A FORMULA FOR THE EQUIVARIANT LOCAL INDEX OF THE REDUCED SPACE IN THE SYMPLECTIC CUTTING

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ABSTRACT. We compute the equivariant local index for the reduced space in a symplectic cut space, provided that the reduced space is compact.

1. INTRODUCTION

In the joint work [2, 3, 4] with Fujita and Furuta we developed an index theory for Dirac-type operators on possibly non-compact Riemannian manifolds. We call the index in our theory the *local index* and also call its equivariant version the *equivariant local index*. The local index has several properties, such as deformation invariance and excision property. In particular, when the manifold is closed, the local index is nothing but the usual index of the Dirac-type operator. Using these properties, for Hamiltonian S^1-actions on prequantizable closed symplectic manifolds we can show that the equivariant Riemann-Roch index is obtained as the sum of the equivariant local indices for the inverse images of the integer lattice points by the moment map. When the lattice point is a regular value of the moment map we can compute its equivariant local index, see [4, 7]. So the problem is how to compute the equivariant local index when the lattice point is a singular value.

In this paper, as a special case of singular lattice points we compute the equivariant local index for the reduced space in a symplectic cut space. Let (M, ω) be a Hamiltonian S^1-manifold with moment map $\mu\colon M \to \mathbb{R}$ and $n \in \mathbb{R}$ a regular value of μ. Suppose the S^1 acts on the level set $\mu^{-1}(n)$ freely. We denote the reduced space $\mu^{-1}(n)/S^1$ by M_n. The symplectic cut is a procedure to make a new Hamiltonian S^1-manifold $\overline{M}_{\mu \leq n}$, called the cut space, from the given (M, ω) and n. This method is introduced by Lerman in [5] and the reduced space M_n is naturally contained in the cut space $\overline{M}_{\mu \leq n}$ as a fixed point set. In particular, M_n is a singular fiber of the moment map of the S^1-action on $\overline{M}_{\mu \leq n}$. Moreover, if the Hamiltonian S^1-manifold (M, ω) is prequantizable and the regular value n is integer, then the cut space $\overline{M}_{\mu \leq n}$ is also prequantizable, and a prequantum line bundle (L, ∇^L) on (M, ω) induces the prequantum line bundle $\left(\overline{L}, \nabla^{\overline{L}}\right)$ on $\overline{M}_{\mu \leq n}$. Suppose also that $\mu^{-1}(n)$ is compact. Let O be an S^1-invariant open neighborhood of M_n in $\overline{M}_{\mu \leq n}$. Then the equivariant local index $\mathrm{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$ is defined. The purpose of this paper is to give a formula for $\mathrm{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$ in Theorem 4.1. This is an equivariant refinement of a partial result of [1, Section 5], in which we give a formula for the local index for a reduced space in a symplectic cut space in order to determine the local index of the zero section of the cotangent bundle of the n-dimensional sphere S^n by using the geodesic flow.

This paper is organized as follows. In Section 2 we recall the equivariant local index in the case of the Hamiltonian S^1-actions. After that we briefly recall the

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symplectic cut in Section 3. The main theorem (Theorem 4.1) is given in Section 4. Finally examples are given in Section 5.

Notation. In this paper we use the notation $\mathbb{C}_{(n)}$ for the irreducible representation of S^1 with weight n.

2. Equivariant local index

Let (M, ω) be a prequantizable Hamiltonian S^1-manifold and (L, ∇^L) an S^1-equivariant prequantum line bundle on (M, ω) with the associated moment map $\mu \colon M \to \mathbb{R}$. We do not assume M is compact. Since all orbits are isotropic the restriction of (L, ∇^L) to each orbit is flat.

Definition 2.1. An orbit $\mathcal{O}$ is said to be *L-acyclic* if $H^0\left(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}\right) = 0$.

Let V be an S^1-invariant open set whose complement is compact and which contains only L-acyclic orbits. For these data we give the following theorem.

Theorem 2.2 ([2, 3, 4]). *There exists an element* $\mathrm{ind}_{S^1}(M, V; L) \in R(S^1)$ *of the representation ring such that* $\mathrm{ind}_{S^1}(M, V; L)$ *satisfies the following properties:*

(1) $\mathrm{ind}_{S^1}(M, V; L)$ *is invariant under continuous deformation of the data.*
(2) *If M is closed, then* $\mathrm{ind}_{S^1}(M, V; L)$ *is equal to the equivariant Riemann-Roch index* $\mathrm{ind}_{S^1}(M; L)$.
(3) *If M' is an S^1-invariant open neighborhood of $M \smallsetminus V$, then* $\mathrm{ind}_{S^1}(M, V; L)$ *satisfies the following excision property*
$$\mathrm{ind}_{S^1}(M, V; L) = \mathrm{ind}_{S^1}(M', M' \cap V; L|_{M'}).$$
(4) $\mathrm{ind}_{S^1}(M, V; L)$ *satisfies a product formula.*

We call $\mathrm{ind}_{S^1}(M, V; L)$ the *equivariant local index*.

Example 2.3. For small positive real number $\varepsilon > 0$ which is less than 1 let $D_\varepsilon\left(\mathbb{C}_{(1)}\right) = \{z \in \mathbb{C}_{(1)} \mid |z| < \varepsilon\}$ be the 2-dimensional disc of radius ε. As $(L, \nabla^L) \to (M, \omega)$ we consider
$$\left(D_\varepsilon\left(\mathbb{C}_{(1)}\right) \times \mathbb{C}_{(m)}, d + \frac{1}{2}(z d\bar{z} - \bar{z} dz)\right) \to \left(D_\varepsilon\left(\mathbb{C}_{(1)}\right), \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}\right).$$

First let us detect non L-acyclic orbits. Suppose the orbit $\mathcal{O}$ through $z \in D_\varepsilon\left(\mathbb{C}_{(1)}\right)$ has a non-trivial parallel section $s \in H^0\left(\mathcal{O}; (L, \nabla)|_{\mathcal{O}}\right)$. Then s satisfies the following equation
$$0 = \nabla^L_{\partial_\theta} s = \frac{\partial s}{\partial \theta} - 2\pi\sqrt{-1}r^2 s,$$
where we use the polar coordinates $z = re^{2\pi\sqrt{-1}\theta}$. Hence s is of the form $s(\theta) = s_0 e^{2\pi\sqrt{-1}r^2\theta}$ for some non-zero constant s_0. Since s is a global section on $\mathcal{O}$ s satisfies $s(0) = s(1)$. This implies $r = 0$.

Next, we put $V = D_\varepsilon\left(\mathbb{C}_{(1)}\right) \smallsetminus \{0\}$ and let us compute $\mathrm{ind}_{S^1}(M, V; L)$. We recall the definition of $\mathrm{ind}_{S^1}(N, V; L)$. For $t \geq 0$ consider the following perturbation of the Spin^c Dirac operator $D \colon \Gamma\left(\wedge^{0,*}T^*M \otimes L\right) \to \Gamma\left(\wedge^{0,*}T^*M \otimes L\right)$ associated with the standard Hermitian structure on $M = D_\varepsilon\left(\mathbb{C}_{(1)}\right)$
$$D_t = D + t\rho D_{S^1},$$
where ρ is a cut-off function of V and $D_{S^1} \colon \Gamma\left((\wedge^* T^* M^{0,1} \otimes L)|_V\right) \to \Gamma\left((\wedge^* T^* M^{0,1} \otimes L)|_V\right)$ is a first order formally self-adjoint differential operator of degree-one that satisfies the following conditions:

(1) D_{S^1} contains only derivatives along orbits.

(2) The restriction $D_{S^1}|_{\mathcal{O}}$ to an orbit $\mathcal{O}$ is the de Rham operator with coefficients in $L|_{\mathcal{O}}$.

(3) For any S^1-equivariant section u of the normal bundle $\nu_{\mathcal{O}}$ of $\mathcal{O}$ in M, D_{S^1} anti-commutes with the Clifford multiplication of u.

See [2, 3, 4] for more details. From the second condition and $\{0\}$ is the unique non L-acyclic orbit we can see $\ker(D_{S^1}|_{\mathcal{O}}) = 0$ for all orbits $\mathcal{O} \neq \{0\}$. Extend the complement of a neighborhood of 0 in $D_\varepsilon(\mathbb{C}_{(1)})$ cylindrically so that all the data are translationally invariant. Then we showed in [2, 3] that for a sufficiently large t D_t is Fredholm, namely, $\ker D_t \cap L^2$ is finite dimensional and its super-dimension is independent of a sufficiently large t. So we define

$$\mathrm{ind}_{S^1}(M, V; L) = \ker D_t^0 \cap L^2 - \ker D_t^1 \cap L^2$$

for a sufficiently large t. In this case, by the direct computation using the Fourier expansion of s with respect to θ, we can show that

$$\ker D_t^0 \cap L^2 \cong \mathbb{C}, \quad \ker D_t^1 \cap L^2 = 0,$$

and $\ker D_t^0 \cap L^2$ is spanned by a certain L^2-function $a_0(r)$ on $D_\varepsilon(\mathbb{C}_{(1)})$ which depends only on $r = |z|$. Since the S^1-action on $\ker D_t^0 \cap L^2$ is given by pull-back and the S^1-action on the fiber is given by $\mathbb{C}_{(m)}$ we obtain

$$\mathrm{ind}_{S^1}(M, V; L) = \mathrm{ind}_{S^1}\left(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \smallsetminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(m)}\right)$$
$$= \mathbb{C}_{(-m)}.$$

For more details see [2, Remark 6.10], or [6, Section 5.3.2].

It is well-known that the lift of S^1-action on M to L defines the moment map $\mu\colon M \to \mathbb{R}$ by the Kostant formula

$$(2.1) \qquad \mathcal{L}_X s = \nabla_X^L s + 2\pi\sqrt{-1}\mu s,$$

where s is a section of L, X is the vector field which generates the S^1-action on (M, ω), and $\mathcal{L}_X s$ is the Lie derivative which is defined by

$$\mathcal{L}_X s(x) = \frac{d}{d\theta}\Big|_{\theta=0} e^{-2\pi\sqrt{-1}\theta} s(e^{2\pi\sqrt{-1}\theta}x).$$

Lemma 2.4. *If an orbit $\mathcal{O}$ is not L-acyclic, namely, $H^0\left(\mathcal{O}; (L, \nabla^L)|_{\mathcal{O}}\right) \neq 0$, then, $\mu(\mathcal{O}) \in \mathbb{Z}$.*

If M is closed, then we have the following localization formula for the equivariant Riemann-Roch index.

Corollary 2.5. *Suppose M is closed. For $i \in \mu(M) \cap \mathbb{Z}$ let V_i be an S^1-invariant open neighborhood of $\mu^{-1}(i)$ such that they are mutually disjoint, namely, $V_i \cap V_j \neq \emptyset$ for all $i \neq j$. Then,*

$$(2.2) \qquad \mathrm{ind}_{S^1}(M; L) = \bigoplus_{i \in \mu(M) \cap \mathbb{Z}} \mathrm{ind}_{S^1}(V_i, V_i \cap V; L|_{V_i}).$$

3. Symplectic cut

In this section let us briefly recall the symplectic cut and its properties we use in this paper. Let (M, ω) be a Hamiltonian S^1-space with moment map $\mu\colon M \to \mathbb{R}$. For a real number n the cut space $\overline{M}_{\mu \leq n}$ of (M, ω) by the symplectic cutting [5] is the reduced space of the diagonal S^1-action on $(M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}\right)$, namely,

$$\overline{M}_{\mu \leq n} = \left\{(x, z) \in (M, \omega) \times \left(\mathbb{C}_{(1)}, \frac{\sqrt{-1}}{2\pi}dz \wedge d\bar{z}\right)\Big| \mu(x) + |z|^2 = n\right\}/S^1.$$

We denote the reduced space $\mu^{-1}(n)/S^1$ by M_n.

Proposition 3.1. (1) If S^1 acts on $\mu^{-1}(n)$ freely, then, $\overline{M}_{\mu \leq n}$ is a smooth Hamiltonian S^1-space. The S^1-action is given as

$$(3.1) \qquad\qquad t[x, z] = [tx, z]$$

for $t \in S^1$ and $[x, z] \in \overline{M}_{\mu \leq n}$.
(2) Under the assumption in (1), the reduced space M_n and $\{x \in M \mid \mu(x) \leq n\}$ are symplectically embedded into $\overline{M}_{\mu \leq n}$ by $M_n \ni [x] \mapsto [x, 0] \in \overline{M}_{\mu \leq n}$ and $\{x \in M \mid \mu(x) \leq n\} \ni x \mapsto \left[x, \sqrt{n - \mu(x)}\right] \in \overline{M}_{\mu \leq n}$, respectively. In particular, $\overline{M}_{\mu \leq n}$ can be identified with the disjoint union $\{x \in M \mid \mu(x) \leq n\} \coprod M_n$ and with this identification M_n is fixed by the S^1-action (3.1).

Suppose that (M, ω) is equipped with a prequantum line bundle $(L, \nabla^L) \to (M, \omega)$ and the S^1-action lifts to (L, ∇^L) in such a way that μ satisfies the Kostant formula (2.1).

Proposition 3.2. If n is an integer and the S^1-action on $\mu^{-1}(n)$ is free, then $\overline{M}_{\mu \leq n}$ is prequantizable. In this case a prequantum line bundle $(\overline{L}, \nabla^{\overline{L}})$ on $\overline{M}_{\mu \leq n}$ is given by

$$(\overline{L}, \nabla^{\overline{L}}) = \left((L, \nabla^L) \otimes \mathbb{C}_{(n)}\right) \boxtimes \left(\mathbb{C}_{(1)} \times \mathbb{C}_{(0)}, d + \frac{1}{2}(zd\bar{z} - \bar{z}dz)\right)\Big|_{\Phi^{-1}(0)} / S^1,$$

where Φ is the moment map $\Phi \colon M \times \mathbb{C}_{(1)} \to \mathbb{R}$ associated to the lift of the diagonal S^1-action which is written as $\Phi(x, z) = \mu(x) + |z|^2 - n$, and the lift of the S^1-action (3.1) on $\overline{M}_{\mu \leq n}$ to $(\overline{L}, \nabla^{\overline{L}})$ is given by

$$(3.2) \qquad\qquad t[u \otimes v \boxtimes (z, w)] = [(tu) \otimes v \boxtimes (z, w)]$$

for $t \in S^1$ and $[u \otimes v \boxtimes (z, w)] \in \overline{L}$. The moment map $\overline{\mu} \colon \overline{M}_{\mu \leq n} \to \mathbb{R}$ associated with the lift (3.2) is written as $\overline{\mu}([x, z]) = \mu(x) = n - |z|^2$.

Remark 3.3. We denote the restriction of $(\overline{L}, \nabla^{\overline{L}})$ to M_n by (L_n, ∇^{L_n}). (L_n, ∇^{L_n}) is a prequantum line bundle on M_n. The S^1-action (3.2) on L_n is given by the fiber-wise multiplication with weight n. Recall that M_n is fixed by the S^1-action (3.1). See Proposition 3.1.

4. MAIN THEOREM

Let (M, ω) be a prequantizable Hamiltonian S^1-manifold and (L, ∇^L) an S^1-equivariant prequantum line bundle on (M, ω) with the associated moment map $\mu \colon M \to \mathbb{R}$. Let n be an integer and we assume the S^1-action on $\mu^{-1}(n)$ is free. Then, the cut space $\overline{M}_{\mu \leq n}$ becomes a prequantizable Hamiltonian S^1-manifold and the S^1-equivariant prequantum line bundle $(\overline{L}, \nabla^{\overline{L}})$ is given by Proposition 3.1 and Proposition 3.2.

Suppose that $\mu^{-1}(n)$ is compact. We take a sufficiently small S^1-invariant open neighborhood O of M_n in $\overline{M}_{\mu \leq n}$ so that the intersection $\overline{\mu}(O) \cap \mathbb{Z}$ consists of the unique point n. Then we can define the equivariant local index $\text{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$ of M_n in $\overline{M}_{\mu \leq n}$. We give the following formula for $\text{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$.

Theorem 4.1. Let (M, ω), (L, ∇^L), and μ be as above. Let n be an integer. Suppose S^1 acts on $\mu^{-1}(n)$ freely and $\mu^{-1}(n)$ is compact. Let O be a sufficiently small S^1-invariant open neighborhood of M_n in $\overline{M}_{\mu \leq n}$ which satisfies $\overline{\mu}(O) \cap \mathbb{Z} = \{n\}$. Then, the equivariant local index is given as

$$\text{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right) = \text{ind}(M_n; L_n)\mathbb{C}_{(n)},$$

where L_n is the prequantum line bundle $L_n = (L|_{\mu^{-1}(n)} \otimes \mathbb{C}_{(n)})/S^1$ on M_n and $\text{ind}(M_n; L_n)$ is the Riemann-Roch number of M_n.

Remark 4.2. By replacing $\mathbb{C}_{(1)}$ with $\mathbb{C}_{(-1)}$ in the above construction we obtain the other cut space $\overline{M}_{\mu \geq n} = \{(x,z) \in M \times \mathbb{C}_{-1} : \mu(x) - |z|^2 = n\}/S^1$. Theorem 4.1 also holds for $\overline{M}_{\mu \geq n}$.

To prove the theorem we need some preliminaries. Let $\operatorname{grad}(\mu)$ be the negative gradient vector field of μ with respect to an S^1-invariant Riemannian metric g of M, namely, the vector field determined by

$$-d\mu = g(\operatorname{grad}(\mu), \cdot\,).$$

Since n is a regular value of μ, $\operatorname{grad}(\mu)$ does not vanish near $\mu^{-1}(n)$. Let ϕ_τ be the flow of the vector field $\frac{1}{\|\operatorname{grad}(\mu)\|^2}\operatorname{grad}(\mu)$. Note that ϕ_τ exists on $\mu^{-1}((n-\varepsilon, n+\varepsilon))$ for a sufficiently small $\varepsilon > 0$ since $\mu^{-1}(n)$ is compact by assumption.

Lemma 4.3. ϕ_τ *commutes with the S^1-action on M and satisfies the following property*

$$\mu(\phi_\tau(x)) = \mu(x) - \tau$$

for $x \in \mu^{-1}((n-\varepsilon, n+\varepsilon))$.

Proof.

$$\begin{aligned}
\frac{d}{d\tau}\mu(\phi_\tau(x)) &= d\mu\left(\frac{1}{\|\operatorname{grad}(\mu)\|^2}\operatorname{grad}(\mu)\right) \\
&= -g\left(\operatorname{grad}(\mu), \frac{1}{\|\operatorname{grad}(\mu)\|^2}\operatorname{grad}(\mu)\right) \\
&= -1.
\end{aligned}$$

$\square$

By the definition of the symplectic cutting, the normal bundle ν of M_n in $\overline{M}_{\mu \leq n}$ is given by

$$\nu = \mu^{-1}(N) \times_{S^1} \mathbb{C}_{(1)}.$$

For a sufficiently small $\varepsilon > 0$ let $D_\varepsilon(\mathbb{C}_{(1)}) = \{z \in \mathbb{C}_{(1)} : |z| < \varepsilon\}$ be the open disc of radius ε. We put $D_\varepsilon(\nu) = \mu^{-1}(n) \times_{S^1} D_\varepsilon(\mathbb{C}_{(1)})$, and define an S^1-action on $D_\varepsilon(\nu)$ by

$$(4.1) \qquad\qquad\qquad\qquad t[x,z] = [tx, z].$$

Let $p\colon D_\varepsilon(\nu) \to M_n$ be the natural projection. We define a complex line bundle $L_{D_\varepsilon(\nu)}$ on $D_\varepsilon(\nu)$ by

$$L_{D_\varepsilon(\nu)} = p^* L_n \otimes \left(\mu^{-1}(n) \times_{S^1} (D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})\right),$$

and define an lift of the S^1-action (4.1) to $L_{D_\varepsilon(\nu)}$ by

$$(4.2) \qquad t\left(([x,z], [u \otimes v]) \otimes [x', z', w]\right) = \left(([tx, z], [(tu) \otimes v]) \otimes [tx', z', w]\right).$$

Then we can show the following lemma.

Lemma 4.4. (1) *For a sufficiently small $\varepsilon > 0$ there exists an S^1-equivariant embedding $f_M\colon D_\varepsilon(\nu) \to \overline{M}_{\mu \leq n}$ with respect to the actions (4.1) and (3.1).*
(2) *Under the assumption in (1) there exists an S^1-equivariant bundle map $f_L\colon L_{D_\varepsilon(\nu)} \to \overline{L}$ with respect to the actions (4.2) and (3.2) such that f_L covers f_M.*

Proof. We define f_M and f_L by

$$f_M([x,z]) = [\phi_{|z|^2}(x), z],$$

$$f_L(([x,z], [u \otimes v]) \otimes [x', z', w]) = \left[\left(\widetilde{\phi}_{|z|^2}(t_0 u) \otimes t_0^n v\right) \otimes (z, w)\right],$$

where $t_0 \in S^1$ is the unique element that satisfies $t_0 u \in L_x$ and $\widetilde{\phi}_\tau(t_0 u)$ is the horizontal lift of $\phi_\tau(x)$ starting from $t_0 u$ and $\widetilde{\phi}_{|z|^2}(t_0 u)$ is its value at $\tau = |z|^2$. Then they are required maps. $\square$

Proof of Theorem 4.1. By Lemma 4.4 and the equivariant version of the product formula [3, Theorem 5.8] we obtain

$$\mathrm{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$$

$$= \mathrm{ind}_{S^1}\left(D_\varepsilon(\nu), \ D_\varepsilon(\nu) \smallsetminus M_n; L_{D_\varepsilon(\nu)}\right)$$

(4.3)

$$= \mathrm{ind}_{S^1}\left(M_n; L_n \otimes \mu^{-1}(n) \times_{S^1} \mathrm{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \smallsetminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})\right).$$

From Example 2.3 the equivariant local index $\mathrm{ind}_{S^1}(D_\varepsilon(\mathbb{C}_{(1)}), D_\varepsilon(\mathbb{C}_{(1)}) \smallsetminus \{0\}; D_\varepsilon(\mathbb{C}_{(1)}) \times \mathbb{C}_{(0)})$ is equal to $\mathbb{C}_{(0)}$. By definition, L_n is naturally identified with the restriction of $\overline{L}$ to M_n. With this identification we can see that the restriction of the S^1-action (3.2) to $L_n \to M_n$ is nothing but the fiberwise multiplication of t^{-n}. Since the S^1-action on $\mathrm{ind}_{S^1}(M_n; L_n)$ is defined by the pull-back, the S^1-action on $\mathrm{ind}_{S^1}(M_n; L_n)$ is given by the multiplication of t^n as we mentioned in Remark 3.3. This proves the theorem. $\square$

5. Examples

Example 5.1 (Complex projective space). As $(L, \nabla) \to (M, \omega)$ we adopt

$$\left((\mathbb{C}_{(1)})^m \times \mathbb{C}_{(0)}, d + \frac{1}{2}\sum_{i=1}^m (z_i d\bar{z}_i - \bar{z}_i dz_i)\right) \to \left((\mathbb{C}_{(1)})^m, \frac{\sqrt{-1}}{2\pi}\sum_{i=1}^m dz_i \wedge d\bar{z}_i\right).$$

For $n = 1$ the obtained $\overline{M}_{\mu \leq n}$, $\overline{L}$, and M_n are $\mathbb{C}P^m$, $\mathcal{O}(1)$, and $\mathbb{C}P^{m-1}$, respectively. The induced S^1-actions on $\mathbb{C}P^m$ and $\mathcal{O}(1)$ are given by

(5.1)

$$t[z_0 : z_1 : \cdots : z_m] = [z_0 : tz_1 : \cdots : tz_m], \quad t[z_0 : z_1 : \cdots : z_m, w] = [z_0 : tz_1 : \cdots : tz_m, w].$$

The moment map $\overline{\mu}$ associated to the S^1-action (5.1) is given by $\overline{\mu}([z_0 : \cdots : z_m]) = \sum_{i=1}^m |z_i|^2$. For $k = 0, 1$ let O_k be a sufficient small S^1-invariant open neighborhood of $\overline{\mu}^{-1}(k)$. Then the equivariant local index $\mathrm{ind}_{S^1}\left(O_k, O_k \smallsetminus \overline{\mu}^{-1}(k); \overline{L}|_{O_k}\right)$ is defined and By Corollary 2.2 the equivariant Riemann-Roch index $\mathrm{ind}_{S^1}\left(\overline{M}_{\mu \leq n}, \overline{L}\right)$ satisfies following equality

(5.2)

$$\mathrm{ind}_{S^1}\left(\overline{M}_{\mu \leq n}, \overline{L}\right) = \mathrm{ind}_{S^1}\left(O_0, O_0 \smallsetminus \overline{\mu}^{-1}(0); \overline{L}|_{O_0}\right) + \mathrm{ind}_{S^1}\left(O_1, O_1 \smallsetminus \overline{\mu}^{-1}(1); \overline{L}|_{O_1}\right).$$

The left hand side is computed as

(5.3) $$\mathrm{ind}_{S^1}\left(\overline{M}_{\mu \leq n}, \overline{L}\right) = \mathrm{ind}_{S^1}\left(\mathbb{C}P^m, \mathcal{O}(1)\right) = \mathbb{C}_{(0)} \oplus m\mathbb{C}_{(1)}.$$

For $k = 1$, since $\overline{\mu}^{-1}(1) = M_n$, by Theorem 4.1 $\mathrm{ind}_{S^1}\left(O_1, O_1 \smallsetminus \overline{\mu}^{-1}(1); \overline{L}|_{O_1}\right)$ is given as

$$\mathrm{ind}_{S^1}\left(O_1, O_1 \smallsetminus \overline{\mu}^{-1}(1); \overline{L}|_{O_1}\right) = \mathrm{ind}_{S^1}\left(O_1, O_1 \smallsetminus M_n; \overline{L}|_{O_1}\right)$$

(5.4) $$= \mathrm{ind}(\mathbb{C}P^{m-1}; \mathcal{O}(1))\mathbb{C}_{(1)}$$

$$= m\mathbb{C}_{(1)}.$$

For $k = 0$, it is easy to see that $\overline{\mu}^{-1}(0) = \{[z_0 : 0 : \cdots : 0]\}$ and $(\overline{L}, \nabla^{\overline{L}})|_{[z_0:0:\cdots:0]} \cong (\mathbb{C}_{(0)}, d + \frac{1}{2}(\bar{z}dz - zd\bar{z}))$. We can take O_0 in such a way that O_0 is identified with a sufficiently small open disc $D = \{(z_1, \ldots, z_m) \in \mathbb{C}^m : \sum_{i=1}^m |z_i|^2 \leq \varepsilon\}$ with S^1-action $t(z_1, \ldots, z_m) = (tz_1, \ldots, tz_m)$. Then, by (5.2), (5.3), and (5.4) we obtain the following formula

$$\mathrm{ind}_{S^1}\left(D, D \smallsetminus \{0\}; D \times \mathbb{C}_0\right) = \mathbb{C}_{(0)}.$$

In the case of $m = 1$ this formula can be obtained in [2, Remark 6.10] and [6, Section 5.3.2].

Example 5.2 (Exceptional divisor)**.** Let n and $(L, \nabla) \to (M, \omega)$ be as in Example 5.1. Then the obtained cut space $\overline{M}_{\mu \geq n}$ is the blow-up $\widetilde{\mathbb{C}}^m$ of the origin in $\mathbb{C}^m$, and M_n and L_n are the exceptional divisor $\mathbb{C}P^{m-1}$ and $\mathcal{O}(n)$, respectively. We take a sufficiently small invariant open neighborhood O of M_n. Then, by Theorem 4.1 the equivariant local index $\mathrm{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right)$ is given by

$$\mathrm{ind}_{S^1}\left(O, O \smallsetminus M_n; \overline{L}|_O\right) = \mathrm{ind}\left(\mathbb{C}P^{m-1}; \mathcal{O}(n)\right) \mathbb{C}_{(n)} = \binom{m-1+n}{m-1} \mathbb{C}_{(n)}.$$

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