frac{\pi}{2n+1}(i-j)}] \tag{A.45}$$

(see Section 3.2 and Lemma 5.2 of KSK (2018)).
By using the Fejé-kernel as the proof of Theorem 5 for $[\sqrt{m_n} c_{ij}]^2$, we can derive the asymptotic variances of the normalized random variables. (See Lemma 5.6 and the

derivation of the asymptotic variance of the SIML estimator in KSK (2018). It may be straight-forward to find the (Lyapunov-type) condition

$$\sum_{j=2}^{n} \mathbf{E}[(U_j^n)^4] \longrightarrow 0 \;, \tag{A.46}$$

as $n \to \infty$.

By using the convergence of each term and applying Theorem 2.2.15 of Jacod and Protter (2012) to the martingale parts, we have the stable convergence for a sequence of random variables. (The derivation of the CLT for the main term in the normalized SIML estimator U_n, which has been given in Chapter 5 of KSK (2018).) We write the normalized SIML estimator in the form of $U_n = \sum_{j=2}^{n} U_j^n$ and it is asymptotically uncorrelated with $V(4)$ $(= \int_0^1 \sigma_s^4 ds)$ (and higher order Brownian functionals). Then, we have the stable convergence of the martingale U_n to the limiting normal random variable given $\int_0^1 [\sigma_x(s)]^4 ds$ and $U_n / \sqrt{\int_0^1 [\sigma_x(s)]^4 ds} \stackrel{w}{\longrightarrow} N(0,1)$ (the standard normal distribution) in the standard weak-convergence.

As a typical example. we have that as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{V}(2) - V(2)\right] \stackrel{\mathcal{L}-s}{\to} N\left[0, W\right] \;, \tag{A.47}$$

where

$$W = 2\int_0^1 \sigma_s^4 ds \;. \tag{A.48}$$

It is tedious, but straight-forward to extend the above arguments to more general cases. (See Jacod ad Protter (2012), and Hausler and Luschgy (2015) for the details of stable convergence.)

Finally, in Theorem 5 for the jump-diffusion model, we need to show that the CLT can be applicable to (A.36) and (A.37) in the stable-converge sense. As (A.39)-(A.41) in the diffusion model, it is possible to re-write it as the sum of martingale different sequences and we can use the similar arguments. It is because both of (A.36) and (A.37) are linear combinations of the underlying Brownian Motions.