EQUIVARIANT LOCAL INDEX

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ABSTRACT. This is an expository article on the equivariant local index developed by Fujita, Furuta, and the author in [7].

1. Background, motivation, and purpose

This is based on a joint work with Hajime Fujita and Mikio Furuta [5, 6, 7, 4]. In [5, 6, 7, 4] we developed an index theory for Dirac-type operators on possibly noncompact Riemannian manifolds and applied the theory to the geometric quantization, in particular, the relationship between the Riemann-Roch index and Bohr-Sommerfeld fibers. The purpose of this note is to explain its equivariant version in a simple symplectic case.

Let us recall the background and our motivation. One of our motivation comes from the geometric quantization. Let (M, ω) be a closed symplectic manifold. Suppose that (M, ω) is prequantizable, namely, the cohomology class represented by ω is in the image of the natural map $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})$. Then, there exists a Hermitian line bundle $L \to M$ with Hermitian connection ∇^L whose curvature form F_{∇^L} is equal to $-2\pi\sqrt{-1}\omega$. (L, ∇^L) is called a *prequantum line bundle*.

It is well known that a symplectic manifold is equipped with an almost complex structure J compatible in the sense that $g(u, v) := \omega(u, Jv)$ is a Riemannian metric. For example see [18]. We take and fix a compatible almost complex structure J. We extend J to $TM \otimes_{\mathbb{R}} \mathbb{C}$ complex linearly, and denote its $-\sqrt{-1}$-eigenspace by $T^{0,1}M$. We put

$$W := \wedge^{0,\bullet}T^*M \otimes L = \wedge^{\bullet}(T^{0,1}M)^* \otimes L.$$

The Riemannian metric g together with the Hermitian connection ∇^L of L induces the canonical connection $\nabla\colon \Gamma(W) \to \Gamma(T^*M \otimes W)$ on W. Moreover, the Clifford module structure $c\colon Cl(T^*M) \to \mathrm{End}(W)$ is defined as

$$c(u) := \sqrt{-2}\left(u^{0,1} \wedge \alpha - u^{0,1} \lrcorner \alpha\right)$$

for $u \in T^*M$ and $\alpha \in W$, where $u^{0,1}$ is the $(0,1)$-factor of $u \otimes 1 \in T^*M \otimes \mathbb{C} \cong (T^{1,0}M)^* \oplus (T^{0,1}M)^*$. Then, the *Spin*^{c} *Dirac operator* is defined to be the composition

$$D := c \circ \nabla\colon \Gamma(W) \to \Gamma(W).$$

It is well known that D is a first order, formally self-adjoint, elliptic differential operator of degree-one, and if (M, ω, J) is Kähler and L is holomorphic, then D is nothing but the Dolbeault operator with coefficients in L up to constant, namely, $D = \sqrt{2}(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L)$.

Let D^0 and D^1 be the degree-zero and degree-one parts of D, namely,

$$D^0 := D|_{\wedge^{0,even}T^*M \otimes L}, \quad D^1 := D|_{\wedge^{0,odd}T^*M \otimes L},$$

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respectively. Since M is closed and D is elliptic, D is Fredholm, namely, both of the kernels of D^0 and D^1 are finite dimensional vector spaces. Then, the index of D is defined by

$$\operatorname{ind} D := \dim\ker D^0 - \dim\ker D^1.$$

$\operatorname{ind} D$ is called the *Riemann-Roch index*. Note that $\operatorname{ind} D$ depends only on ω and does not depend on the choice of J and ∇^L since the index is homotopy invariant and the space of compatible almost complex structures of (M,ω) is contractible. By the Atiyah-Singer index theorem, $\operatorname{ind} D$ can be expressed as

$$\operatorname{ind} D = \int_M e^{\omega} Td(TM, J),$$

where $Td(TM,J)$ is the Todd class of the complex vector bundle TM with complex structure J. Moreover, if (M,ω,J) is Kähler and L is holomorphic, then $\operatorname{ind} D$ is equal to the Euler-Poincaré characteristic

$$\operatorname{ind} D = \sum_{q\geq 0}(-1)^q \dim H^q(M, \mathcal{O}_L).$$

For the Spin^c Dirac operators see [16].

A *Lagrangian fibration* is a fiber bundle $\pi\colon (M,\omega)\to B$ from (M,ω) to a manifold B whose fiber is a Lagrangian submanifold of (M,ω). Note that for a Lagrangian fibration $\pi\colon (M,\omega)\to B$, the restriction $(L,\nabla^L)|_{\pi^{-1}(b)}$ to each fiber $\pi^{-1}(b)$ is a flat line bundle since $F_{\nabla^L} = -2\pi\sqrt{-1}\omega$ and a fiber is Lagrangian. A fiber $\pi^{-1}(b)$ of a Lagrangian fibration $\pi\colon (M,\omega)\to B$ is said to be *Bohr-Sommerfeld* if $(L,\nabla^L)|_{\pi^{-1}(b)}$ has a non-trivial global parallel section. The Bohr-Sommerfeld condition is equivalent to that the degree zero cohomology $H^0\left(\pi^{-1}(b);(L,\nabla^L)|_{\pi^{-1}(b)}\right)$ with coefficients in the local system $(L,\nabla^L)|_{\pi^{-1}(b)}$ is non-trivial. It is known that the Bohr-Sommerfeld fibers appear discretely. Then, in [1] Andersen showed that for a Lagrangian fibration $\pi\colon (M,\omega)\to B$ the Riemann-Roch index is equal to the number of Bohr-Sommerfeld fibers.

A completely integrable system can be thought of as a Lagrangian fibration with singular fibers. Similar results are known for several completely integrable systems, such as the polygon space [14], the Gelfand-Cetlin completely integrable system on a complex flag variety [12] and the Goldman completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemann surface [13].

If a compact Lie group G acts effectively on (M,ω) and the G-action lifts to L that preserves all the data, then, $\ker D^0$ and $\ker D^1$ become G-representations. In this case the equivariant Riemann-Roch index is defined as

$$\operatorname{ind}_G D := \ker D^0 - \ker D^1 \in R(G),$$

where $R(G)$ is the representation ring of G. In the case where G is a torus $(S^1)^n$, M is a complex n-dimensional nonsingular projective toric variety, and L is an ample line bundle, it is known by Danilov [2] that $\operatorname{ind}_G D$ has the following irreducible decomposition

$$\operatorname{ind}_G D = \bigoplus_{\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{\gamma^*},$$

where μ is the moment map associated to M and $\mathbb{C}_{\gamma^*}$ is the irreducible representation with weight γ^*. The moment map μ can be thought of as a Lagrangian fibration with singular fibers. All singular fibers of μ are smooth tori. Hence, in this case, the notion of a Bohr-Sommerfeld fiber makes sense even for singular fibers. Moreover, elements of $\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$ correspond one-to-one to Bohr-Sommerfeld fibers. In particular, the Danilov formula can be thought of as a refinement of Andersen's result. The Danilov formula is generalized to non-symplectic cases, such as, presymplectic toric manifolds [15], Spin^c manifolds [9], and torus manifolds [17].

In the Geometric quantization the Riemann-Roch index and the number of Bohr-Sommerfeld fibers correspond to the dimensions of the quantum Hilbert spaces obtained by the Spin^c quantization and the geometric quantization using a real polarization, respectively. From the viewpoint of the geometric quantization it is fundamental to investigate the relationship between these two quantizations.

The above results are localization phenomena of the Riemann-Roch index to the Bohr-Sommerfeld fibers. So we have a natural question: We wonder whether all of these localization phenomena might be caused by the same mechanism. If it is true, make clear the mechanism of the phenomena.

For this question we gave a partial answer in [5, 6, 7, 8]. Namely, we developed an index theory for Dirac-type operators on possibly noncompact Riemannian manifolds, which we call the *local index*, and some of the above results, such as the relationship between the Riemann-Roch index and Bohr-Sommerfeld fibers for nonsingular Lagrangian fibrations and the Danilov formula for a toric variety, were obtained as consequences of the excision property for the local index.

In this note, for simplicity, we will explain the equivariant version of the local index for the Hamiltonian S^1-actions though the local index is defined not only for symplectic manifolds but also for Riemannian manifolds.

This note is organized as follows. In Section 2 we will explain two versions of local indices and their localization formulas. In Section 3, we will consider a special case in detail.

2. Equivariant local index

2.1. $\mathrm{ind}_{S^1}(M, V; L)$. In this subsection let us recall the equivariant local index in the symplectic case. Let (M, ω) be a possibly non-compact symplectic manifold and $(L, \nabla^L) \to (M, \omega)$ a prequantum line bundle on it. Suppose M is equipped with an effective Hamiltonian S^1-action and the S^1-action lifts to L which preserves all the data. Note that each orbit $\mathcal{O}$ is isotropic in the sense that $\omega|_{\mathcal{O}} \equiv 0$. In particular, the restriction of (L, ∇^L) to each orbit is a flat line bundle because of $\frac{\sqrt{-1}}{2\pi} F_{\nabla^L} = \omega$.

In order to define the equivariant local index we introduce the following notion.

Definition 2.1. An orbit $\mathcal{O}$ is said to be *L-acyclic* if it satisfies the condition $H^0\left(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}\right) = 0$.

Note that the non L-acyclic condition is a generalization of the Bohr-Sommerfeld condition for Lagrangian submanifolds.

The following lemma is one of the key points to define the equivariant local index.

Lemma 2.2. *Let $\mathcal{O}$ be an orbit of the S^1-action on M. Then, the following conditions are equivalent:*

1. *$\mathcal{O}$ is L-acyclic.*
2. *$(L, \nabla^L)|_{\mathcal{O}}$ admits no non-trivial global parallel section.*
3. *$H^\bullet\left(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}\right) = 0$.*
4. *The kernel of the de Rham operator of $\mathcal{O}$ with coefficients in L vanishes.*

Remark 2.3. An orbit consisting of a fixed point is not L-acyclic since on such an orbit (L, ∇^L) always has a non-trivial global parallel section.

Proof of Lemma 2.2. It is clear that the first two conditions are equivalent. Since an L-acyclic orbit $\mathcal{O}$ is a circle the first and third conditions are equivalent. See [6, Lemma 2.29]. Moreover, by the Hodge theory, the third condition is also equivalent to the fourth condition. □

Example 2.4 (Non L-acyclic orbits in $\mathbb{C}P^1$)**.** Let k be a positive integer. Define (M,ω) and (L,∇^L) to be the following quotient spaces by the equivalence relations

$$(M,\omega) := \left(S^3_k, \frac{\sqrt{-1}}{2\pi}\sum_{j=0}^{1} dz_j\wedge d\bar{z}_j\right)/_{(z_0,z_1)\sim(hz_0,hz_1)\quad (h\in S^1)},$$

$$(L,\nabla^L) := \left(S^3_k\times\mathbb{C}, d+\frac{1}{2}\sum_{j=0}^{1}(z_jd\bar{z}_j-\bar{z}_jdz_j)\right)/_{(z_0,z_1,v)\sim(hz_0,hz_1,h^kv)},$$

where $S^3_k := \{z=(z_0,z_1)\in\mathbb{C}^2\colon \|z\|^2=k\}$. Namely, (M,ω) is the one-dimensional complex projective space $\mathbb{C}P^1$ with $k\omega_{FS}$, where ω_{FS} is the Fubini-Study form which represents the generator of $H^\bullet(\mathbb{C}P^1;\mathbb{Z})$, and L is the kth tensor power of the hyperplane line bundle $H^{\otimes k}$.

Take and fix an integer m. Let us consider the toric S^1-action on M and its lift on L which is defined by

$$g[z_0:z_1,v] := [z_0:gz_1,g^mv]$$

for $g\in S^1$ and $[z_0:z_1,v]\in L$. In this example we have the following exactly $k+1$ non L-acyclic orbits

$$\mathcal{O}_i := \{[z_0:z_1]\in M\colon |z_1|^2=i\}\ (i=0,1,\dots,k).$$

In fact, for an orbit $\mathcal{O}$ take and fix an element $[z_0:z_1]\in\mathcal{O}$. Then, $\mathcal{O}$ can be written as $\mathcal{O}=\{[z_0:hz_1]\colon h\in S^1\}$. Suppose $s\in H^0(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}})$ is a non-trivial global parallel section. Then, it is easy to show that s should be of the form

$$s([z_0:hz_1]) = [z_0:hz_1,h^{|z_1|^2}s_0] \tag{2.1}$$

for some complex number $s_0\in\mathbb{C}$. In particular, by (2.1), $|z_1|^2$ should be integer since s is a global section on $\mathcal{O}$. Conversely, suppose $|z_1|^2$ is an integer. Then, (2.1) defines a non-trivial global parallel section on $\mathcal{O}$.

In [6, 7] we obtained the following theorem.

Theorem 2.5 ([6, 7])**.** *Let (M,ω) be a possibly non-compact symplectic manifold with effective Hamiltonian S^1-action and $(L,\nabla^L)\to(M,\omega)$ an S^1-equivariant pre-quantum line bundle on it. Let $V\subset M$ be an S^1-invariant open set which contains only L-acyclic orbits and whose complement $M\setminus V$ is compact. For these data, there exists an element* $\mathrm{ind}_{S^1}(M,V;L)\in R(S^1)$ *of the representation ring $R(S^1)$ of S^1 that satisfies the following properties:*

1. $\mathrm{ind}_{S^1}(M,V;L)$ *is invariant under continuous deformation of the data.*
2. *If M is closed, then,* $\mathrm{ind}_{S^1}(M,V;L)$ *is equal to the equivariant index of a Spin^c Dirac operator.*
3. *If V' is an S^1-invariant open subset of V with complement $M\setminus V'$ compact, then we have*

$$\mathrm{ind}_{S^1}(M,V;L) = \mathrm{ind}_{S^1}(M,V';L).$$

4. *If M' is an S^1-invariant open neighborhood of $M\setminus V$, then* $\mathrm{ind}_{S^1}(M,V;L)$ *has the following excision property*

$$\mathrm{ind}_{S^1}(M,V;L) = \mathrm{ind}_{S^1}(M',V\cap M';L|_{M'}).$$

5. *If M is a disjoint union $M=M_1\coprod M_2$, then we have the following sum formula*

$$\mathrm{ind}_{S^1}(M,V;L) = \mathrm{ind}_{S^1}(M_1,V\cap M_1;L|_{M_1})\oplus\mathrm{ind}_{S^1}(M_2,V\cap M_2;L|_{M_2}).$$

6. *We have a product formula for* $\mathrm{ind}_{S^1}(M,V;L)$*. For the precise statement see* [6, Theorem 5.8].

We call $\mathrm{ind}_{S^1}(M, V; L)$ *an equivariant local index.*

Remark 2.6. An orbifold version is available which will be necessary in Section 3.

See [6] for a proof. Let us briefly recall the construction of $\mathrm{ind}_{S^1}(M, V; L)$. The idea used here is the following infinite dimensional analog of the Witten deformation. Let $D\colon \Gamma(W) \to \Gamma(W)$ be the S^1-invariant Spin^c Dirac operator. For $t \geq 0$ consider the following perturbation of D

$$D_t := D + t\rho D_{\mathrm{fiber}},$$

where ρ is an S^1-invariant cut-off function on M with $\rho|_{M\setminus V} \equiv 0$ and $\rho \equiv 1$ outside a compact neighborhood of $M \setminus V$, and D_{fiber} is the S^1-invariant de Rham operator on V along orbits in the following sense, namely,

1. $D_{\mathrm{fiber}}\colon \Gamma\left(W|_V\right) \to \Gamma\left(W|_V\right)$ is an order-one, formally self-adjoint S^1-invariant differential operator of degree-one.
2. D_{fiber} contains only derivatives along orbits.
3. For each orbit $\mathcal{O}$ in V $D_{\mathrm{fiber}}|_{\mathcal{O}}$ is the de Rham operator of $\mathcal{O}$ with coefficients in $(L, \nabla^L)|_{\mathcal{O}}$.
4. For each orbit $\mathcal{O}$ in V let $u \in \Gamma(TV|_{\mathcal{O}})$ be an S^1-invariant section perpendicular to the orbit direction. u acts on $\Gamma\left(W|_{\mathcal{O}}\right)$ as a Clifford multiplication $c(u)$. Then, D_{fiber} anti-commutes with $c(u)$.

It is possible to take such D_{fiber}. Note that the kernel of $D_{\mathrm{fiber}}|_{\mathcal{O}}$ is trivial for any orbit $\mathcal{O}$ in V since V contains only L-acyclic orbits, by the third condition for D_{fiber} and Lemma 2.2.

First we give a definition of $\mathrm{ind}_{S^1}(M, V; L)$ for the special case where M has a cylindrical end.

Proposition 2.7. *Under the assumption in Theorem 2.5 suppose that* M *has a cylindrical end* $V = N \times (0, \infty)$ *and all the data are translationally invariant on the end. Then for a sufficiently large* $t \gg 0$*, the space of* L^2*-solutions of* $D_t s = 0$ *is finite dimensional and its super-dimension is independent of a sufficiently large* $t \gg 0$ *and any other continuous deformations of data.*

Definition 2.8. In the case of Proposition 2.7 we define the $\mathrm{ind}(M, V, W)$ to be the super-dimension of the space of L^2-solutions of $D_t s = 0$, namely,

$$\mathrm{ind}_{S^1}(M, V; L) := \dim \ker D_t^0 \cap L^2 - \dim \ker D_t^1 \cap L^2$$

for a sufficiently large $t \gg 0$.

For the general end case, we deform V cylindrically so that all the data are translationally invariant on the end, and come down to the cylindrical end case. We can show that $\mathrm{ind}_{S^1}(M, V, W)$ is well-defined, namely, it does not depend on

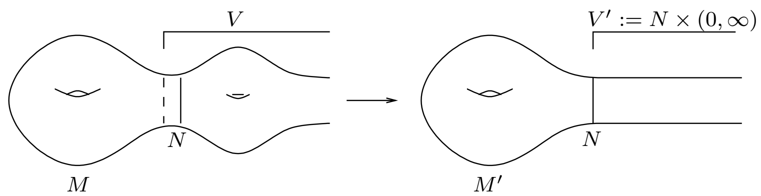

FIGURE 1. Deform V cylindrically

various choice of the construction.

Remark 2.9. To obtain a product formula we need to formulate and define $\mathrm{ind}_{S^1}(M, V; L)$ for a manifold whose end is the total space of a fiber bundle such that both of its base space and its fiber are manifolds with cylindrical end.

See [6, 7] for more details.

Let $(L, \nabla^L) \to (M, \omega)$ and V be the data as in Theorem 2.5. Suppose that there exist finitely many mutually disjoint S^1-invariant open sets $V_1, \ldots, V_n$ of M such that $V_1, \ldots, V_n$, and V form an open covering of M, namely, $M = V \cup (\cup_{i=1}^n V_i)$. Then, for each $i = 1, \ldots, n$ the equivariant local index $\mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}) \in R(S^1)$ is defined, and as a corollary of Theorem 2.5 we have the following localization formula for $\mathrm{ind}_{S^1}(M, V; L)$.

Corollary 2.10. $\mathrm{ind}_{S^1}(M, V; L)$ *is written as the sum of* $\mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i})$*'s, namely,*

$$\mathrm{ind}_{S^1}(M, V; L) = \bigoplus_{i=1}^{n} \mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}). \tag{2.2}$$

Corollary 2.10 implies that $\mathrm{ind}_{S^1}(M, V; L)$ can be described in terms of the data restricted to the neighborhood V_i of non L-acyclic orbits.

Proof of Corollary 2.10. Since $\cup_{i=1}^n V_i$ is an S^1-invariant open neighborhood of $M \setminus V$ the excision property shows

$$\mathrm{ind}_{S^1}(M, V; L) = \mathrm{ind}_{S^1}\left(\cup_{i=1}^n V_i, \cup_{i=1}^n \cap V; L|_{\cup_{i=1}^n V_i}\right).$$

Moreover, since V_i's are mutually disjoint, by the sum formula, we obtain the equality (2.2). □

Remark 2.11. We can prove this theorem in the torus action case. In that case we need to construct an additional geometric structure named "strongly acyclic compatible system" on V. See [6, 7]. As an application of the theorem for the torus action we can obtain the Danilov formula for a nonsingular projective toric variety M [2]. It will be explained in [8].

Example 2.12 (Equivariant localization formula for $\mathbb{C}P^1$)**.** Let us consider the case of Example 2.4. Recall that we have the exactly $k+1$ non L-acyclic orbits $\mathcal{O}_0, \ldots, \mathcal{O}_k$. For each $i = 0, 1, \ldots, k$ we take a sufficiently small positive real number $\varepsilon_i > 0$ and define V_i by

$$V_i := \{[z_0 : z_1] \in M : i - \varepsilon_i < |z_1|^2 < i + \varepsilon_i\}.$$

We put

$$V := \{[z_0 : z_1] \in M : |z_1|^2 \notin \mathbb{Z}\}.$$

Then, for each $i = 0, 1, \ldots, k$ the local index $\mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i})$ is defined. Now we show the following formula

$$\mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}) = \mathbb{C}_{i-m}. \tag{2.3}$$

For each $i = 1, \ldots, k-1$, $(L, \nabla^L)|_{V_i} \to (V_i, \omega|_{V_i})$ is equivariantly isomorphic to the trivial line bundle on the cylinder $S^1 \times (i - \varepsilon_i, i + \varepsilon_i)$

$$\left(S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C}, d - 2\pi\sqrt{-1} r d\theta\right) \to \left(S^1 \times (i - \varepsilon_i, i + \varepsilon_i), dr \wedge d\theta\right)$$

with S^1-action

$$g(e^{2\pi\sqrt{-1}\theta}, r, v) := (g e^{2\pi\sqrt{-1}\theta}, r, g^m v) \tag{2.4}$$

for $(e^{2\pi\sqrt{-1}\theta}, r, v) \in S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C}$. The isomorphism $f_i \colon S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C} \to (L, \nabla^L)|_{V_i}$ is given as

$$f_i(e^{2\pi\sqrt{-1}\theta}, r, v) := \left[\sqrt{k - r} : e^{2\pi\sqrt{-1}\theta}\sqrt{r}, v\right].$$

For each $i = 0, k$, $(L, \nabla^L)|_{V_i} \to (V_i, \omega|_{V_i})$ is equivariantly isomorphic to the trivial line bundle on the disc $D_{\varepsilon_i} := \{w \in \mathbb{C} \colon |w|^2 < \varepsilon_i\}$

$$\left(D_{\varepsilon_i} \times \mathbb{C}, d + \frac{1}{2}(wd\bar{w} - \bar{w}dw)\right) \to \left(D_{\varepsilon_i}, \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w}\right)$$

with S^1-action

$$g(w, v) := \begin{cases} (gw, g^m v) & \text{if } i = 0 \\ (g^{-1}w, g^{m-k}v) & \text{if } i = k \end{cases} \tag{2.5}$$

for $(w, v) \in D_{\varepsilon_i} \times \mathbb{C}$. The isomorphism $f_i \colon D_{\varepsilon_i} \times \mathbb{C} \to (L, \nabla^L)|_{V_i}$ is given as

$$f_i(w, v) := \begin{cases} [\sqrt{k - |w|^2} : w, v] & \text{if } i = 0 \\ [w : \sqrt{k - |w|^2}, v] & \text{if } i = k. \end{cases}$$

With the above identifications f_i, we can compute $\operatorname{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i})$. According to [5, Remark 6.10] both of $\dim \ker D_t^0 \cap L^2$ and $\dim \ker D_t^1 \cap L^2$ in Definition 2.8 for $\operatorname{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i})$ are computed as

$$\dim \ker D_t^0 \cap L^2 = 1, \ \dim \ker D_t^1 \cap L^2 = 0,$$

and a generator of $\ker D_t^0 \cap L^2$ is given as

$$s_i(e^{2\pi\sqrt{-1}\theta}, r) = \left(e^{2\pi\sqrt{-1}\theta}, r, a_i(r)e^{2\pi\sqrt{-1}i\theta}\right) \tag{2.6}$$

for $i = 1, \ldots, k-1$ and

$$s_i(w) = (w, b_i(|w|)) \tag{2.7}$$

for $i = 0, k$, where $a_i(r)$ and $b_i(|w|)$ are some functions on r and $|w|$, respectively. See [5, Remark 6.10],or [25, Section 5.3] for more details. The S^1 acts on $\ker D_t^0 \cap L^2$ by pull-back. For $g \in S^1$ we denote by φ_g and ψ_g the S^1-action on M and L, respectively. By using the explicit expressions for the S^1-actions (2.4), (2.5) and the generators (2.6), (2.7) the S^1-actions on the generators are written as

$$(\psi_{g^{-1}} \circ s_i \circ \varphi_g)(e^{2\pi\sqrt{-1}\theta}, r) = g^{i-m} s_i(e^{2\pi\sqrt{-1}\theta}, r)$$

for $i = 1, \ldots, k-1$, and

$$(\psi_{g^{-1}} \circ s_i \circ \varphi_g)(w) = \begin{cases} g^{-m}\,(s_i(w)) & \text{if } i = 0 \\ g^{k-m}\,(s_i(w)) & \text{if } i = k. \end{cases}$$

Hence, we can obtain the formula (2.3).

By the second property for $\operatorname{ind}_{S^1}(M, V; L)$ in Theorem 2.5 and the formula (2.3) for $\operatorname{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i})$, the equivariant localization formula (2.2) is written as

$$\begin{aligned} H^0(M; \mathcal{O}_L) &= \operatorname{ind}_{S^1} D \\ &= \operatorname{ind}_{S^1}(M, V; L) \\ &= \bigoplus_{i=0}^{k} \operatorname{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}) \\ &= \bigoplus_{i=0}^{k} \mathbb{C}_{i-m}. \end{aligned}$$

2.2. $\mathrm{ind}_{S^1}^{\gamma^*}(M,V;L)$**.** Let $\mathfrak{t}_{\mathbb{Z}}^*$ be the weight lattice of S^1. For each $\gamma^* \in \mathfrak{t}_{\mathbb{Z}}^*$ and an element $U \in R(S^1)$ let us denote by U^{γ^*} the multiplicity of the irreducible representation with weight γ^* in U. By taking the multiplicities of the irreducible representations with weight γ^* in the both sides of the equivariant localization formula (2.2), we obtain the following localization formula for $\mathrm{ind}_{S^1}(M,V;L)^{\gamma^*}$.

$$\mathrm{ind}_{S^1}\left(M,V;L\right)^{\gamma^*} = \bigoplus_{i=1}^{n} \mathrm{ind}_{S^1}\left(V_i, V_i\cap V; L|_{V_i}\right)^{\gamma^*}. \tag{2.8}$$

In this subsection, for each $\gamma^* \in \mathfrak{t}_{\mathbb{Z}}^*$, we define an (L,γ^*)-acyclic condition which is a milder condition than the L-acyclic condition. By using the (L,γ^*)-acyclic condition we obtain a version of a local index, which is denoted by $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$, and its localization formula. In particular, (2.8) is obtained as a special case of the localization formula for $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$.

Since the S^1-action preserves all the data, for each orbit $\mathcal{O}$, S^1 acts on $H^0(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}})$ by pull-back.

Definition 2.13. For each $\gamma^* \in \mathfrak{t}_{\mathbb{Z}}^*$ an orbit $\mathcal{O}$ is said to be (L,γ^*)-acyclic if $\mathcal{O}$ does not consist of a fixed point and satisfies the condition $H^0(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}})^{\gamma^*}=0$.

Remark 2.14. By definition, any L-acyclic orbit are (L,γ^*)-acyclic orbit.

The following lemma is a version of Lemma 2.2 for (L,γ^*)-acyclic orbits.

Lemma 2.15. *Let $\mathcal{O}$ be an orbit of the S^1-action on M. Then, the following conditions are equivalent:*

(1) *$\mathcal{O}$ is (L,γ^*)-acyclic.*
(2) $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\gamma^*}=0$.
(3) *The irreducible component with weight γ^* of the kernel of the de Rham operator of $\mathcal{O}$ with coefficients in L vanishes.*

The proof is similar to that of Lemma 2.2.

Example 2.16 (Non (L,γ^*)-acyclic orbits in $\mathbb{C}P^1$)**.** Let us find non (L,γ^*)-acyclic orbits for Example 2.4. By definition, non (L,γ^*)-acyclic orbits are orbits consisting of a fixed point, or orbits with $H^0(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}})^{\gamma^*}\neq 0$. The orbits of the former type are

$$\mathcal{O}_0=\{[z_0:0]\},\ \mathcal{O}_k=\{[0:z_1]\}.$$

We can show that there exists an orbit of the latter type if and only if $0\le m+\gamma^*\le k$, and in that case we have the unique orbit of the latter type which is

$$\mathcal{O}_{m+\gamma^*}=\{[z_0:z_1]\in M\colon |z_1|^2=m+\gamma^*\}.$$

Recall that $\mathcal{O}_i$s are the only orbits which satisfy $H^0(\mathcal{O}_i;(L,\nabla^L)|_{\mathcal{O}_i})\neq 0$, and in that case an element $s\in H^0(\mathcal{O}_i;(L,\nabla^L)|_{\mathcal{O}_i})$ has the form (2.1). S^1 acts on $H^0(\mathcal{O}_i;(L,\nabla^L)|_{\mathcal{O}_i})$ by pull-back. For $g\in S^1$ and $s([z_0:hz_1])=[z_0:hz_1,h^{|z_1|^2}s_0]\in H^0(\mathcal{O}_i;(L,\nabla^L)|_{\mathcal{O}_i})$ the S^1-action can be written as

$$(\psi_{g^{-1}}\circ s\circ\varphi_g)([z_0:hz_1])=[z_0:hz_1,g^{|z_1|^2-m}h^{|z_1|^2}s_0].$$

Thus, $(\psi_{g^{-1}}\circ s\circ\varphi_g)=g^{\gamma^*}s$ if and only if $|z_1|^2-m=\gamma^*$.

Now we have a version of Theorem 2.5. See [7] in case of $\gamma^*=0$.

Theorem 2.17. *Let (M,ω) be a possibly non-compact symplectic manifold with effective Hamiltonian S^1-action and $(L,\nabla^L)\to(M,\omega)$ an S^1-equivariant prequantum line bundle on it. Let $O\subset M$ be an S^1-invariant open set which contains only (L,γ^*)-acyclic orbits and whose complement $M\setminus O$ is compact. For these data,* *there exists an integer* $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)\in\mathbb{Z}$ *that satisfies the same properties as in Theorem 2.5.*

Remark 2.18. In order to define $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)\in\mathbb{Z}$ we replace the L-acyclic condition by the (L,γ^*)-acyclic condition in the construction of the equivariant local index, and consider the multiplicity of the irreducible representation with weight γ^* in $\ker D_t^0\cap L^2-\ker D_t^1\cap L^2$ for the perturbed Spin^c Dirac operator D_t instead of $\ker D_t^0\cap L^2-\ker D_t^1\cap L^2$ itself. In particular, since L-acyclic orbits are (L,γ^*)-acyclic, V in Theorem 2.5 can be taken as O in Theorem 2.17. In this case, by definition, $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$ is equal to $\mathrm{ind}_{S^1}(M,V;L)^{\gamma^*}$.

Let $(L,\nabla^L)\to(M,\omega)$ and O be the data as in Theorem 2.17. Suppose that there exist finitely many mutually disjoint S^1-invariant open sets O_1, …, O_l of M such that O_1, …, O_l, and O form an open covering of M, namely, $M=O\cup\left(\cup_{i=1}^l O_i\right)$. Then, for each $i=1,\dots,l$ $\mathrm{ind}_{S^1}^{\gamma^*}(O_i,O_i\cap O;L|_{O_i})\in\mathbb{Z}$ is well defined, and we have the following localization formula for $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$.

Corollary 2.19. $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$ *is written as the sum of* $\mathrm{ind}_{S^1}^{\gamma^*}(O_i,O_i\cap O;L|_{O_i})$*'s, namely,*

$$\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)=\bigoplus_{i=1}^{l}\mathrm{ind}_{S^1}^{\gamma^*}(O_i,O_i\cap O;L|_{O_i}). \tag{2.9}$$

The formula 2.9 implies that $\mathrm{ind}_{S^1}^{\gamma^*}(M,O;L)$ can be described in terms of the data restricted to a sufficiently neighborhood of the fixed point set and orbits with $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\gamma^*}\neq 0$.

Remark 2.20. By Remark 2.18, in Corollary 2.19, if we take V and V_i's in Theorem 2.10 as O and O_i's, respectively, then, (2.8) is obtained by (2.9).

Example 2.21 (Localization formula for multiplicities in $\mathbb{C}P^1$)**.** In Example 2.16 we showed that for each $\gamma^*\in\mathfrak{t}^*_{\mathbb{Z}}$ with $0<m+\gamma^*<k$ there are exactly three non (L,γ^*)-acyclic orbits $\mathcal{O}_0$, $\mathcal{O}_k$, and $\mathcal{O}_{m+\gamma^*}$, otherwise we have exactly two non (L,γ^*)-acyclic orbits $\mathcal{O}_0$ and $\mathcal{O}_k$. We put

$O_0:=V_0$, $O_k:=V_k$, $O_{m+\gamma^*}:=V_{m+\gamma^*}$, and $O:=\{[z_0:z_1]\in M\colon |z_1|^2\neq 0,k,m+\gamma^*\}$.

Then, for each i $\mathrm{ind}_{S^1}^{\gamma^*}(O_i,O_i\cap O;L|_{O_i})$ is defined. By definition, $(O_i,O_i\cap O)$ is equal to $(V_i,V_i\cap V)$. Hence, by Remark 2.18 and the formula (2.3) we obtain

$$\begin{aligned}\mathrm{ind}_{S^1}^{\gamma^*}(O_i,O_i\cap O;L|_{O_i})&=\mathrm{ind}_{S^1}^{\gamma^*}(V_i,V_i\cap V;L|_{V_i})\\&=\mathrm{ind}_{S^1}(V_i,V_i\cap V;L|_{V_i})^{\gamma^*}\\&=\begin{cases}1 & \text{if } 0<m+\gamma^*<k \text{ and } i=m+\gamma^*\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

3. A special case

Let $(L,\nabla^L)\to(M,\omega)$ be as above. For $g\in S^1$ we denote by φ_g and ψ_g the S^1-action on M and L, respectively. It is well known that corresponding to the infinitesimal lift of the S^1-action on M to L, the moment map $\mu\colon M\to\mathfrak{t}^*$ is determined uniquely by the following Kostant formula

$$\frac{d}{dt}\Big|_{t=0}\psi_{e^{-t\xi}}\circ s\circ\varphi_{e^{t\xi}}=\nabla_{X_\xi}s+2\sqrt{-1}\pi\langle\mu,\xi\rangle s \tag{3.1}$$

for $\xi\in\mathfrak{t}$ and $s\in\Gamma(L)$, where $\mathfrak{t}$ is the Lie algebra of S^1, $\langle\ ,\ \rangle$ is the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$, and $X_\xi\in\Gamma(TM)$ is the infinitesimal action of ξ. We have the following relationship between non L-acyclic orbits, (L,γ^*)-acyclic orbits, and the values of μ.

Lemma 3.1. (1) *Non L-acyclic orbits are contained in $\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$. In particular, fixed points are contained in $\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$.*
(2) *Orbits with $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\gamma^*}\neq 0$ are contained in $\mu^{-1}(\gamma^*)$.*

Proof. Let $\mathcal{O}$ be a non L-acyclic orbit with $\mu(\mathcal{O})=\eta^*\in\mathfrak{t}^*$. Then, by definition, there exists a non-trivial global parallel section $s\in H^0\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)$. For any element ξ in the integral lattice $\mathfrak{t}_{\mathbb{Z}}$ we put

$$s_t:=\psi_{e^{-t\xi}}\circ s\circ\varphi_{e^{t\xi}}.$$

By (3.1), we have

$$\frac{d}{dt}s_t(x)=2\sqrt{-1}\pi\left\langle\eta^*,\xi\right\rangle s_t(x)$$

for $x\in\mathcal{O}$. Then, s_t has the form

$$s_t=e^{2\sqrt{-1}\pi\langle\eta^*,\xi\rangle t}s.$$

Since $\xi\in\mathfrak{t}_{\mathbb{Z}}$, by putting $t=1$,

$$s=s_1=e^{2\sqrt{-1}\pi\langle\eta^*,\xi\rangle}s.$$

Thus, $\langle\eta^*,\xi\rangle$ should be integer for arbitrary $\xi\in\mathfrak{t}_{\mathbb{Z}}$. This implies the first part.

Let $\mathcal{O}$ be an orbit with $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\gamma^*}\neq 0$. Then, there exists a non-trivial global parallel section $s\in H^0\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\gamma^*}$. For any element $\xi\in\mathfrak{t}$, by (3.1), we have

$$\begin{aligned}2\pi\sqrt{-1}\left\langle\gamma^*,\xi\right\rangle s(x)&=\frac{d}{dt}\Big|_{t=0}\psi_{e^{-t\xi}}\circ s\circ\varphi_{e^{t\xi}}(x)\\&=2\pi\sqrt{-1}\left\langle\mu(x),\xi\right\rangle s(x)\end{aligned}$$

for $x\in\mathcal{O}$. Since s is non-trivial this implies the second part. □

In the rest of this section we assume that μ is proper and the cardinality of $\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$ is finite. For each $\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$ let V_{γ^*} be a sufficiently small S^1-invariant neighborhood of $\mu^{-1}(\gamma^*)$ so that $\{V^*_\gamma\}_{\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}}$ are mutually disjoint. Let V be the complement of $\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$, namely, $V:=M\setminus\mu^{-1}(\mathfrak{t}^*_{\mathbb{Z}})$. By Lemma 3.1 V contains only L-acyclic orbits. Moreover, by assumption, $V_{\gamma^*}\setminus V$ is compact for each $\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$. Hence, for each $\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$ the equivariant local index $\mathrm{ind}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)\in R(S^1)$ is defined. By applying Corollary 2.10 to this case we have the following localization formula for $\mathrm{ind}_{S^1}(M,V;L)$.

$$\mathrm{ind}_{S^1}(M,V;L)=\bigoplus_{\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}}\mathrm{ind}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right).\tag{3.2}$$

We show the following theorem.

Theorem 3.2. *For each $\gamma^*\in\mu(M)\cap\mathfrak{t}^*_{\mathbb{Z}}$ and $\sigma^*\in\mathfrak{t}^*_{\mathbb{Z}}$ with $\gamma^*\neq\sigma^*$*

$$\mathrm{ind}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)^{\sigma^*}=0.$$

Proof. Since L-acyclic orbits are (L,σ^*)-acyclic $\mathrm{ind}^{\sigma^*}_{S^1}(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}})$ is also defined, and by Remark 2.18 it is equal to $\mathrm{ind}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)^{\sigma^*}$. $\mathrm{ind}^{\sigma^*}_{S^1}(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}})$ is described in terms of the data restricted to a sufficiently neighborhood of the fixed point set and orbits with $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^{\sigma^*}\neq 0$. By Lemma 3.1 and the definition of V_{γ^*}, if $\gamma^*\neq\sigma^*$, then, V_{γ^*} contains no orbits of the latter type.

Suppose V_{γ^*} contains fixed points. By Lemma 3.1 the fixed point set $(V_{\gamma^*})^{S^1}$ is contained in $\mu^{-1}(\gamma^*)$. In particular, $(V_{\gamma^*})^{S^1}$ is compact since μ is proper. Suppose $(V_{\gamma^*})^{S^1}$ has the exactly l connected components $(V_{\gamma^*})^{S^1}_1, \dots, (V_{\gamma^*})^{S^1}_l$. For each $i = 1, \dots, l$ we take a sufficiently small S^1-invariant neighborhood O_i of $(V_{\gamma^*})^{S^1}_i$, and also put $O := V_{\gamma^*} \setminus (V_{\gamma^*})^{S^1}$. Then, for each $i = 1, \dots, l$ $\mathrm{ind}^{\sigma^*}_{S^1}(O_i, O_i \cap O; L|_{O_i})$ is defined, and by the third property in Theorem 2.17 and Corollary 2.19 for $\mathrm{ind}^{\sigma^*}_{S^1}\left(V_{\gamma^*}, O; L|_{V_{\gamma^*}}\right)$ we have

$$\begin{aligned}\mathrm{ind}^{\sigma^*}_{S^1}\left(V_{\gamma^*}, V_{\gamma^*} \cap V; L|_{V_{\gamma^*}}\right) &= \mathrm{ind}^{\sigma^*}_{S^1}\left(V_{\gamma^*}, O; L|_{V_{\gamma^*}}\right) \\ &= \bigoplus_{i=1}^{l} \mathrm{ind}^{\sigma}_{S^1}(O_i, O_i \cap O; L|_{O_i}).\end{aligned}$$

Now, for each fixed point $x_0 \in V_{\gamma^*}$, the fiber L_{x_0} becomes a representation of S^1. By the Kostant formula (3.1), for $\xi \in \mathfrak{t}$ and $s \in \Gamma(L)$ we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \psi_{e^{-t\xi}}\left(s(x_0)\right) &= \left.\frac{d}{dt}\right|_{t=0} \psi_{e^{-t\xi}} \circ s \circ \varphi_{e^{t\xi}}(x_0) \\ &= \left(\nabla_{X_\xi} s\right)(x_0) + 2\sqrt{-1}\pi \left\langle \mu(x_0), \xi \right\rangle s(x_0) \\ &= 2\sqrt{-1}\pi \left\langle \gamma^*, \xi \right\rangle s(x_0).\end{aligned}$$

This implies $\psi_g(v) = g^{-\gamma^*} v$ for $g \in S^1$ and $v \in L_{x_0}$. By definition it is easy to see that

$$\mathrm{ind}^{\sigma^*}_{S^1}(O_i, O_i \cap O; L|_{O_i}) = \mathrm{ind}^{0}_{S^1}\left(O_i, O_i \cap O; L \otimes \mathbb{C}_{\sigma^*}|_{O_i}\right).$$

Thus, by [7, Theorem 4.1], for $\sigma^* \neq \gamma^*$ we obtain

$$\mathrm{ind}^{\sigma}_{S^1}(O_i, O_i \cap O; L|_{O_i}) = 0$$

for each $i = 1, \dots, l$. This proves the theorem. □

As a corollary we have the following formula.

Corollary 3.3.

$$\mathrm{ind}_{S^1}(M, V; L)^{\gamma^*} = \begin{cases} \mathrm{ind}^{\gamma^*}_{S^1}(V_{\gamma^*}, V_{\gamma^*} \cap V; L|_{V_{\gamma^*}}) & \text{if } \gamma^* \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}} \\ 0 & \text{otherwise} \end{cases}.$$

So, we have a natural question:

Question 3.4. *How to compute* $\mathrm{ind}^{\gamma^*}_{S^1}\left(V_{\gamma^*}, V_{\gamma^*} \cap V; L|_{V_{\gamma^*}}\right)$*?*

We give a partial answer of this question. First let us consider the case where $\gamma^* = 0$. Suppose $0 \in \mathfrak{t}^*_{\mathbb{Z}}$ is a regular value of μ. Then, a new symplectic orbifold (M_0, ω_0) with prequantum line bundle (L_0, ∇^{L_0}) is obtained by the symplectic reduction, namely,

$$(M_0, \omega_0) := \left(\mu^{-1}(0), \omega|_{\mu^{-1}(0)}\right)/S^1, \ \ (L_0, \nabla^{L_0}) := \left((L, \nabla^L)|_{\mu^{-1}(0)}\right)/S^1.$$

Since μ is proper M_0 is compact. Let D_0 be the Spin^c Dirac operator on (M_0, ω_0) with coefficients in L_0. Then, in [7, Section 5.2] we showed the following formula.

Theorem 3.5 ([7])**.** *Let $\gamma^* = 0 \in \mathfrak{t}^*_{\mathbb{Z}}$ be a regular value of μ. Then,* $\mathrm{ind}^0_{S^1}\left(V_0, V_0 \cap V; L|_{V_0}\right)$ *is equal to the index of D_0.*

Remark 3.6. Note that if 0 is a regular value of μ, V_0 contains no fixed points. In fact, by Lemma 3.1 and the definition of V_0, if fixed points exist in V_0, they should be contained in $\mu^{-1}(0)$. But, by assumption, 0 is a regular value of μ. Thus V_0 contains no fixed point. In particular, $\mathrm{ind}^0_{S^1}(V_0, V_0 \cap V; L|_{V_0})$ is described in terms of the data restricted to a sufficiently neighborhood of the orbits with $H^\bullet\left(\mathcal{O};(L,\nabla^L)|_{\mathcal{O}}\right)^0\neq 0$.

Next let us consider the general case. For a general regular value $\gamma^*\in\mathfrak{t}^*_{\mathbb{Z}}$ of μ we use the shifting trick. By tensoring $\mathbb{C}_{\gamma^*}$ with (L,∇^L) we obtain the prequantum line bundle $(L,\nabla^L)\otimes\mathbb{C}_{\gamma^*}$ on (M,ω) with shifted S^1-action. Then, the moment map associated to the shifted S^1-action, which we denote by μ_{γ^*}, is written as

$$\mu_{\gamma^*}=\mu-\gamma^*. \tag{3.3}$$

Since γ^* is a regular value of μ 0 is a regular value of μ_{γ^*}. Hence, by the symplectic reduction for the shifted S^1-action, a new compact symplectic orbifold $(M_{\gamma^*},\omega_{\gamma^*})$ with prequantum line bundle $(L_{\gamma^*},\nabla^{L_{\gamma^*}})$ is obtained as

$$(M_{\gamma^*},\omega_{\gamma^*}):=\left(\mu_{\gamma^*}^{-1}(0),\omega|_{\mu_{\gamma^*}^{-1}(0)}\right)/S^1,\ (L_{\gamma^*},\nabla^{L_{\gamma^*}}):=\left((L,\nabla^L)\otimes\mathbb{C}_{\gamma^*}|_{\mu_{\gamma^*}^{-1}(0)}\right)/S^1.$$

Let D_{γ^*} be the Spin^c Dirac operator on $(M_{\gamma^*},\omega_{\gamma^*})$ with coefficients in L_{γ^*}. Then, as a corollary of Theorem 3.5 we obtain the following formula.

Corollary 3.7. *For a regular value $\gamma^*\in\mathfrak{t}^*_{\mathbb{Z}}$ of μ $\mathrm{ind}^{\gamma^*}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)$ is equal to the index of D_{γ^*}.*

Proof. By (3.3) V_{γ^*} is a sufficiently small S^1-invariant neighborhood of $\mu_{\gamma^*}^{-1}(0)$. Thus, by Theorem 3.5 for the prequantum line bundle $(L,\nabla^L)\otimes\mathbb{C}_{\gamma^*}$ on (M,ω) with shifted S^1-action we obtain

$$\mathrm{ind}^0_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L\otimes\mathbb{C}_{\gamma^*}|_{V_{\gamma^*}}\right)=\mathrm{ind}\,D_{\gamma^*}.$$

On the other hand, it is easy to see that

$$\mathrm{ind}^{\gamma^*}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)=\mathrm{ind}^0_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L\otimes\mathbb{C}_{\gamma^*}|_{V_{\gamma^*}}\right).$$

This proves the corollary. □

In particular, for a closed M, we obtain the the quantization conjecture for the S^1-action.

Corollary 3.8 ([11, 10, 3, 19, 23, 24, 20, 22], etc.)**.** *Let $(L,\nabla^L)\to(M,\omega)$ be as above. Assume M is closed. If $\gamma^*\in\mathfrak{t}^*_{\mathbb{Z}}$ is a regular value of μ, then,*

$$(\mathrm{ind}_{S^1}D)^{\gamma^*}=\mathrm{ind}\,D_{\gamma^*}.$$

Proof. From the second property in Theorem 2.5 $\mathrm{ind}_{S^1}(M,V;L)$ is equal to the equivariant index $\mathrm{ind}_{S^1}D$ for the Spin^c Dirac operator D on M with coefficients in L. Then, this is a consequence of Corollaries 3.3 and 3.7. □

If γ^* is a critical value of μ, the reduced space $(M_{\gamma^*},\omega_{\gamma^*})$ has more complicated singularities than orbifold singularities in general. Even in this case Meinrenken and Sjamaar defined the Riemann-Roch index for $(M_{\gamma^*},\omega_{\gamma^*})$, and showed that the quantization conjecture still holds for a closed M [21].

We conclude this note with the following question:

Question 3.9. *For a critical value γ^*, is $\mathrm{ind}^{\gamma^*}_{S^1}\left(V_{\gamma^*},V_{\gamma^*}\cap V;L|_{V_{\gamma^*}}\right)$ equal to Meinrenken and Sjamaar's Riemann-Roch index for $(M_{\gamma^*},\omega_{\gamma^*})$?*

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