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Optimal control problem for Allen-Cahn type equation associated with total variation energy

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Dedicated to Professor Mitsuharu Ôtani on the Occasion of his 60th Birthday

Abstract. In this paper we study an optimal control problem for a singular diffusion equation associated with total variation energy. The singular diffusion equation is derived as an Allen-Cahn type equation, and then the observing optimal control problem corresponds to a temperature control problem in the solid-liquid phase transition. We show the existence of an optimal control for our singular diffusion equation by applying the abstract theory. Next we consider our optimal control problem from the view-point of numerical analysis. In fact we consider the approximating problem of our equation, and we show the relationship between the original control problem and its approximating one. Moreover we show the necessary condition of an approximating optimal pair, and give a numerical experiment of our approximating control problem.

Key Words. Optimal control problems, Allen-Cahn type equation, numerical experiments.

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1 Introduction

In this paper we consider an optimal control problem for the following singular diffusion equation:

$$w_t - \kappa \operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) + \partial I_{[-1,1]}(w) \ni w + u \quad \text{a.e. in } Q_T := (0,T) \times \Omega, \tag{1.1}$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{a.e. on } \Sigma_T := (0,T) \times \Gamma, \tag{1.2}$$

$$w(0,x) = w_0(x) \quad \text{for a.a. } x \in \Omega, \tag{1.3}$$

where $T > 0$ is a fixed finite time, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\Gamma := \partial\Omega$, κ is a (small) positive constant, $\partial I_{[-1,1]}(\,\cdot\,)$ is the subdifferential of the indicator function $I_{[-1,1]}(\,\cdot\,)$ on the closed interval $[-1,1]$, $u = u(t,x)$ is a given forcing term on Q_T, $\partial/\partial n$ is the outward normal derivative on Γ, and w_0 is a given initial datum.

The main focus of this paper is to study the following optimal control problem (OP) for our singular diffusion equation (P):={(1.1), (1.2), (1.3)}:

Problem (OP). Find a function $u_* \in L^2(0,T;L^2(\Omega))$, the so-called *optimal control*, that realizes the minimization:

$$J(u_*) = \inf_{u \in L^2(0,T;L^2(\Omega))} J(u);$$

of the following cost function:

$$J(u) := \frac{\alpha}{2}\int_0^T |(w - w_d)(t)|^2_{L^2(\Omega)}\,dt + \frac{1}{2}\int_0^T |u(t)|^2_{L^2(\Omega)}\,dt \tag{1.4}$$
$$\text{for any } u \in L^2(0,T;L^2(\Omega));$$

where α is a positive constant, w is the unique solution of (P) for each forcing (control) term u, w_d is the given target profile in $L^2(0,T;L^2(\Omega))$, and $|\cdot|_{L^2(\Omega)}$ is the standard norm in $L^2(\Omega)$.

Note that Problem (P) is derived as the L^2-gradient flow for the following functional:

$$w \in L^2(\Omega) \mapsto \kappa \int_\Omega |\nabla w| + \int_\Omega I_{[-1,1]}(w)\,dx - \frac{1}{2}\int_\Omega |w+u|^2\,dx \qquad (u \in L^2(\Omega));$$

including the total variation $\int_\Omega |\nabla w|$ of parameter w. Accordingly, the singular diffusion $-\mathrm{div}(\frac{\nabla w}{|\nabla w|})$ is just a formal phrase to describe the first variation (subdifferential) of the total variation. The above functional is known as a possible expression of *free energy*, governing phase field dynamics, and in the context u denotes the relative temperature, and w denotes the non-conserved order parameter. Hence we figure out that Problem (P) is a kind of Allen-Cahn equation, and Problem (OP) is a kind of temperature control problem in the observing solid-liquid phase transition.

The main difficulty of (OP) is in the singularity, arising from the total variation and the indicator function. Recently the singular diffusion equations, kindred to (P), were studied by a lot of mathematicians (cf. [1, 2, 4, 9, 12, 18, 20]) from various viewpoints.

Especially in the case when the space dimension of Ω is one, the authors [16, 17] showed the necessary condition of (OP) and proposed the numerical scheme to find the optimal control of the approximating problem, although their theories were made for slightly different singular diffusion equation with ours.

In this paper we will demonstrate some theorems, which extend the available situation (one-dimensional situation) of the line of foregoing results [15, 16, 17], into general multi-dimensional situations. Additionally we perform the numerical experiment for approximating control problem in two dimensional space, to support the validity of the resulted theorems. Consequently the main novelties found in this paper are:

(a) to show the existence of optimal controls, and to prove the necessary condition for the optimal controls in Problem (OP);

(b) to construct an effective approximating method for Problem (OP) under multi-dimensional setting of Ω;

(c) to propose the numerical scheme to find the approximating optimal control of (OP), and to show the convergence of our numerical algorithm;

(d) to give a numerical experiment of the approximating optimal control problem of (OP) in two-dimensional space.

The plan of this paper is as follows. In Section 2 we recall the fundamentals of the theory of functions of bounded variation, including the exact definition of the total variation functional. In Section 3 we study the problems (P) and (OP) by applying the abstract theory. In Section 4 we consider the approximating problems of (P) and (OP), and prove the necessary condition of an optimal pair to the approximating problem of (OP). In Section 5 we prove the main result (Theorem 5.1) in this paper, which is concerned with the necessary condition of the optimal control of (OP). In Section 6 we propose the numerical scheme to find the optimal control of approximating control problem for (OP), and show Theorem 6.2 which is concerned with the convergence of our numerical algorithm. Furthermore we give a numerical experiment of the approximating control problem for (OP) in two-dimensional space.

Notations and basic assumptions

Throughout this paper we use the following notations.

For any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B' the dual space of B. Additionally we denote by $\langle\cdot,\cdot\rangle_{B',B}$ the duality pairing between B' and B.

In particular we put $H := L^2(\Omega)$ with usual real Hilbert space structures. The inner product and norm in H are denoted by $(\cdot,\cdot)$ and by $|\cdot|_H$, respectively. Also we put $X := H^1(\Omega)$ with usual norm $|\cdot|_X$, and denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X' and X.

Let us here prepare some notations and definitions. For a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function $\psi : H \to \mathbb{R} \cup \{\infty\}$, the effective domain $D(\psi)$ of ψ is defined by

$$D(\psi) := \{z \in H;\ \psi(z) < \infty\}.$$

The subdifferential of ψ is a possibly multi-valued operator in H, and it is defined that $z^* \in \partial\psi(z)$ if and only if

$$z \in D(\psi) \quad \text{and} \quad (z^*, y - z) \leq \psi(y) - \psi(z) \quad \text{for all } \ y \in H.$$

For various properties and related notions of the proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to a monograph by Brézis [5].

Let us now give some assumptions on data. Throughout this paper we assume the following conditions (A1)–(A2):

(A1) T, α and κ are the fixed positive constants in $\mathbb{R}$.

(A2) w_d is the given target profile in $L^2(0, T; H)$.

Finally, throughout this paper we use $N_i, i = 1, 2, 3, \cdots$ to denote positive (or nonnegative) constants depending only on the argument(s).

2 Preliminaries

We begin by recalling the definitions of functions of bounded variation and their total variation.

Definition 2.1. (I) Let $f \in L^1(\Omega)$. Then f is called a function of bounded variation (or simply BV-function), if and only if:

$$\int_\Omega |\nabla f| := \sup \left\{ \int_\Omega f \ \mathrm{div}\boldsymbol{\varphi} \ dx \ ; \begin{array}{l} \boldsymbol{\varphi} \in C^1(\Omega; \mathbb{R}^N) \text{ with a compact support,} \\ |\boldsymbol{\varphi}| \leq 1 \text{ in } \Omega \end{array} \right\} < \infty.$$

Here we call $\displaystyle\int_\Omega |\nabla f|$ the total variation of f.

(II) We denote by $BV(\Omega)$ the space of all BV-functions.

Now we recall the important property of the space $BV(\Omega)$ as follows (cf. [7]):

Proposition 2.1 (cf. [7, Chapter 5])**.** (I) *(Lower semicontinuity) Let $\{f_j\} \subset BV(\Omega)$, and let $f \in L^1(\Omega)$. If $f_j \longrightarrow f$ in $L^1(\Omega)$ as $j \to \infty$, then*

$$\liminf_{j\to\infty} \int_\Omega |\nabla f_j| \geq \int_\Omega |\nabla f|.$$

(II) *(Compactness) The space $BV(\Omega)$ is a Banach space endowed with the norm*

$$|z|_{BV(\Omega)} := |z|_{L^1(\Omega)} + \int_\Omega |\nabla z| \quad \text{for any } z \in BV(\Omega).$$

Moreover $BV(\Omega)$ is compactly embedded into $L^1(\Omega)$. Hence $BV(\Omega) \cap L^\infty(\Omega)$ is compactly embedded into the space $L^p(\Omega)$ for any $1 \leq p < \infty$.

Proposition 2.2 (cf. [7, Chapter 5]). *Let $f \in BV(\Omega)$. Then there exists a Radon measure $|\nabla f|$ on Ω, and $|\nabla f|$-measurable function $\boldsymbol{\nu}_f : \Omega \to \mathbb{R}^N$ such that*

(i) $|\boldsymbol{\nu}_f| = 1, \quad |\nabla f|$*-a.e. on* Ω*;*

(ii) $\displaystyle\int_\Omega f \, \mathrm{div}\boldsymbol{\varphi} \, dx = -\int_\Omega \boldsymbol{\varphi} \cdot \boldsymbol{\nu}_f |\nabla f|$ *for all* $\boldsymbol{\varphi} \in C^1(\Omega; \mathbb{R}^N)$ *with a compact support.*

Remark 2.1. If f belongs to the Sobolev space $W^{1,1}(\Omega)$, then $|\nabla f|$ is absolutely continuous with respect to the Lebesgue measure, and it follows that:

$$\int_U |\nabla f| = \int_U |\nabla f(x)| dx \quad \text{for any Borel subset } U \subset \Omega$$

and

$$\boldsymbol{\nu}_f(x) = \begin{cases} \dfrac{\nabla f(x)}{|\nabla f(x)|} & \text{if } \nabla f(x) \neq \mathbf{0}, \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{a.a. } x \in \Omega.$$

Now we define a functional V on H by

$$V(z) := \begin{cases} \displaystyle\int_\Omega |\nabla z| & \text{if } z \in BV(\Omega) \text{ with } |z| \leq 1 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that the effective domain $D(V)$ of V is of the form:

$$D(V) = \{z \in BV(\Omega) \cap H \ ; \ |z| \leq 1 \text{ a.e. in } \Omega\}.$$

Clearly V is proper, l.s.c. and convex on H.

On the other hand let V_0 be the total variation functional on H without constraint, namely

$$V_0(z) := \begin{cases} \displaystyle\int_\Omega |\nabla z| & \text{if } z \in BV(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Also we define the proper, l.s.c. and convex functional $\mathcal{I}_{[-1,1]}$ of H by

$$\mathcal{I}_{[-1,1]}(z) := \int_\Omega I_{[-1,1]}(z) dx \quad \text{for any } z \in H,$$

where $I_{[-1,1]}$ is the indicator function on the closed interval $[-1,1]$. Then the total variation functional V can be formulated as in the form

$$V(z) = V_0(z) + \mathcal{I}_{[-1,1]}(z) \quad \text{for any } z \in H.$$

Here we recall the representation result of ∂V_0 obtained in [1, 2].

Proposition 2.3 (cf. [1, 2]). *$w^* \in \partial V_0(w)$ if and only if there is a vector field $\boldsymbol{\nu}_w \in L^\infty(\Omega;\mathbb{R}^N)$ such that*

$$|\boldsymbol{\nu}_w| \le 1 \quad a.e.\ in\ \Omega, \qquad \mathrm{div}\boldsymbol{\nu}_w \in H, \qquad w^* = -\mathrm{div}\boldsymbol{\nu}_w \quad in\ \mathcal{D}'(\Omega),$$

$$-\int_\Omega \mathrm{div}\,\boldsymbol{\nu}_w\, z\, dx = \int_\Omega \boldsymbol{\nu}_w \cdot \nabla z\, dx \quad for\ any\ z \in W^{1,1}(\Omega) \cap H \tag{2.2}$$

and

$$V_0(w) = -\int_\Omega \mathrm{div}\,\boldsymbol{\nu}_w\, w\, dx.$$

Moreover we see the following decomposition property of the subdifferential ∂V. For the detailed proof, we refer to [20, Theorem 3.1]:

Proposition 2.4 (cf. [20, Theorem 3.1]). *The subdifferential ∂V of V is decomposed into the following form:*

$$\partial V(z) = \partial V_0(z) + \partial \mathcal{I}_{[-1,1]}(z) \ in\ H \qquad for\ any\ z \in H.$$

3 Problems (P) and (OP)

We begin by giving the notion of a solution to (P):={(1.1), (1.2), (1.3)}.

Definition 3.1. Let $u \in L^2(0,T;H)$ and $w_0 \in H$. Then a function $w : [0,T] \longrightarrow H$ is called a solution of (P), or (P;u, w_0) when the data are specified, on $[0,T]$, if the following conditions are satisfied:

(i) $w \in W^{1,2}(0,T;H)$ with $V(w) \in L^1(0,T)$.

(ii) There is a function $w^* \in L^2(0,T;H)$ such that $w^*(t) \in \partial V(w(t))$ and

$$w'(t) + \kappa w^*(t) = w(t) + u(t) \quad \text{ in } H \text{ a.a. } t \in (0,T),$$

where $w' := \frac{dw}{dt}$.

(iii) $w(0) = w_0$ in H.

Remark 3.1. By Proposition 2.4, the condition (ii) in Definition 3.1 is equivalent to the following condition (ii)':

(ii)' There is a function $w_0^* \in L^2(0,T;H)$ and a function $\xi \in L^2(0,T;H)$ such that

$$w_0^*(t) \in \partial V_0(w(t)) \text{ in } H, \qquad \xi(t) \in \partial \mathcal{I}_{[-1,1]}(w(t)) \text{ in } H,$$

$$w'(t) + \kappa w_0^*(t) + \xi(t) = w(t) + u(t) \text{ in } H$$

for a.a. $t \in (0,T)$.

Thus the subdifferential ∂V corresponds to the rigorous formulation of the singular term $-\mathrm{div}\left(\frac{\nabla w}{|\nabla w|}\right) + \partial I_{[-1,1]}(w)$ as in (1.1). Furthermore, in the light of (2.2), we infer that the homogeneous Neumann type boundary condition is implicitly inherent in (ii) and (ii)'.

Remark 3.2. It follows from (ii) of Definition 3.1 that the equation (1.1) is equivalent to the following variational inequality:

$$(w'(t) - w(t) - u(t), w(t) - z) + \kappa V(w(t)) - \kappa V(z) \leq 0$$

$$\text{for any } z \in D(V) \text{ and a.a. } t \in (0, T).$$

Here we mention the result of the existence-uniqueness of solutions for (P).

Proposition 3.1 (cf. [5, 12])**.** *Assume* (A1)*. Then, for each* $u \in L^2(0,T;H)$ *and* $w_0 \in D(V)$ *there is a unique solution* w *of* $(\mathrm{P}; u, w_0)$ *on* $[0,T]$*.*

Proof. We easily see that Problem (P) can be reformulated as the following Cauchy problem $(\mathrm{CP}; u, w_0)$:

$$(\mathrm{CP}; u, w_0) \begin{cases} w'(t) + \kappa \partial V(w(t)) - w(t) \ni u(t) & \text{in } H \text{ for a.a. } t \in (0,T), \\ w(0) = w_0, \end{cases}$$

of the evolution equation, that is governed by the subdifferential ∂V of the convex function V on H, given in (2.1). Therefore, by applying the abstract theory established by Brézis [5], the Cauchy problem $(\mathrm{CP}; u, w_0)$ has one and only one solution $w \in W^{1,2}(0,T;H)$, in the sense as in Definition 3.1, for each $u \in L^2(0,T;H)$ and $w_0 \in D(V)$. Hence $(\mathrm{P}; u, w_0)$ has a unique solution on $[0,T]$. □

Recently Yamazaki [21] considered the optimal control problems of nonlinear evolution equation governed by subdifferential operator in a real Hilbert space. So by applying the abstract result in [21], we can get the existence of an optimal control for (OP) as follows:

Proposition 3.2 (cf. [21, Section 5.2])**.** *Assume* (A1)–(A2)*. Let* $w_0 \in D(V)$*. Then Problem* (OP) *has at least one optimal control* $u_* \in L^2(0,T;H)$ *so that*

$$J(u_*) = \inf_{u \in L^2(0,T;H)} J(u),$$

where $J(\cdot)$ *is the cost functional defined in* (1.4)*.*

As is mentioned in Proposition 3.1, we see that (P) can be reformulated as the Cauchy problem $(\mathrm{CP}; u, w_0)$. The evolution equation, as in (CP; u, w_0), just corresponds to a special case of the nonlinear evolution equation, treated in [21]. Thus the existence of our optimal control problem (OP) will turn out a direct consequence of the abstract theory, obtained in [21]. For the detailed argument, we refer to [21], and omit the proof of Proposition 3.2.

Remark 3.3. The above Proposition 3.2 does not cover the uniqueness of optimal controls. So, throughout this paper, we have to note the situation that Problem (OP) may have more than two optimal controls.

We get the optimal control of (OP) in Proposition 3.2. But it is very difficult to show the necessary condition of the optimal control for (OP) since the subdifferential $\partial V(\cdot)$ is not smooth. Hence the optimality condition of (OP) will be derived by constructing some effective approximating method for the original problem (OP).

4 Approximating problems of (P) and (OP)

In this section we study the approximating problems of (P) and (OP).

With regard to Problem (P), we consider the following approximating problem $(\mathrm{P})^\varepsilon$, prescribed for each $\varepsilon \in (0,1]$:

Problem $(\mathrm{P})^\varepsilon$. Find a function $w^\varepsilon : [0,T] \to H$ which fulfills the following equations:

$$w^\varepsilon_t - \kappa \,\mathrm{div}\,(\boldsymbol{a}^\varepsilon(\nabla w^\varepsilon)) + F^\varepsilon(w^\varepsilon) = w^\varepsilon + u \quad \text{a.e. in } Q_T, \tag{4.1}$$

$$\nu \cdot \boldsymbol{a}^\varepsilon(\nabla w^\varepsilon) = 0 \quad \text{a.e. on } \Sigma_T, \tag{4.2}$$

$$w^\varepsilon(0,x) = w^\varepsilon_0(x) \quad \text{for a.a. } x \in \Omega, \tag{4.3}$$

where $\boldsymbol{a}^\varepsilon(\boldsymbol{\eta}) = (a^\varepsilon_1(\boldsymbol{\eta}), a^\varepsilon_2(\boldsymbol{\eta}), \cdots, a^\varepsilon_N(\boldsymbol{\eta}))$ is a vector filed on $\mathbb{R}^N$ of the form:

$$\boldsymbol{a}^\varepsilon(\boldsymbol{\eta}) = \frac{\boldsymbol{\eta}}{\sqrt{|\boldsymbol{\eta}|^2 + \varepsilon^2}} + \varepsilon \boldsymbol{\eta} \quad \text{for any } \boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_N) \in \mathbb{R}^N, \tag{4.4}$$

and ν is the outward unit normal vector on Γ. Also we define a nondecreasing function F^ε on $\mathbb{R}$ by

$$F^\varepsilon(r) := \mathrm{sign}(r) \int_0^{|r|} \min\left\{ \frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2} \right\} ds \quad \text{for } r \in \mathbb{R}, \tag{4.5}$$

where $[\cdot]^+$ denotes the positive part of functions. Clearly F^ε is a C^1-function with derivative $(F^\varepsilon)' \in W^{1,\infty}(\mathbb{R})$, such that

$$0 \le (F^\varepsilon)'(r) \le \frac{1}{\varepsilon} \quad \text{for any } r \in \mathbb{R} \tag{4.6}$$

and

$$|F^\varepsilon(r)| \ge \frac{1}{\varepsilon}\left([r-1]^+ + [-1-r]^+\right) - \frac{1}{2} \quad \text{for any } r \in \mathbb{R}. \tag{4.7}$$

We fix a primitive $\widehat{F^\varepsilon}$ of F^ε such that

$$\widehat{F^\varepsilon}(0) = 0 \quad \text{and} \quad \widehat{F^\varepsilon}(r) \ge 0 \quad \text{for any } r \in \mathbb{R}. \tag{4.8}$$

In the rest we denote $(\mathrm{P})^\varepsilon$ by $(\mathrm{P};u, w^\varepsilon_0)^\varepsilon$ when the data of the control u and the initial value w^ε_0 are specified. Note that for each $\varepsilon \in (0,1]$ the singular diffusion term $\mathrm{div}\left(\frac{\nabla w}{|\nabla w|}\right)$ and the constraint $\partial I_{[-1,1]}(w)$ as in (1.1) are approximated by $\mathrm{div}\,(\boldsymbol{a}^\varepsilon(\nabla w^\varepsilon))$ and $F^\varepsilon(w^\varepsilon)$, respectively.

Next for each $\varepsilon \in (0,1]$ we consider the approximating optimal control problem $(\mathrm{OP})^\varepsilon$ of (OP) as follows:

Problem $(\mathbf{OP})^\varepsilon$. Find a function (optimal control) $u_*^\varepsilon \in L^2(0,T;H)$ that realizes the minimization:

$$J^\varepsilon(u_*^\varepsilon) = \inf_{u \in L^2(0,T;H)} J^\varepsilon(u);$$

of the following cost function:

$$J^\varepsilon(u) := \frac{\alpha}{2}\int_0^T |(w^\varepsilon - w_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |u(t)|_H^2 dt \qquad \text{for any } u \in L^2(0,T;H); \tag{4.9}$$

where w^ε is a unique solution of the approximating problem $(\mathrm{P};u, w_0^\varepsilon)^\varepsilon$ for each control $u \in L^2(0,T;H)$, and w_d is the given target profile in $L^2(0,T;H)$.

Here we mention the result of the existence-uniqueness of a solution for $(\mathrm{P})^\varepsilon$.

Proposition 4.1. *Assume* (A1). *For each $\varepsilon \in (0,1]$ let $u \in L^2(0,T;H)$ and $w_0^\varepsilon \in X$. Then there is a unique function w^ε, called a solution of $(\mathrm{P};u,w_0^\varepsilon)^\varepsilon$ on $[0,T]$, which solves the equations* (4.1)–(4.3) *in the following sense:*

(i) $w^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$;

(ii) (4.1) *holds in the variational sense, i.e.,*

$$((w^\varepsilon)'(t), z) + \kappa \int_\Omega \boldsymbol{a}^\varepsilon(\nabla w^\varepsilon(t)) \cdot \nabla z \, dx + (F^\varepsilon(w^\varepsilon(t)), z) - (w^\varepsilon(t), z) = (u(t), z) \quad \textit{for all } z \in X \textit{ and a.a. } t \in (0,T);$$

(iii) $w^\varepsilon(0) = w_0^\varepsilon$ *in* H.

Proof. For each $\varepsilon \in (0,1]$ we define an approximating energy functional V^ε for V on H of the form:

$$V^\varepsilon(z) := \begin{cases} \displaystyle\int_\Omega \left(\sqrt{|\nabla z|^2 + \varepsilon^2} + \frac{\varepsilon}{2}|\nabla z|^2\right) dx + \frac{1}{\kappa}\int_\Omega \widehat{F}^\varepsilon(z) dx & \text{if } z \in X, \\ \infty & \text{otherwise.} \end{cases} \tag{4.10}$$

Clearly V^ε is proper, l.s.c. and convex on H. Also we observe that the approximating problem $(\mathrm{P})^\varepsilon$ is considered as the Cauchy problem $(\mathrm{CP};u,w_0^\varepsilon)^\varepsilon$ of the form:

$$(\mathrm{CP};u,w_0^\varepsilon)^\varepsilon \begin{cases} (w^\varepsilon)'(t) + \kappa \partial V^\varepsilon(w^\varepsilon(t)) - w^\varepsilon(t) = u(t) \text{ in } H \text{ for a.a. } t \in (0,T), \\ w^\varepsilon(0) = w_0^\varepsilon. \end{cases}$$

Hence by applying the abstract theory of evolution equations governed by maximal monotone operators and Lipschitz perturbation (cf. Brézis [5]), the problem $(\mathrm{P})^\varepsilon$ has a unique solution $w^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$ for each $\varepsilon \in (0,1]$, $u \in L^2(0,T;H)$ and $w_0^\varepsilon \in X$. Thus the proof of Proposition 4.1 has been completed. □

Here we give the following important lemma, which is key one to showing the relation between (P) and $(\mathrm{P})^\varepsilon$.

Lemma 4.1 (cf. [18, Lemma 3.1]). *$V^\varepsilon \to V$ on H in the sense of Mosco [14] as $\varepsilon \to 0$, namely the following two conditions are satisfied:*

(i) *For any subsequence $\{V^{\varepsilon_k}\} \subset \{V^\varepsilon\}$, if $z_k \to z$ weakly in H and $\varepsilon_k \to 0$ as $k \to \infty$, then $\liminf_{k\to\infty} V^{\varepsilon_k}(z_k) \geq V(z)$.*

(ii) *For any $z \in D(V)$ and any sequence $\{\varepsilon_n\} \subset (0,\infty)$ satisfying $\varepsilon_n \searrow 0$ as $n \to \infty$, there is a sequence $\{z_n\} \subset H$ such that $z_n \to z$ in H, and $\lim_{n\to\infty} V^{\varepsilon_n}(z_n) = V(z)$.*

Proof. The result follows by a slight modification of the proof of [18, Lemma 3.1]. For the calculation details we refer to [18, Lemma 3.1]. □

Now we mention the result of the continuous dependence between (P) and $(\mathrm{P})^\varepsilon$ as follows.

Proposition 4.2. *Assume* (A1). *For each $\varepsilon \in (0,1]$, let $u^\varepsilon \in L^2(0,T;H)$, $w_0^\varepsilon \in X$, and let w^ε be the unique solution of the approximating problem $(\mathrm{P};u^\varepsilon, w_0^\varepsilon)^\varepsilon$ on $[0,T]$. Then there is a positive constant $N_1 > 0$, independent of $\varepsilon \in (0,1]$, such that*

$$\begin{aligned} &\sup_{t\in[0,T]} |w^\varepsilon(t)|_H^2 + \kappa \sup_{t\in[0,T]} V^\varepsilon(w^\varepsilon(t)) + \int_0^T |(w^\varepsilon)'(\tau)|_H^2 d\tau \\ \leq\ & N_1 \left(V^\varepsilon(w_0^\varepsilon) + |w_0^\varepsilon|_H^2 + |u^\varepsilon|_{L^2(0,T;H)}^2 \right). \end{aligned} \tag{4.11}$$

Furthermore assume $w_0 \in D(V)$, $\{w_0^\varepsilon\} \subset X$, $u \in L^2(0,T;H)$, $\{u^\varepsilon\} \subset L^2(0,T;H)$ and

$$w_0^\varepsilon \to w_0 \text{ in } H, \qquad V^\varepsilon(w_0^\varepsilon) \to V(w_0), \tag{4.12}$$

$$u^\varepsilon \to u \text{ weakly in } L^2(0,T;H) \tag{4.13}$$

as $\varepsilon \to 0$. Then w^ε converges to the unique solution w of $(\mathrm{P};u,w_0)$ on $[0,T]$ in the sense that

$$w^\varepsilon \to w \text{ in } C([0,T];L^1(\Omega)) \text{ and in } L^2(0,T;H) \quad \text{as} \quad \varepsilon \to 0. \tag{4.14}$$

Proof. Taking $w^\varepsilon(t)$ and $(w^\varepsilon)'(t)$ as the test functions in (ii) of Proposition 4.1, and applying Gronwall's inequality, we get the boundedness (4.11). Such calculations are standard one, so we omit the detailed proof.

Now let us show (4.14). From (4.11)–(4.13) we infer that

$$w^\varepsilon \text{ is bounded in } W^{1,2}(0,T;H) \text{ uniformly in } \varepsilon \in (0,1] \tag{4.15}$$

and

$$\sup_{t\in[0,T]} V^\varepsilon(w^\varepsilon(t)) \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.16}$$

Also we observe from (4.16), the definitions of $V_0(\cdot)$ and $V^\varepsilon(\cdot)$ and Remark 2.1 that

$$\sup_{t\in[0,T]} V_0(w^\varepsilon(t)) \text{ is bounded uniformly in } \varepsilon \in (0,1],$$

hence,

$$\sup_{t\in[0,T]} |w^\varepsilon(t)|_{BV(\Omega)} \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.17}$$

Since $H \subset L^1(\Omega)$ and the imbedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact, it follows from (4.15) and (4.17) that there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $w \in W^{1,2}(0,T;H) \cap L^\infty(0,T;BV(\Omega)) \cap C([0,T];L^1(\Omega))$ satisfying $\varepsilon_k \to 0$ and

$$\left.\begin{array}{ll} w^{\varepsilon_k} \to w & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{in } C([0,T];L^1(\Omega)), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;BV(\Omega)), \\ & \text{in the pointwise sense, a.e. in } Q_T, \end{array}\right\} \tag{4.18}$$

as $k \to \infty$.

Now we show that w is the unique solution to (P;u, w_0) on $[0,T]$. To do so, we give the bounded estimates of $F^{\varepsilon_k}(w^{\varepsilon_k})$ uniformly in $k \in \mathbb{N}$.

Here by taking $F^{\varepsilon_k}(w^{\varepsilon_k}(t))$ as the test function in (ii) of Proposition 4.1, and by using Schwarz's inequality, (4.4) and (4.6), we get

$$\frac{d}{d\tau}\int_\Omega \widehat{F}^{\varepsilon_k}(w^{\varepsilon_k}(\tau,x))dx + \frac{1}{2}|F^{\varepsilon_k}(w^{\varepsilon_k}(\tau))|_H^2 \le |w^{\varepsilon_k}(\tau)|_H^2 + |u^{\varepsilon_k}(\tau)|_H^2 \quad \text{for a.a. } \tau \in (0,T).$$

By integrating this inequality in τ over $[0,t]$, we have

$$\begin{array}{rl} & \displaystyle\int_\Omega \widehat{F}^{\varepsilon_k}(w^{\varepsilon_k}(t,x))dx + \frac{1}{2}\int_0^t |F^{\varepsilon_k}(w^{\varepsilon_k}(\tau))|_H^2 d\tau \\ \le & \displaystyle\int_\Omega \widehat{F}^{\varepsilon_k}(w_0^{\varepsilon_k}(x))dx + |w^{\varepsilon_k}|^2_{L^2(0,T;H)} + |u^{\varepsilon_k}|^2_{L^2(0,T;H)} \\ & \text{for all } t \in (0,T). \end{array} \tag{4.19}$$

Therefore by taking account of (4.8), (4.11)–(4.13) and (4.19), there is a positive constant N_2, independent of ε_k, such that

$$\int_0^T |F^{\varepsilon_k}(w^{\varepsilon_k}(\tau))|_H^2 d\tau \le N_2. \tag{4.20}$$

Hence it follows from (4.7) and (4.20) that

$$\begin{array}{rl} & \displaystyle\int_0^T\int_\Omega |[w^{\varepsilon_k}(\tau,x)-1]^+|^2 dxd\tau \\ \le & \displaystyle\varepsilon_k^2 \int_0^T\int_\Omega \left(|F^\varepsilon(w^{\varepsilon_k}(\tau,x))| + \frac{1}{2}\right)^2 dxd\tau \\ \le & \displaystyle 2\varepsilon_k^2 \int_0^T\int_\Omega \left(|F^\varepsilon(w^{\varepsilon_k}(\tau,x))|^2 + \frac{1}{4}\right) dxd\tau \\ \le & \displaystyle 2\varepsilon_k^2\left(N_2 + \frac{1}{4}T|\Omega|\right), \end{array}$$

which implies that

$$|[w^{\varepsilon_k}-1]^+|^2_{L^2(0,T;H)} \longrightarrow 0 \quad \text{ as } k\to\infty, \tag{4.21}$$

where $|\Omega|$ is the volume (N-dimensional Lebesgue measure) of Ω.

Similarly it follows from (4.7) and (4.20) that

$$|[-1-w^{\varepsilon_k}]^+|^2_{L^2(0,T;H)} \longrightarrow 0 \quad \text{ as } k\to\infty. \tag{4.22}$$

Here let $K := \{\zeta \in L^2(0,T;H)\ ;\ |\zeta|\le 1 \text{ a.e. on } Q_T\}$, and let π_K be the orthogonal projection from $L^2(0,T;H)$ onto K. Then by applying Lebesgue's dominated convergence theorem, we easily see from (4.18) that

$$\pi_K w^{\varepsilon_k} \longrightarrow \pi_K w \quad \text{ in } L^2(0,T;H) \quad \text{ as } k\to\infty. \tag{4.23}$$

Therefore it follows from (4.18) and (4.21)–(4.23) that

$$\begin{aligned} w^{\varepsilon_k} &= [w^{\varepsilon_k}-1]^+ + \pi_K w^{\varepsilon_k} - [-1-w^{\varepsilon_k}]^+ \\ &\to \pi_K w = w\ (\in K) \quad \text{ in } L^2(0,T;H) \quad \text{ as } k\to\infty. \end{aligned} \tag{4.24}$$

Now we show that w is the solution to (P;u, w_0) on $[0,T]$. Since w^{ε_k} is the unique solution of the approximating problem (P;$u^{\varepsilon_k}, w_0^{\varepsilon_k}$)$^{\varepsilon_k}$ on $[0,T]$, we see that (cf. [5, Example 2.1.3 and Proposition 2.16]):

$$-(w^{\varepsilon_k})' + w^{\varepsilon_k} + u^{\varepsilon_k} \in \kappa\partial\Phi^{\varepsilon_k}(w^{\varepsilon_k}) \ \text{ in } L^2(0,T;H), \quad \forall k\in\mathbb{N},$$

where $\partial\Phi^{\varepsilon_k}$ is the subdifferential of the proper, l.s.c. and convex function Φ^{ε_k} on $L^2(0,T;H)$ defined by

$$\Phi^{\varepsilon_k}(z) := \int_0^T V^{\varepsilon_k}(z(t))dt, \quad \forall z\in L^2(0,T;H).$$

Here we note from Lemma 4.1 that Φ^{ε_k} converges to Φ on $L^2(0,T;H)$ in the sense of Mosco as $k\to\infty$ (cf. [10, Proposition.7]), where Φ is the proper, l.s.c. and convex function Φ on $L^2(0,T;H)$ defined by

$$\Phi(z) := \int_0^T V(z(t))dt, \quad \forall z\in L^2(0,T;H).$$

Therefore, by the general theory of subdifferentials (cf. [3, Theorem 3.66]), we see that $\partial\Phi^{\varepsilon_k}$ converges to $\partial\Phi$ in the graph sense (cf. [3, Definiton 3.58]). Hence we observe from [3, Proposition 3.59] that:

($\star$) if $z^*_{\varepsilon_k} \in \partial\Phi^{\varepsilon_k}(z_{\varepsilon_k})$, $z^*_{\varepsilon_k} \to z^*$ weakly in $L^2(0,T;H)$ and $z_{\varepsilon_k}\to z$ in $L^2(0,T;H)$, then, $z^*\in\partial\Phi(z)$.

For the detailed definition and various properties of graph convergence of maximal monotone operators, we refer to a monograph by Attouch [3].

Thus it follows from the above general theory, (4.13), (4.18) and (4.24) that $-w' + w + u \in \kappa\partial\Phi(w)$ in $L^2(0,T;H)$, which implies that

$$w'(t) + \kappa\partial V(w(t)) - w(t) \ni u(t) \quad \text{in } H \text{ for a.a. } t\in(0,T).$$

Also we observe from (4.12) and (4.18) that $w(0) = w_0$ in H. Therefore w is the solution to (P;u, w_0) on $[0, T]$. Clearly w is the unique solution to (P;u, w_0) on $[0, T]$ (cf. Proposition 3.1), whence (4.14) holds without extracting any subsequence from $\{\varepsilon\}$. Thus the proof of Proposition 4.2 has been completed. □

By a similar argument to [21, Sections 3–4], we get the following Proposition 4.3, which is concerned with the relation between (OP) and (OP)$^\varepsilon$.

Proposition 4.3. *Suppose* (A1)–(A2). *Then, for each* $\varepsilon \in (0, 1]$ *and* $w_0^\varepsilon \in X$ *the approximating problem* (OP)$^\varepsilon$ *has at least one optimal control* $u_*^\varepsilon \in L^2(0, T; H)$ *so that*

$$J^\varepsilon(u_*^\varepsilon) = \inf_{u \in L^2(0,T;H)} J^\varepsilon(u).$$

Furthermore, fix any sequence $\{u_*^\varepsilon\}$ *in* $L^2(0, T; H)$ *such that* u_*^ε *is the optimal control of* (OP)$^\varepsilon$. *Assume* $w_0 \in D(V)$, $\{w_0^\varepsilon\} \subset X$,

$$w_0^\varepsilon \to w_0 \text{ in } H \quad \text{and} \quad V^\varepsilon(w_0^\varepsilon) \to V(w_0) \quad \text{as } \varepsilon \to 0. \tag{4.25}$$

Then there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ *and a function* $u_{**} \in L^2(0, T; H)$ *such that* u_{**} *is the optimal control of* (OP), $\varepsilon_k \to 0$ *and*

$$u_*^{\varepsilon_k} \to u_{**} \quad \text{weakly in } L^2(0, T; H) \text{ as } k \to \infty. \tag{4.26}$$

Proof. Note that the approximating problem (P)$^\varepsilon$ is considered as the Cauchy problem (CP; u, w_0^ε)$^\varepsilon$ (cf. Proposition 4.1). Therefore, by applying the abstract result in [21], we can get the existence of an optimal control u_*^ε of (OP)$^\varepsilon$ for each $\varepsilon \in (0, 1]$.

Now we show (4.26). Let us fix any sequence $\{u_*^\varepsilon\}$ in $L^2(0, T; H)$ such that u_*^ε is the optimal control of (OP)$^\varepsilon$. Let u be any function in $L^2(0, T; H)$. Also, let w^ε be a unique solution for (P;u, w_0^ε)$^\varepsilon$ on $[0, T]$, and let w be a unique solution for (P;u, w_0) on $[0, T]$. Then we observe from (4.25) and Proposition 4.2 that

$$w^\varepsilon \to w \text{ in } C([0, T]; L^1(\Omega)) \text{ and in } L^2(0, T; H) \ \text{ as } \ \varepsilon \to 0. \tag{4.27}$$

Since u_*^ε is the optimal control of (OP)$^\varepsilon$, we see that

$$J^\varepsilon(u_*^\varepsilon) \le J^\varepsilon(u) = \frac{\alpha}{2} \int_0^T |(w^\varepsilon - w_d)(t)|_H^2 dt + \frac{1}{2} \int_0^T |u(t)|_H^2 dt. \tag{4.28}$$

Clearly it follows from (4.9), (4.27) and (4.28) that $\{u_*^\varepsilon\}$ is bounded in $L^2(0, T; H)$ with respect to $\varepsilon \in (0, 1]$. Thus there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $u_{**} \in L^2(0, T; H)$ such that $\varepsilon_k \to 0$ and

$$u_*^{\varepsilon_k} \to u_{**} \quad \text{weakly in } L^2(0, T; H) \quad \text{as } k \to \infty. \tag{4.29}$$

For any $k \in \mathbb{N}$, let $w_*^{\varepsilon_k}$ be a unique solution of (P;$u_*^{\varepsilon_k}, w_0^{\varepsilon_k}$)$^{\varepsilon_k}$ on $[0, T]$. Then, by (4.25), (4.29) and Proposition 4.2, we see that $w_*^{\varepsilon_k}$ converges to the unique solution w_{**} of (P;u_{**}, w_0) on $[0, T]$ in the sense that

$$w_*^{\varepsilon_k} \to w_{**} \text{ in } C([0, T]; L^1(\Omega)) \text{ and in } L^2(0, T; H) \ \text{ as } k \to \infty. \tag{4.30}$$

Now, by using (4.27)–(4.30) and the weak lower semicontinuity of L^2-norm, we see that

$$J(u_{**}) \leq \liminf_{k\to\infty} J^{\varepsilon_k}(u_*^{\varepsilon_k}) \leq J(u).$$

Since u is any function in $L^2(0,T;H)$, we infer from the above inequality that u_{**} is the optimal control of (OP) satisfying the convergence (4.29) (i.e. (4.26)). Thus the proof of Proposition 4.3 has been completed. $\square$

Now we give the necessary condition of an optimal pair $(w_*^\varepsilon, u_*^\varepsilon)$ for $(\mathrm{OP})^\varepsilon$, where w_*^ε is the unique solution for $(\mathrm{P};u_*^\varepsilon, w_0^\varepsilon)^\varepsilon$, and u_*^ε is the optimal control of $(\mathrm{OP})^\varepsilon$ obtained in Proposition 4.3.

Proposition 4.4. *Suppose* (A1)–(A2). *For the fixed number $\varepsilon \in (0,1]$, let $w_0^\varepsilon \in X$, and let $u_*^\varepsilon \in L^2(0,T;H)$ be the optimal control of $(\mathrm{OP})^\varepsilon$ obtained in Proposition 4.3. Let w_*^ε be the unique solution of $(\mathrm{P};u_*^\varepsilon, w_0^\varepsilon)^\varepsilon$. Then there exists a unique solution p^ε of the adjoint equation on $[0,T]$ as follows:*

$$p^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H); \tag{4.31}$$

$$\begin{aligned} &\int_0^T \langle -(p^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau + \kappa \int_0^T \int_\Omega \left(\left[\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w_*^\varepsilon(\tau)) \right]^T \nabla p^\varepsilon(\tau) \right) \cdot \nabla \zeta(\tau)\, dx d\tau \\ &\qquad + \int_0^T ((F^\varepsilon)'(w_*^\varepsilon(\tau)) p^\varepsilon(\tau), \zeta(\tau))\, d\tau - \int_0^T (p^\varepsilon(\tau), \zeta(\tau))\, d\tau \\ &= \int_0^T (\alpha(w_*^\varepsilon(\tau) - w_d(\tau)), \zeta(\tau))\, d\tau \qquad \text{for any } \zeta \in L^2(0,T;X); \end{aligned} \tag{4.32}$$

subject to:

$$p^\varepsilon(T,x) = 0 \quad \text{for a.a. } x \in \Omega; \tag{4.33}$$

where $\left[\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\cdot)\right]^T \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$ *denotes the transpose of the gradient* $\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta} \in L^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$. *Moreover p^ε satisfies the equation*

$$p^\varepsilon + u_*^\varepsilon = 0 \quad \text{in } L^2(0,T;H). \tag{4.34}$$

We prove Proposition 4.4 by showing the result of Gâteaux differentiability of the cost functional $J^\varepsilon(\cdot)$. Here we define the solution operator Λ^ε of $(\mathrm{P})^\varepsilon$.

Definition 4.2. (I) Let $w_0^\varepsilon \in X$. Then we denote by $\Lambda^\varepsilon : L^2(0,T;H) \to L^2(0,T;X) \subset L^2(0,T;H)$ a solution operator of $(\mathrm{P})^\varepsilon$ which assigns to any control $u \in L^2(0,T;H)$ the unique solution $w^\varepsilon := \Lambda^\varepsilon(u)$ of the state problem $(\mathrm{P};u, w_0^\varepsilon)^\varepsilon$.
(II) Let $w_0^\varepsilon \in X$, and let $u_*^\varepsilon \in L^2(0,T;H)$ be the optimal solution of $(\mathrm{OP})^\varepsilon$. Then $(w_*^\varepsilon, u_*^\varepsilon) = (\Lambda^\varepsilon(u_*^\varepsilon), u_*^\varepsilon)$ is called the optimal pair of the control problem $(\mathrm{OP})^\varepsilon$.

For a moment we often omit the subscript $\varepsilon \in (0,1]$. For any $\lambda \in [-1,1] \setminus \{0\}$, any $u \in L^2(0,T;H)$ and any $\tilde{u} \in L^2(0,T;H)$, we put $w_\lambda := \Lambda^\varepsilon(u + \lambda\tilde{u})$, $w := \Lambda^\varepsilon(u)$ and $\chi_\lambda := \frac{w_\lambda - w}{\lambda}$. Then we easily see that χ_λ satisfies the following system:

$$(\chi_\lambda)_t - \kappa \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\sum_{j=1}^N \overline{a}_{ij}^\lambda(t,x)\frac{\partial \chi_\lambda}{\partial x_j}\right) + \overline{F}_\lambda^\varepsilon(t,x)\chi_\lambda = \chi_\lambda + \tilde{u} \quad \text{a.e. in } Q_T, \tag{4.35}$$

$$\sum_{i=1}^N \nu_i \left(\sum_{j=1}^N \overline{a}_{ij}^\lambda(t,x)\frac{\partial \chi_\lambda}{\partial x_j}\right) = 0 \quad \text{a.e. on } \Sigma_T, \tag{4.36}$$

$$\chi_\lambda(0,x) = 0 \quad \text{for a.a. } x \in \Omega, \tag{4.37}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$ is the outward unit normal vector on Γ, and we define

$$\begin{aligned}
\overline{a}_{ij}^\lambda(t,x) &:= \int_0^1 \frac{\partial a_i^\varepsilon}{\partial \eta_j}(\nabla w(t,x) + s(\nabla w_\lambda(t,x) - \nabla w(t,x)))ds \quad \text{for a.a. } (t,x) \in Q_T,\\
\overline{F}_\lambda^\varepsilon(t,x) &:= \int_0^1 (F^\varepsilon)'(w(t,x) + s(w_\lambda(t,x) - w(t,x)))ds \qquad \text{for a.a. } (t,x) \in Q_T.
\end{aligned}$$

Here $\frac{\partial a_i^\varepsilon}{\partial \eta_j}$ is the partial derivative of $a_i^\varepsilon(\boldsymbol{\eta})$ with respect to η_j, where $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_N) \in \mathbb{R}^N$ and $\boldsymbol{a}^\varepsilon(\boldsymbol{\eta}) = (a_1^\varepsilon(\boldsymbol{\eta}), a_2^\varepsilon(\boldsymbol{\eta}), \cdots, a_N^\varepsilon(\boldsymbol{\eta}))$ is a vector filed on $\mathbb{R}^N$ defined in (4.4).

Now we give the uniform estimate of solutions χ_λ with respect to $\lambda \in [-1,1] \setminus \{0\}$.

Lemma 4.3. *Suppose all the same conditions in Proposition 4.4. Then there is a positive constant $N_3 > 0$ independent of $\lambda \in [-1,1] \setminus \{0\}$ such that*

$$\sup_{t\in[0,T]} |\chi_\lambda(t)|_H^2 + \varepsilon\kappa \int_0^T |\nabla \chi_\lambda(t)|_H^2 dt + \int_0^T |\chi_\lambda'(t)|_{X'}^2 dt \le N_3 |\tilde{u}|_{L^2(0,T;H)}^2 \tag{4.38}$$

for any $\tilde{u} \in L^2(0,T;H)$.

Proof. Clearly we see that $a_i^\varepsilon(\cdot) \in C^1(\mathbb{R}^N)$ for $i = 1, \cdots, N$,

$$\sum_{i,j=1}^N \frac{\partial a_i^\varepsilon}{\partial \eta_j}(\boldsymbol{\eta})\xi_i\xi_j \ge \varepsilon |\boldsymbol{\xi}|^2, \tag{4.39}$$

$$\left|\frac{\partial a_i^\varepsilon}{\partial \eta_j}(\boldsymbol{\eta})\right| \le \frac{1}{\varepsilon} + \varepsilon \quad \text{for } i,j = 1, \cdots, N \tag{4.40}$$

for any $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_N) \in \mathbb{R}^N$ and any $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$.

Also note that the function F^ε satisfies (4.6). Therefore by (4.6) and (4.39)–(4.40), we can get a priori estimate (4.38). Such calculations are standard one, so, we omit the detailed proof. □

Now let us mention the result of the Gâteaux differentiability of Λ^ε and J^ε.

Lemma 4.4. *Under the same conditions as in Proposition 4.4, the following two statements hold.*

(I) *The solution operator Λ^ε admits the Gâteaux derivative $D_{\tilde{u}}\Lambda^\varepsilon(u)$ at any $u \in L^2(0,T;H)$ and any direction $\tilde{u} \in L^2(0,T;H)$. More precisely, for any $u \in L^2(0,T;H)$ there exists a bounded and linear operator $\mathcal{X}_u : L^2(0,T;H) \longrightarrow L^2(0,T;H)$ such that:*

$$\mathcal{X}_u(\tilde{u}) = D_{\tilde{u}}\Lambda^\varepsilon(u) := \lim_{\lambda\to 0}\frac{\Lambda^\varepsilon(u+\lambda\tilde{u}) - \Lambda^\varepsilon(u)}{\lambda} \quad \text{for all direction } \tilde{u} \in L^2(0,T;H). \tag{4.41}$$

Moreover, for arbitrary $u, \tilde{u} \in L^2(0,T;H)$ the function $\chi := \mathcal{X}_u(\tilde{u})$ fulfills that:

$$\chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H); \tag{4.42}$$

$$\langle \chi'(t), z\rangle + \kappa\int_\Omega\Big(\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w(t))\nabla\chi(t)\Big)\cdot\nabla z(x)dx + ((F^\varepsilon)'(w(t))\chi(t), z) = (\chi(t), z) + (\tilde{u}(t), z) \quad \text{for a.a. } t \in (0,T) \text{ and all } z \in X; \tag{4.43}$$

$$\chi(0,x) = 0 \quad \text{for a.a. } x \in \Omega; \tag{4.44}$$

where $w = \Lambda^\varepsilon(u)$.

(II) *The cost function J^ε admits the Gâteaux derivative $D_{\tilde{u}}J^\varepsilon(u)$ at any $u \in L^2(0,T;H)$ and any direction $\tilde{u} \in L^2(0,T;H)$. More precisely,*

$$\begin{aligned} D_{\tilde{u}}J^\varepsilon(u) &:= \lim_{\lambda\to 0}\frac{J^\varepsilon(u+\lambda\tilde{u}) - J^\varepsilon(u)}{\lambda} \\ &= \int_0^T(\alpha(w(t)-w_d(t)), \chi(t))dt + \int_0^T(u(t),\tilde{u}(t))dt, \end{aligned} \tag{4.45}$$

for any $u \in L^2(0,T;H)$ and any direction $\tilde{u} \in L^2(0,T;H)$, where $w = \Lambda^\varepsilon(u)$, w_d is the given target profile in $L^2(0,T;H)$, and χ $(= \mathcal{X}_u(\tilde{u}))$ is the Gâteaux derivative as in the assertion (I).

Proof. We show (I). We put $w_\lambda := \Lambda^\varepsilon(u+\lambda\tilde{u})$, $w := \Lambda^\varepsilon(u)$ and $\chi_\lambda := \frac{w_\lambda - w}{\lambda}$ for all $u, \tilde{u} \in L^2(0,T;H)$ and $\lambda \in [-1,1]\setminus\{0\}$. Then by the uniform estimate (4.38) of χ_λ, there is a subsequence $\{\lambda_n\} \subset \{\lambda\}$ and a function $\chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X)$ such that $\lambda_n \to 0$,

$$\chi_{\lambda_n} \to \chi \quad \text{weakly-}* \text{ in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;X), \tag{4.46}$$

$$\chi_{\lambda_n} \to \chi \quad \text{in } L^2(0,T;H) \text{ and in } C([0,T];X'), \tag{4.47}$$

$$\chi'_{\lambda_n} \to \chi' \quad \text{weakly in } L^2(0,T;X') \tag{4.48}$$

as $n \to \infty$, and the following estimate holds:

$$|\chi|^2_{L^\infty(0,T;H)} + \varepsilon\kappa|\nabla\chi|^2_{L^2(0,T;H)} + |\chi'|^2_{L^2(0,T;X')} \le N_3|\tilde{u}|^2_{L^2(0,T;H)}; \tag{4.49}$$

where $N_3 > 0$ is the same constant as in Lemma 4.3.

Next we show that the limit function χ of $\{\chi_{\lambda_n}\}$ fulfills (4.43)–(4.44). Thanks to (4.38), we see that

$$\begin{aligned} |w_{\lambda_n} - w|_{L^2(0,T;X)} &= \lambda_n |\chi_{\lambda_n}|_{L^2(0,T;X)} \\ &\le \lambda_n N_3^{\frac{1}{2}} \left(T + \varepsilon^{-1}\kappa^{-1}\right)^{\frac{1}{2}} |\tilde{u}|_{L^2(0,T;H)} \\ &\to 0 \ \text{ as } n \to \infty. \end{aligned} \tag{4.50}$$

So, taking a subsequence if necessary, we see from the definition of $\overline{F}^{\varepsilon}_{\lambda}$ and Lipschitz continuity of $(F^{\varepsilon})'$ that :

$$\overline{F}^{\varepsilon}_{\lambda_n}(t,x) \to (F^{\varepsilon})'(w(t,x)) \text{ for a.a. } (t,x) \in Q_T,$$
$$\text{in the pointwise sense, as } n \to \infty.$$

Since the function $\overline{F}^{\varepsilon}_{\lambda}$ ($\lambda \in [-1,1] \setminus \{0\}$) is bounded (cf. (4.6)), we can apply Lebesgue's dominated convergence theorem to show that

$$\overline{F}^{\varepsilon}_{\lambda_n} \to (F^{\varepsilon})'(w) \text{ in } L^2(0,T;H) \ \text{ as } n \to \infty. \tag{4.51}$$

Similarly we observe from (4.40), (4.46), (4.50) that for each $i, j = 1, 2, \cdots, N$,

$$\overline{a}^{\lambda_n}_{ij}(t,x)\frac{\partial \chi_{\lambda_n}}{\partial x_j} \to \frac{\partial a^{\varepsilon}_i}{\partial \eta_j}(\nabla w)\frac{\partial \chi}{\partial x_j} \quad \text{weakly in } L^2(0,T;H) \tag{4.52}$$

as $n \to \infty$.

Note that the solution χ_{λ_n} of (4.35)–(4.37) satisfies the following variational identity:

$$\begin{aligned} &\int_0^T \langle \chi'_{\lambda_n}(t), z\rangle dt + \kappa \int_0^T \int_\Omega \sum_{i=1}^N \Big(\sum_{j=1}^N \overline{a}^{\lambda_n}_{ij}(t,x) \frac{\partial \chi_{\lambda_n}}{\partial x_j} \Big) \frac{\partial z}{\partial x_i} dx dt \\ &+ \int_0^T (\overline{F}^{\varepsilon}_{\lambda_n}(t)\chi_{\lambda_n}(t), z) dt = \int_0^T (\chi_{\lambda_n}(t), z) dt + \int_0^T (\tilde{u}(t), z) dt \\ &\qquad \text{for all } z \in X \text{ and } n = 1, 2, 3, \cdots. \end{aligned} \tag{4.53}$$

Taking account of (4.46)–(4.52), we get the variational form (4.43) by taking the limits in (4.53) as $n \to \infty$. Also we infer from (4.47) that

$$\chi(0,\cdot) = \lim_{n\to\infty} \chi_{\lambda_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X',$$

which implies (4.44).

Furthermore we see that the solutions of the Cauchy problem $\{$(4.43), (4.44)$\}$ are uniquely determined within (4.42), which guarantees the uniqueness of cluster points of the sequence $\{\chi_\lambda\}$ as $\lambda \to 0$.

From the linearity inherent in (4.43) and the estimate (4.49) it follows that each operator $\mathcal{X}_u$ ($u \in L^2(0,T;H)$) is a bounded and linear operator from $L^2(0,T;H)$ into itself, and the operator Λ^{ε} is Gâteaux differentiable in $L^2(0,T;H)$. Thus we conclude the assertion (I) of this lemma.

By the differentiability of Λ^{ε}, we easily prove the assertion (II) of this lemma. Hence the proof of Lemma 4.4 has been completed. □

By taking account of Lemma 4.4, we easily prove Proposition 4.4, which is concerned with the necessary condition of an optimal pair $(w_*^\varepsilon, u_*^\varepsilon) = (\Lambda^\varepsilon(u_*^\varepsilon), u_*^\varepsilon)$.

Proof of Proposition 4.4. Taking account of (4.6) and (4.39), we see from [13, Chapter 3] that there exists the unique solution $p^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$ of the variational problem $\{(4.32), (4.33)\}$.

Now let $(w_*^\varepsilon, u_*^\varepsilon) = (\Lambda^\varepsilon(u_*^\varepsilon), u_*^\varepsilon)$ be the optimal pair of the problem $(\mathrm{OP})^\varepsilon$. Let us fix any $\tilde{u} \in L^2(0,T;H)$, and let χ_*^ε be the directional derivative $D_{\tilde{u}}\Lambda^\varepsilon(u_*^\varepsilon)$. Since u_*^ε is a minimizer for $J^\varepsilon(\cdot)$, we have

$$\begin{aligned}
0 \;\le\; & \liminf_{\lambda \to 0} \frac{J^\varepsilon(u_*^\varepsilon + \lambda \tilde{u}) - J^\varepsilon(u_*^\varepsilon)}{\lambda} \\
= \; & \int_0^T (\alpha(w_*^\varepsilon(t) - w_d(t)), \chi_*^\varepsilon(t))dt + \int_0^T (u_*^\varepsilon(t), \tilde{u}(t))dt \\
= \; & \int_0^T \langle -(p^\varepsilon)'(t), \chi_*^\varepsilon(t) \rangle dt + \kappa \int_0^T \int_\Omega \left(\left[\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w_*^\varepsilon(t)) \right]^T \nabla p^\varepsilon(t) \right) \cdot \nabla \chi_*^\varepsilon(t) dx dt \\
& + \int_0^T ((F^\varepsilon)'(w_*^\varepsilon(t)) p^\varepsilon(t), \chi_*^\varepsilon(t))dt - \int_0^T (p^\varepsilon(t), \chi_*^\varepsilon(t))\, dt + \int_0^T (u_*^\varepsilon(t), \tilde{u}(t))dt \\
= \; & \int_0^T \langle (\chi_*^\varepsilon)'(t), p^\varepsilon(t) \rangle dt + \kappa \int_0^T \int_\Omega \Big(\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w_*^\varepsilon(t)) \nabla \chi_*^\varepsilon(t) \Big) \cdot \nabla p^\varepsilon(t) dx dt \\
& + \int_0^T ((F^\varepsilon)'(w_*^\varepsilon(t)) \chi_*^\varepsilon(t), p^\varepsilon(t))dt - \int_0^T (\chi_*^\varepsilon(t), p^\varepsilon(t))\, dt + \int_0^T (u_*^\varepsilon(t), \tilde{u}(t))dt \\
= \; & \int_0^T (p^\varepsilon(t) + u_*^\varepsilon(t), \tilde{u}(t))\, dt, \qquad (4.54)
\end{aligned}$$

where we have used the variational identities (4.32) and (4.43) for p^ε and χ_*^ε, respectively. Since $\tilde{u} \in L^2(0,T;H)$ is arbitrary, we infer from (4.54) that the optimal control u_*^ε satisfies (4.34). Thus the proof of Proposition 4.4 has been completed. □

Remark 4.1. Casas–Fernández–Yong [6] and Fernández [8] have already studied the optimal control of quasilinear parabolic equations. By the same arguments in [6, 8], we show Proposition 4.4 which is concerned with the necessary condition of the optimal control for $(\mathrm{OP})^\varepsilon$.

5 Optimality condition for (OP)

In this section we show the main result (Theorem 5.1) in this paper, which is concerned with the necessary condition of the optimal control of (OP).

Theorem 5.1. *Suppose the same conditions in Proposition 4.3. Let u_{**} be the optimal control of* (OP) *obtained in Proposition 4.3. Let w_{**} be the unique solution to* $(\mathrm{P}; u_{**}, w_0)$ *on $[0,T]$, and we set*

$$W := \{ z \in H^1(Q_T) \; ; \; z(0,x) = 0 \textit{ for a.a. } x \in \Omega \}.$$

Then there is a function $p \in L^\infty(0,T;H)$ and an element $\mu \in W'$ satisfying the following:

$$\begin{aligned} &\int_0^T (p(\tau), z'(\tau))d\tau + \langle \mu, z\rangle_{W',W} - \int_0^T (p(\tau), z(\tau))d\tau \\ = &\int_0^T (\alpha(w_{**}(\tau) - w_d(\tau)), z(\tau))d\tau \qquad \text{for any } z \in W. \end{aligned} \tag{5.1}$$

Moreover p satisfies the equation

$$p + u_{**} = 0 \quad \text{in } L^2(0,T;H). \tag{5.2}$$

Proof. It is very difficult to show the necessary condition of the optimal control for (OP) directly. So by using Propositions 4.2–4.4, we prove Theorem 5.1.

Let u_{**} be the optimal control of (OP) obtained in Proposition 4.3. Namely, there are a subsequence of ε (which we also denote ε for simplicity) and a sequence $\{u_*^\varepsilon\}$ of optimal controls u_*^ε of $(\mathrm{OP})^\varepsilon$, for every ε, such that

$$u_*^\varepsilon \to u_{**} \quad \text{weakly in } L^2(0,T;H) \quad \text{as } \varepsilon \to 0. \tag{5.3}$$

Then, applying Proposition 4.2 under (4.25) and (5.3), we see that

$$w_*^\varepsilon \to w_{**} \text{ in } C([0,T];L^1(\Omega)) \text{ and in } L^2(0,T;H) \ \text{ as } \varepsilon \to 0, \tag{5.4}$$

where w_*^ε is the unique solution to $(\mathrm{P};u_*^\varepsilon, w_0^\varepsilon)^\varepsilon$ on $[0,T]$, and w_{**} is the unique solution of $(\mathrm{P};u_{**}, w_0)$ on $[0,T]$.

Now, through the limiting observation of p^ε as $\varepsilon \searrow 0$, we prove (5.1)–(5.2). To this end we give a priori estimate of the solution p^ε for the adjoint equation (4.32)–(4.33).

Note that the function p^ε satisfies the following variational identity:

$$\begin{aligned} &\int_{T-t}^T \langle -(p^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau + \kappa \int_{T-t}^T \int_\Omega \left(\left[\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w_*^\varepsilon(\tau))\right]^T \nabla p^\varepsilon(\tau)\right) \cdot \nabla \zeta(\tau) dx d\tau \\ &\qquad + \int_{T-t}^T ((F^\varepsilon)'(w_*^\varepsilon(\tau))p^\varepsilon(\tau), \zeta(\tau))\, d\tau - \int_{T-t}^T (p^\varepsilon(\tau), \zeta(\tau))\, d\tau \\ = &\int_{T-t}^T (\alpha(w_*^\varepsilon(\tau) - w_d(\tau)), \zeta(\tau))\, d\tau \end{aligned} \tag{5.5}$$

$$\text{for any } t \in [0,T] \text{ and any } \zeta \in L^2(T-t,T;X).$$

Let us assign p^ε to the test function ζ as in (5.5). Then it follows from (4.6) and (4.39) that

$$|p^\varepsilon(T-t)|_H^2 \le 3\int_{T-t}^T |p^\varepsilon(\tau)|_H^2 d\tau + \alpha^2 \int_{T-t}^T |w_*^\varepsilon(\tau) - w_d(\tau)|_H^2 d\tau \tag{5.6}$$

for any $t \in [0,T]$. So applying Gronwall's lemma, the convergence (5.4) implies the existence of a positive constant N_4 independent of $\varepsilon \in (0,1]$ such that:

$$\sup_{t\in[0,T]} |p^\varepsilon(t)|_H^2 \le N_4 \left(\int_0^T |w_{**}(\tau) - w_d(\tau)|_H^2 d\tau + 1 \right). \tag{5.7}$$

Here for any $\varepsilon \in (0,1]$ let us define a bounded and linear functional $\mu^\varepsilon \in W'$ on W by putting:

$$
\begin{aligned}
& \langle \mu^\varepsilon, \zeta \rangle_{W',W} \\
:= \ & \int_0^T \Big\{ \kappa \int_\Omega \left(\left[\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta} (\nabla w_*^\varepsilon(t)) \right]^T \nabla p^\varepsilon(t) \right) \cdot \nabla \zeta(t) dx + ((F^\varepsilon)'(w_*^\varepsilon(t)) p^\varepsilon(t), \zeta(t)) \Big\} dt;
\end{aligned}
$$

for all $\zeta \in W$. Then by (5.4) and (5.7) there exists a positive constant N_5 independent of $\varepsilon \in (0,1]$ such that:

$$
\begin{aligned}
& |\langle \mu^\varepsilon, \zeta \rangle_{W',W}| \\
\leq \ & \left| \int_0^T (\alpha(w_*^\varepsilon(t) - w_d(t)), \zeta(t))\, dt \right| + \left| \int_0^T \langle (p^\varepsilon)'(t), \zeta(t) \rangle dt \right| + \left| \int_0^T (p^\varepsilon(t), \zeta(t))\, dt \right| \\
= \ & \left| \int_0^T (\alpha(w_*^\varepsilon(t) - w_d(t)), \zeta(t))\, dt \right| + \left| \int_0^T (-p^\varepsilon(t), \zeta'(t))\, dt \right| + \left| \int_0^T (p^\varepsilon(t), \zeta(t))\, dt \right| \\
\leq \ & N_5 \left(|w_{**} - w_d|_{L^2(0,T;H)} + 1 \right) |\zeta|_W \qquad (5.8)
\end{aligned}
$$

$$\text{for any } \zeta \in W := \{ z \in H^1(Q_T)\,;\, z(0,x) = 0 \text{ for a.a. } x \in \Omega \}.$$

Therefore we get

$$|\mu^\varepsilon|_{W'} \leq N_5 \left(|w_{**} - w_d|_{L^2(0,T;H)} + 1 \right) \quad \text{for all } \varepsilon \in (0,1]. \tag{5.9}$$

By virtue of (5.7) and (5.9), we find a function $p \in L^\infty(0,T;H)$ and an element $\mu \in W'$ such that

$$p^\varepsilon \to p \qquad \text{weakly-}* \text{ in } L^\infty(0,T;H), \tag{5.10}$$
$$\mu^\varepsilon \to \mu \qquad \text{weakly in } W', \tag{5.11}$$

as $\varepsilon \to 0$, by taking a subsequence if necessary. In the light of (4.34), (5.3) and (5.10), we deduce that

$$p + u_{**} = 0 \quad \text{in } L^2(0,T;H). \tag{5.12}$$

Finally, since p^ε also solves that:

$$
\begin{aligned}
& \int_0^T (p^\varepsilon(\tau), z'(\tau))\, d\tau + \langle \mu^\varepsilon, z \rangle_{W',W} - \int_0^T (p^\varepsilon(\tau), z(\tau))\, d\tau \\
= \ & \int_0^T (\alpha(w_*^\varepsilon(\tau) - w_d(\tau)), z(\tau))\, d\tau, \qquad \forall z \in W, \qquad (5.13)
\end{aligned}
$$

we infer from (5.4), (5.10)–(5.11) and (5.13) that the identity (5.1) holds. Thus the proof of Theorem 5.1 has been completed. □

6 Numerical Scheme for $(\mathrm{OP})^\varepsilon$

In this section we fix the parameter $\varepsilon \in (0,1]$ and the element $w_0^\varepsilon \in X$. Then we study the problem $(\mathrm{OP})^\varepsilon$ from the view-point of numerical analysis.

For a moment we often omit the subscript $\varepsilon \in (0,1]$.

Here we define the solution operator of the adjoint equation $\{(4.32)–(4.33)\}$.

Definition 6.1. We denote by $\Lambda_{ad}^{\varepsilon}$ the solution operator which maps any control $u \in L^2(0,T;H)$ to the unique solution $p^{\varepsilon} := \Lambda_{ad}^{\varepsilon}(u)$ of the adjoint equation $\{$(4.32)–(4.33)$\}$ under $w_*^{\varepsilon} = \Lambda^{\varepsilon}(u)$.

Now by using the necessary condition (4.34) of $(\mathrm{OP})^{\varepsilon}$ obtained in Proposition 4.4, we propose the following numerical algorithm, denoted by (NA), to find the optimal control of $(\mathrm{OP})^{\varepsilon}$.

Numerical Algorithm (NA) of $(\mathrm{OP})^{\varepsilon}$

(Step 0) Give the stop parameter σ, and choose the initial data $w_0^{\varepsilon} \in X$;

(Step 1) Choose the initial control function $u_0 \in L^2(0,T;H)$, and put $u_n = u_0$;

(Step 2) Solve the problem $(\mathrm{P};u_n, w_0^{\varepsilon})^{\varepsilon}$, and let $w_n = \Lambda^{\varepsilon}(u_n)$;

(Step 3) Solve the adjoint equation $\{$(4.32)–(4.33)$\}$ under $w_*^{\varepsilon} = w_n = \Lambda^{\varepsilon}(u_n)$, and let $p_n = \Lambda_{ad}^{\varepsilon}(u_n)$;

(Step 4) Test: If $|u_n + p_n|_{L^2(0,T;H)} < \sigma$, then STOP; Otherwise go to (Step 5);

(Step 5) Prepare constants $\beta, \delta \in (0,1)$, and set

$$\rho_n := \beta^{l_n} \quad \text{and} \quad u_{n+1} := u_n - \rho_n(u_n + p_n),$$

by finding the minimal constant $l_n \in \mathbb{N} \cup \{0\}$, to realize that:

$$J^{\varepsilon}(u_n - \beta^{l_n}(u_n + p_n)) - J^{\varepsilon}(u_n) \leq -\delta\beta^{l_n}|u_n + p_n|^2_{L^2(0,T;H)}.$$

The minimal constant l_n is actually found by using an appropriate line search method;

(Step 6) Set $n = n + 1$, and go to (Step 2).

Now we mention our final main result in this paper, which is concerned with the convergence of the numerical algorithm (NA).

Theorem 6.2 (cf. [17, Theorem 4.1])**.** *Assume* (A1)–(A2), $\varepsilon \in (0,1]$ *and* $w_0^{\varepsilon} \in X$. *Let* $\{u_n\}$ *be a sequence in* $L^2(0,T;H)$ *defined by the numerical algorithm* (NA). *Also let* $p_n = \Lambda_{ad}^{\varepsilon}(u_n)$. *Then:*

(I) $\lim_{n\to\infty} J^{\varepsilon}(u_n)$ *exists.*

(II)
$$\lim_{n\to\infty}(u_n + p_n) = 0 \quad \text{in } L^2(0,T;H). \tag{6.1}$$

(III) *There are functions* $u_{**}^{\varepsilon} \in L^2(0,T;H)$ *and* $p_{**}^{\varepsilon} \in L^2(0,T;H)$*, and a subsequence* $\{n_k\} \subset \{n\}$ *such that* p_{**}^{ε} *is a unique solution of the adjoint equation* $\{(4.32)\text{–}(4.33)\}$ *under* $w_*^{\varepsilon} = \Lambda^{\varepsilon}(u_{**}^{\varepsilon})$*,*

$$u_{n_k} \to u_{**}^{\varepsilon}, \qquad p_{n_k} \to p_{**}^{\varepsilon} \qquad \text{in } L^2(0,T;H) \text{ as } k \to \infty,$$
$$u_{**}^{\varepsilon} + p_{**}^{\varepsilon} = 0 \qquad \text{in } L^2(0,T;H),$$

hence, $D_{\tilde{u}} J^{\varepsilon}(u_{**}^{\varepsilon}) := \lim_{\lambda \to 0} \dfrac{J^{\varepsilon}(u_{**}^{\varepsilon} + \lambda \tilde{u}) - J^{\varepsilon}(u_{**}^{\varepsilon})}{\lambda} = 0$ *for any* $\tilde{u} \in L^2(0,T;H)$*.*

Remark 6.1. The above (NA) is actually derived by using an analogy from the algorithm, proposed and studied by the authors [17], although the referred algorithm is made for the case when the space dimension of Ω is one and the term $w + u$, as in the right hand side of (1.1), is replaced by $w - \nu w^3 + u$ ($\nu \geq 0$).

By the arguments similar to those in [17, Section 4], we can show Theorem 6.2. Here we give the following key lemma.

Lemma 6.3. *Assume the same conditions as in Theorem 6.2. Let* $\xi \in [-1,1] \setminus \{0\}$*. Then the Gâteaux derivative of the solution operator* Λ^{ε} *is continuous in the following sense:*

$$\chi_{\xi} = D_{\tilde{u}} \Lambda^{\varepsilon}(u + \xi z) := \lim_{\lambda \to 0} \frac{\Lambda^{\varepsilon}(u + \xi z + \lambda \tilde{u}) - \Lambda^{\varepsilon}(u + \xi z)}{\lambda}$$
$$\longrightarrow \quad \chi = D_{\tilde{u}} \Lambda^{\varepsilon}(u) := \lim_{\lambda \to 0} \frac{\Lambda^{\varepsilon}(u + \lambda \tilde{u}) - \Lambda^{\varepsilon}(u)}{\lambda} \qquad \text{in } L^2(0,T;H) \tag{6.2}$$

for any $u \in L^2(0,T;H)$*, any* $z \in L^2(0,T;H)$ *and any direction* $\tilde{u} \in L^2(0,T;H)$*,*

as $\xi \to 0$*.*

Proof. For any $u \in L^2(0,T;H)$, $z \in L^2(0,T;H)$ and $\xi \in [-1,1] \setminus \{0\}$, we put $w_{\xi} := \Lambda^{\varepsilon}(u + \xi z)$ and $w := \Lambda^{\varepsilon}(u)$. Then by the quite standard calculation we get the following estimate:

$$\sup_{t \in [0,T]} |w_{\xi}(t) - w(t)|_H^2 + \varepsilon \kappa \int_0^T |\nabla w_{\xi}(t) - \nabla w(t)|_H^2 dt \leq N_6 \xi^2 |z|_{L^2(0,T;H)}^2 \tag{6.3}$$

for some constant $N_6 > 0$ independent of $\xi \in [-1,1] \setminus \{0\}$. Thus we infer from (6.3) that

$$w_{\xi} \to w \text{ in } L^2(0,T;X) \text{ as } \xi \to 0. \tag{6.4}$$

Now we show (6.2) by using the convergence (6.4). Note from (I) of Lemma 4.4 that $\chi_{\xi} = D_{\tilde{u}} \Lambda^{\varepsilon}(u + \xi z)$ satisfies the following variational identity:

$$\int_0^T \langle \chi_{\xi}'(t), \zeta(t) \rangle dt + \kappa \int_0^T \int_{\Omega} \Big(\frac{\partial \boldsymbol{a}^{\varepsilon}}{\partial \eta} (\nabla w_{\xi}(t)) \nabla \chi_{\xi}(t) \Big) \cdot \nabla \zeta(t) dx dt$$
$$+ \int_0^T ((F^{\varepsilon})'(w_{\xi}(t)) \chi_{\xi}(t), \zeta(t)) dt = \int_0^T (\chi_{\xi}(t), \zeta(t)) dt + \int_0^T (\tilde{u}(t), \zeta(t)) dt \tag{6.5}$$
$$\text{for all } \zeta \in L^2(0,T;X) \text{ and any direction } \tilde{u} \in L^2(0,T;H),$$

$$\chi_{\xi}(0,x) = 0 \qquad \text{for a.a. } x \in \Omega. \tag{6.6}$$

By (6.5)–(6.6) and the standard calculation, we get the uniform estimate of solutions χ_ξ with respect to $\xi \in [-1,1] \setminus \{0\}$ as follows:

$$\sup_{t\in[0,T]} |\chi_\xi(t)|_H^2 + \varepsilon\kappa \int_0^T |\nabla \chi_\xi(t)|_H^2 dt \le N_7 |\tilde{u}|^2_{L^2(0,T;H)} \tag{6.7}$$

for some positive constant N_7 independent of $\xi \in [-1,1] \setminus \{0\}$. Thus we infer from (4.6), (4.40), (6.5) and (6.7) that

$$|\chi_\xi'|_{L^2(0,T;X')} \le N_8 |\tilde{u}|_{L^2(0,T;H)} \tag{6.8}$$

for some positive constant $N_8 > 0$ independent of $\xi \in [-1,1] \setminus \{0\}$. Therefore by the uniform estimates (6.7) and (6.8) of χ_ξ, there is a subsequence $\{\xi_n\} \subset \{\xi\}$ and a function $\overline{\chi} \in W^{1,2}(0,T;X') \cap L^2(0,T;X)$ such that $\xi_n \to 0$,

$$\chi_{\xi_n} \to \overline{\chi} \ \text{ weakly-}* \text{ in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;X), \tag{6.9}$$

$$\chi_{\xi_n} \to \overline{\chi} \ \text{ in } L^2(0,T;H) \text{ and in } C([0,T];X'), \tag{6.10}$$

$$\chi_{\xi_n}' \to \overline{\chi}' \text{ weakly in } L^2(0,T;X'), \tag{6.11}$$

as $n \to \infty$.

Here from (4.6), (6.4), (6.9), Lipschitz continuity of function $(F^\varepsilon)'$, and Lebesgue's dominated convergence theorem, we infer that:

$$(F^\varepsilon)'(w_{\xi_n})\chi_{\xi_n} \to (F^\varepsilon)'(w)\overline{\chi} \quad \text{in } L^2(0,T;H) \ \text{ as } n \to \infty. \tag{6.12}$$

Similarly we observe from (4.40), (6.4), (6.9) that for each $i, j = 1, 2, \cdots, N$,

$$\frac{\partial a_i^\varepsilon}{\partial \eta_j}(\nabla w_{\xi_n}) \frac{\partial \chi_{\xi_n}}{\partial x_j} \to \frac{\partial a_i^\varepsilon}{\partial \eta_j}(\nabla w) \frac{\partial \overline{\chi}}{\partial x_j} \quad \text{weakly in } L^2(0,T;H) \tag{6.13}$$

as $n \to \infty$.

By (6.9)–(6.13), and by taking the limits in (6.5)–(6.6) as $n \to \infty$, we observe that $\overline{\chi}$ satisfies the following system:

$$\int_0^T \langle \overline{\chi}'(t), \zeta(t) \rangle dt + \kappa \int_0^T \int_\Omega \Big(\frac{\partial \boldsymbol{a}^\varepsilon}{\partial \eta}(\nabla w(t)) \nabla \overline{\chi}(t) \Big) \cdot \nabla \zeta(t) dx dt$$
$$+ \int_0^T ((F^\varepsilon)'(w(t))\overline{\chi}(t), \zeta(t)) dt = \int_0^T (\overline{\chi}(t), \zeta(t)) dt + \int_0^T (\tilde{u}(t), \zeta(t)) dt, \tag{6.14}$$

for all $\zeta \in L^2(0,T;X)$ and any direction $\tilde{u} \in L^2(0,T;H)$;

$$\overline{\chi}(0,\cdot) = \lim_{n\to\infty} \chi_{\xi_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X'. \tag{6.15}$$

Since the solutions of the Cauchy problem $\{$(6.14)–(6.15)$\}$ are uniquely determined, we see that $\overline{\chi} = \chi$ and the convergence (6.2) holds, i.e.,

$$\chi_\xi = D_{\tilde{u}} \Lambda^\varepsilon(u + \xi z) \to \chi = D_{\tilde{u}} \Lambda^\varepsilon(u) \quad \text{in } L^2(0,T;H) \quad \text{as } \xi \to 0$$

for any $u \in L^2(0,T;H)$, $z \in L^2(0,T;H)$ and any direction $\tilde{u} \in L^2(0,T;H)$. Thus the proof of this lemma has been completed. □

Taking account of (4.45), (6.2) and (6.4), we easily see that the following corollary holds.

Corollary 6.1. *Assume the same conditions as in Theorem 6.2. Let $\xi \in [-1,1] \setminus \{0\}$. Then the Gâteaux derivative of the cost functional J^ε is continuous in the following sense:*

$$\begin{aligned} & D_{\tilde{u}} J^\varepsilon(u+\xi z) := \lim_{\lambda \to 0} \frac{J^\varepsilon(u+\xi z+\lambda \tilde{u}) - J^\varepsilon(u+\xi z)}{\lambda} \\ \longrightarrow \quad & D_{\tilde{u}} J^\varepsilon(u) := \lim_{\lambda \to 0} \frac{J^\varepsilon(u+\lambda \tilde{u}) - J^\varepsilon(u)}{\lambda} \qquad (6.16) \\ & \text{for any } u \in L^2(0,T;H), \text{ any } z \in L^2(0,T;H) \text{ and any direction } \tilde{u} \in L^2(0,T;H), \end{aligned}$$

as $\xi \to 0$.

By the arguments similar to those in Lemma 6.3, we can show the following lemma, so we omit the detailed proof.

Lemma 6.4. *Suppose the same conditions in Theorem 6.2. For any $\xi \in [-1,1] \setminus \{0\}$, $u \in L^2(0,T;H)$ and $z \in L^2(0,T;H)$, let $p_\xi = \Lambda^\varepsilon_{ad}(u+\xi z)$. Then $p_\xi = \Lambda^\varepsilon_{ad}(u+\xi z)$ converges to $p = \Lambda^\varepsilon_{ad}(u)$ in $L^2(0,T;H)$ as $\xi \to 0$.*

Also we get the following lemmas by the same proof of [17, Lemmas 4.5, 4.6]. For the detailed proofs we refer to [17, Section 4].

Lemma 6.5 (cf. [17, Lemma 4.5]). *Assume the same conditions as in Theorem 6.2. Let $n \in \mathbb{N}$, and let $\{u_k; k = 1, 2, \cdots, n\}$ be a sequence in $L^2(0,T;H)$ defined by the numerical algorithm* (NA). *Let $p_n = \Lambda^\varepsilon_{ad}(u_n)$, $\beta \in (0,1)$ and $\delta \in (0,1)$. Assume that $u_n + p_n \neq 0$ in $L^2(0,T;H)$. Then there is a minimal constant $l_n \in \mathbb{N} \cup \{0\}$ such that*

$$J^\varepsilon(u_n - \beta^{l_n}(u_n+p_n)) - J^\varepsilon(u_n) \leq -\delta \beta^{l_n} |u_n + p_n|^2_{L^2(0,T;H)}. \qquad (6.17)$$

Lemma 6.6 (cf. [17, Lemma 4.6]). *Assume the same conditions as in Theorem 6.2. Let $n \in \mathbb{N}$, and let $\{u_k; k = 1, 2, \cdots, n\}$ be a sequence in $L^2(0,T;H)$ defined by the numerical algorithm* (NA). *Let $p_n = \Lambda^\varepsilon_{ad}(u_n)$, $\beta \in (0,1)$ and $\delta \in (0,1)$. Assume that $u_n + p_n \neq 0$ in $L^2(0,T;H)$. Also, let l_n be the constant obtained in Lemma 6.5. Then we have*

$$\beta \gamma \left((1-\delta) |u_n + p_n|_{L^2(0,T;H)} \right) \leq \beta^{l_n} |u_n + p_n|_{L^2(0,T;H)}, \qquad (6.18)$$

where $\gamma : [0,\infty) \to [0,\infty)$ is the function defined by

$$\gamma(t) := \inf \left\{ |\xi z|_{L^2(0,T;H)} \ ; \ |u + \xi z + p_\xi - (u+p)|_{L^2(0,T;H)} \geq t \right\} \qquad (6.19)$$

with $p_\xi = \Lambda^\varepsilon_{ad}(u+\xi z)$ and $p = \Lambda^\varepsilon_{ad}(u)$ for $\xi \in [-1,1] \setminus \{0\}$, $u \in L^2(0,T;H)$ and $z \in L^2(0,T;H)$.

Taking account of Lemmas 6.3–6.6, we can prove Theorem 6.2 by using a similar demonstration method, adopted in [17, Theorem 4.1]. For the detailed proof, see [17, Theorem 4.1].

Finally we give a numerical experiment of $(\mathrm{OP})^\varepsilon$ in two dimension of space. We performed the numerical simulation of $(\mathrm{OP})^\varepsilon$ under the setting parameter: $\alpha = 100.0$, $\kappa = 0.01$, $\varepsilon = 0.01$, and the stopping parameter $\delta = 10^{-6}$ in the numerical algorithm (NA). For the numerics we change the variables from (t, x) to $(t/c, x/c) = (s, y)$ with $c = 0.01$. We consider the domain $(0, T) \times \Omega \ni (s, y)$ by $T = 0.01$ and $\Omega = (-1, 1) \times (-1, 1)$, and making a lattice for numerics with space mesh size $\Delta h = 0.01$ and time mesh span $\Delta t = 0.00001 = 0.1 \times \Delta h^2$.

With regard to the target profile w_d, we suppose that:

$$w_d(t,x) := \begin{cases} 1, & \text{if } |x| < 0.5, \\ -1, & \text{if } |x| \geq 0.5, \end{cases} \quad \text{for a.a. } (t,x) \in Q_T.$$

Also, for simplicity, we set that the given initial data $w_0^\varepsilon \equiv 0$ a.e. in Ω, and the initial control function $u_0 \equiv 0$ a.e. in Q_T. For the detailed profiles of given data w_d and w_0^ε, see Figure 1.

We do a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the explicit finite difference scheme similar to [15]. For the detailed scheme, we refer to [15].

Figure 2 is the numerical result of the solution for $(\mathrm{P})^\varepsilon$ with initial data $w_0^\varepsilon \equiv 0$ at $T = 0.01$ in the case of the iteration number $n = 20$. Figure 3 is the graph of the value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$. We observe from Figures 1–3 that the solution of $(\mathrm{P})^\varepsilon$ has the similar profiles to the desired one w_d and the cost functional J^ε almost takes the minimal value.

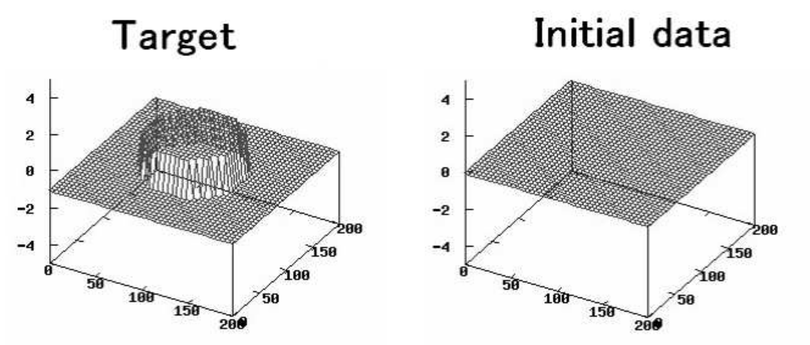

Figure 1: Given target and initial data for $(\mathrm{P})^\varepsilon$.

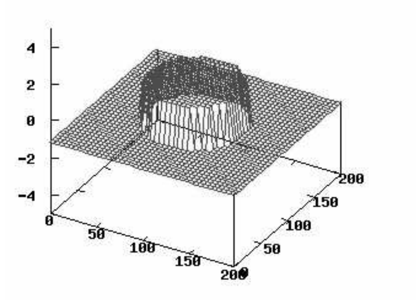

Figure 2: Solution for $(\mathrm{P})^\varepsilon$ at $T = 0.01$ and iteration number $n = 20$.

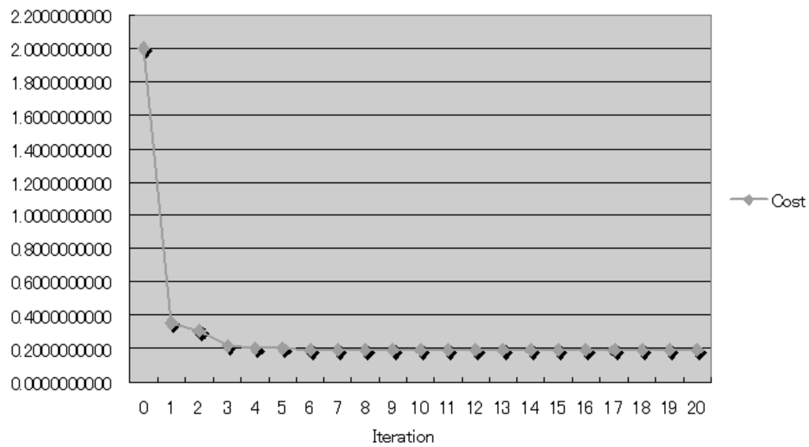

Figure 3: The graph of the value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

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