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Trend, Seasonality, and Economic Time Series : A New Approach Using Non-stationary Errors-in-Variables Models

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Trend, Seasonality, and Economic Time Series : A New Approach Using Non-stationary Errors-in-Variables Models *

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Abstract

The use of seasonally adjusted (official) data may introduce statistical problem, particularly the use of *X-12-ARIMA* in *the official seasonal adjustment*, which adopts univariate ARIMA (autoregressive integrated moving average) time series modeling with some refinements. Instead of using seasonally adjusted data for estimating the structural parameters and relationships among non-stationary economic time series with seasonality and noise, we propose a new method called the Separating Information Maximum Likelihood (SIML) estimation. We use an additive decomposition of components of multivariate time series to handle the measurement errors with non-stationary trends and seasonality. We will show that the SIML estimation can identify the non-stationary trend, the seasonality, and the noise components, and recover statistical relationships among the nonstationary trend and seasonality. The SIML estimator is consistent, and it has asymptotic normality when the sample size is large. Since the SIML estimator has also reasonable finite sample properties, it would be useful for practice.

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Key Words

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1 Introduction

There is a vast amount of published research on the use of statistical time series analysis for analyzing macroeconomic time series. One important distinction of macroeconomic time series from the standard time series analysis in other areas has been the mixture of non-stationarity and measurement errors, including apparent seasonality; however, the analysis of seasonality of economic time series has often been brief. (See Hayashi (2000) for instance.) Although there have been many attempts to deal with stationarity, non-stationarity and seasonality separately in macroeconomic time series analysis, there remains some need to incorporate these different aspects of economic time series in a unifying manner.

For expository purposes, we illustrate two macro time series in Figure 1-1, which displays the original quarterly data of the real GDP and fixed investment series published by Japan's Cabinet Office. We have standardized two time series such that the data in scale have similar values and we can observe clear common trends, common seasonality, and noise in two important time series, which are quite typical in Japanese quarterly macro time series data. An interesting empirical question here would be to find reasonable estimates of correlations of trends and seasonalities between two non-stationary macro time series we observe quarterly.

The use of seasonally adjusted data has been a common practice among many economists in macroeconomics and business practice, however, we must cope with problems of *the official seasonal adjustments method* that generates the published data used for macroeconomic variables. It has been a common practice to use *X-12-ARIMA* in many official agencies, including the U.S. Census Bureau and Cabinet office of the Japanese Government (i.e., they produce the official gross domestic product (GDP) and other macro time series in Japan), but they use the univariate seasonal ARIMA (autoregressive integrated moving average) time series modeling with some refinements, which is called *Reg-ARIMA* modeling. (See Nerlove et al. (1995) on economic analysis of seasonality and Findley et al. (1998) on the *X-12-ARIMA* program.)

In this study, instead of using seasonally adjusted (published) data and investigating the statistical relationships among macro time series, we propose to use the separating information maximum likelihood (SIML) estimation method, which is new to macro time series analysis, although it was originally developed as an estimation method for high frequency econometrics by Kunitomo and Sato (2013), and there are important differences from their analysis. For instance, to handle high frequency financial data in the finance SIML method, we use an asymptotic theory when the observation intervals become smaller with more observations while the underlying hidden process is a continuous (time) stochastic process, including the diffusion or jump-diffusion type processes. The more relevant asymptotic theory for macroeconomic time series should be the one in which the observation intervals are fixed while the number of observations becomes large, which is standard in discrete time series analysis. We will investigate the macro-SIML method in the latter asymptotic framework and show that it is useful to identify trends, seasonals, cycles, and irregular-noise components in the non-stationary errors-in-variables model. The conditions for the consistency and asymptotic normality of the macro SIML estimator in this study are new because of the relevant asymptotic theory. We will use the additive decomposition model of components of time series because it gives a simple way to represent the non-stationary time series with *measurement errors*. It can be regarded as an extension of the univariate decomposition of its components by Kitagawa and Gersch (1984) and Kitagawa (2010) with different perspectives; that is, their main interests were the statistical filtering of non-stationary state variables from a discrete time series [1].

There have been many studies on errors-in-variables models that are closely related to the classical multivariate analysis, including the factor models and simultaneous equations models. [see Anderson (1984, 2003) and Fuller (1987) for discussions on the related classical issues.] It has been known that serious identification problems occur in classical errors-in-variables models when we have independent observations with homogenous measurement errors, and the estimation problem of unknown parameters for the underlying hidden variables has some difficulty. In the the standard approach of time series analysis it is not easy to handle measurement errors with non-stationary trends and seasonals and instead we shall use the errors-in-variables representation of multivariate time series. In this study we will show that in the mixture of non-stationary and stationary components, including seasonal factors,

[1] They have developed the computer program *DECOMP*, which has been available at Institute of Statistical Mathematics (ISM).

we can identify the unknown parameters generating the hidden time series components. The typical examples are the variance-covariance matrices of the hidden trend variables, and the variance-covariance matrix of hidden seasonal components and noise components. We will show that SIML estimation can estimate the trend, the seasonality and noise components from the observed time series, and recover the structural relationships between the non-stationary trend and seasonality. We also show that SIML estimation provides consistency and asymptotic normality, when the sample size is large in the standard asymptotic theory. Based on a set of simulations, we find that the SIML estimator has reasonable finite sample properties and thus it would be useful for practice.

A motivation of our study is the fact that it is not a trivial task to handle the exact likelihood function and calculate the exact maximum likelihood (ML) method for estimating structural relationships among trends from non-stationary time series data when the observed time series contain seasonality, noise, and measurement errors in the non-stationary errors-in-variables models (see Section 3 for an illustration). This aspect is quite important for the analysis of multivariate macroeconomic time series because modeling the seasonality and noise could lead to possible misspecifications. In this study we regard seasonality and noise as measurement errors. Instead of calculating the Gaussian likelihood function, we try to separate the information of the signal part and the measurement errors part from the likelihood function, and then use each separately. This procedure approximates the maximization of the likelihood function and makes the estimation procedure applicable to multivariate non-stationary time series data in a straight-forward manner. We denote our estimation method as the separating information maximum likelihood (SIML) estimator because it extends the standard ML estimation method. The main merit of SIML estimation is its simplicity and its use in practical applications for multivariate non-stationary economic time series.

Earlier and related literature in econometrics are Engle and Granger (1987) and Johansen (1995), which dealt with multivariate non-stationary and stationary time series and developed the notion of co-integration, but importantly without measurement errors. The problem of the present study is related to their work, but it has different aspects due to the fact that the main focus of our analysis would be the non-stationary trend, seasonality and stationary measurement errors in the non-stationary errors-in-variable model. The existing literature on non-stationary (econometric) time series analysis may have a problem of handling *measurement errors* and stochastic *seasonality* of economic time series data.

In Section 2 we will present a general formulation of the problem and give simple examples to illustrate the problem in this study. Then in Section 3, we will develop the non-stationary multivariate time series model with a common factor case and in Section 4 we will develop the macro SIML estimation method. Section 5 discusses our method to analyze the seasonal components. In Section 6, we will discuss some simulation results and then present some concluding remarks in Section 7. The proofs will be given in Appendix and the technical methods of proofs in this study are extensions of the results reported in Kunitomo and Sato (2013).

2 The general problem and some examples

2.1 The Decomposion Model

Let y_{ij} be the $i-$th observation of the $j-$th time series at i for $i = 1, \cdots, n; j = 1, \cdots, p$. We set $\mathbf{y}_i = (y_{1i}, \cdots, y_{pi})'$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i') \ (= (y_{ij}))$ be an $n \times p$ matrix of observations and we denote $\mathbf{y}_0$ as the initial $p \times 1$ vector. We consider the situation when the underlying non-stationary trends $\mathbf{x}_i \ (= (x_{ji})) \ (i = 1, \cdots, n)$ are not necessarily the same as the observed time series and let $\mathbf{s}_i' = (s_{1i}, \cdots, s_{pi})$ and $\mathbf{v}_i' = (v_{1i}, \cdots, v_{pi})$ be the vectors of the seasonal components, and the stationary components, respectively, which are independent of $\mathbf{x}_i$. Then we use the additive decomposition form

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{s}_i + \mathbf{v}_i \quad (i = 1, \cdots, n), \tag{2.1}$$

where a sequence of non-stationary trend components $\mathbf{x}_i \ (i = 1, \cdots, n)$ satisfies

$$\Delta \mathbf{x}_i = (1 - \mathcal{L})\mathbf{x}_i = \mathbf{w}_i^{(x)} \tag{2.2}$$

with $\mathcal{L}\mathbf{x}_i = \mathbf{x}_{i-1}$, $\Delta = 1 - \mathcal{L}$, $\mathcal{E}(\mathbf{w}_i^{(x)}) = \mathbf{0}$, $\mathcal{E}(\mathbf{w}_i^{(x)}\mathbf{w}_i^{(x)'}) = \mathbf{\Sigma}_x$, and a sequence of seasonal components $\mathbf{s}_i \ (i = 1, \cdots, n)$ satisfies

$$(1 + \mathcal{L} + \cdots + \mathcal{L}^{s-1})\mathbf{s}_i = \mathbf{w}_i^{(s)} \tag{2.3}$$

with $\mathcal{L}^s\mathbf{s}_i = \mathbf{s}_{i-s}$, $\mathcal{E}(\mathbf{w}_i^{(s)}) = \mathbf{0}$, $\mathcal{E}(\mathbf{w}_i^{(s)}\mathbf{w}_i^{(s)'}) = \mathbf{\Sigma}_s$, and a sequence of stationary components satisfyies $\mathbf{v}_i \ (i = 1, \cdots, n)$ with $\mathcal{E}(\mathbf{v}_i\mathbf{v}_i') = \mathbf{\Sigma}_v$ and

$$\mathbf{v}_i = \sum_{j=-\infty}^{\infty} \mathbf{C}_j \mathbf{e}_{i-j} \ , \tag{2.4}$$

with absolutely summable coefficients $\mathbf{C}_j$ and a sequence of i.i.d. random vectors with $\mathcal{E}(\mathbf{e}_i) = \mathbf{0}$, $\mathcal{E}(\mathbf{e}_i\mathbf{e}_i^{'}) = \mathbf{\Sigma}_e$.
We assume that $\mathbf{w}_i^{(x)}, \mathbf{w}_i^{(s)}$ and $\mathbf{e}_i$ are the sequence of i.i.d. random vectors with $\mathbf{\Sigma}_e$ being positive-semi-definite, and the random vectors $\mathbf{w}_i^{(x)}, \mathbf{w}_i^{(s)}$ and $\mathbf{e}_i$ are mutually independent. When $\mathbf{v}_i = \mathbf{e}_i$, we can interpret that it is a sequence of independent measurement errors. The present additive decomposition is similar to the one given by Kitagawa and Gersch (1984) and Kitagawa (2010).

The main purpose of this study is to estimate structural parameters and structural relationships among the hidden random variables; the trend components and seasonal components in the the non-stationary errors-in-variables models. Let $\boldsymbol{\beta}$ be a $p \times 1$ (non-zero) vector and we want to estimate the statistical relationship as

$$\boldsymbol{\beta}^{'}\mathbf{y}_i = O_p(1) \ \ (i = 1, \cdots, n), \tag{2.5}$$

when we have the observations of $p \times 1$ vectors $\mathbf{y}_i$ $(i = 1, \cdots, n)$. More generally, let $\mathbf{B}^{'}$ be a $r_x \times p$ $(1 \leq r_x \leq p)$ non-zero matrix and we want to estimate a set of statistical relationships

$$\mathbf{B}^{'}\mathbf{y}_i = O_p(1) \ \ (i = 1, \cdots, n) \tag{2.6}$$

when we have the observations of $p \times 1$ vectors $\mathbf{y}_i$ $(i = 1, \cdots, n)$. Also some structural relations among seasonal components can be written as

$$\mathbf{B}_s^{'}\mathbf{s}_i = \mathbf{0} \ \ (i = 1, \cdots, n) \ , \tag{2.7}$$

where $\mathbf{B}_s^{'}$ is a non-zero $r_s \times p$ matrix $(1 \leq r_s \leq p)$ and they imply that the observed multivariate time series have common seasonality.

2.2 Some examples

We give simple examples when $p = 2$ for illustrating the problem of non-stationary errors-in-variables models, which have different representations.
Example 1 : Assume that for the sequence of observable random vectors $\mathbf{y}_i = (y_{1i}, y_{2i})^{'}$, the random variables $x_{1i} = \mu_i$ and $x_{2i} = -\beta_2\mu_i$ satisfy $\mu_i = \mu_{i-1} + w_{1i}^{(x)}$ $(i = 1, \cdots, n)$ and $w_{1i}^{(x)}$ are i.i.d. random variables with $\mathcal{E}(w_{1i}^{(x)}) = 0$ and $\mathcal{E}(w_{1i}^{(x)2}) = \sigma_\mu^2$, $\mathbf{w}_i^{(x)} = (w_{1i}^{(x)}, w_{2i}^{(x)})^{'} = (1, -\beta_2)^{'}\Delta\mu_i$. We take the case when $\mathbf{s}_i = \mathbf{0}$ and $\mathbf{v}_i$ is a sequence of i.i.d. random vectors.

Then we can write

$$\mathbf{y}_i = \begin{pmatrix} 1 \\ -\beta_2 \end{pmatrix} \mu_i + \mathbf{v}_i \ , \tag{2.8}$$

where $\mathbf{v}_i$ is a sequence of 2×1 noise vectors and we will denote $\boldsymbol{\pi} = (1, -\beta_2)^{'}$. Since μ_i follows the random walk model, the invariance principle (or CLT) says that as $n \to \infty$, $(1/n^2) \sum_{i=1}^{n} \mu_i^2 \xrightarrow{w} \sigma_\mu^2 \int_0^1 B_s^2 ds$ and B_s is the standard Brownian Motion on $[0,1]$. Let also $\mathbf{z}_i = (z_{1i}, z_{2i})^{'}$ and $\boldsymbol{\Omega}_z = \mathcal{E}[\mathbf{z}_i \mathbf{z}_i^{'}]$, where $z_{1i} = w_{1i}^{(x)} + v_{1i}$ and $z_{2i} = -\beta_2 w_{1i}^{(x)} + v_{2i}$. Then we have the representation

$$\mathbf{y}_i = \mathbf{y}_{i-1} + \mathbf{z}_i - \boldsymbol{\Theta} \mathbf{z}_{i-1} \ , \tag{2.9}$$

where $\mathbf{z}_{i-1} = \boldsymbol{\Omega}_z^{1/2} \boldsymbol{\Sigma}_v^{-1/2} \mathbf{v}_{i-1}$, $\boldsymbol{\Theta} = \boldsymbol{\Sigma}_v^{1/2} \boldsymbol{\Omega}_z^{-1/2}$ and $\boldsymbol{\Omega}_z = (1, -\beta_2)^{'}(1, -\beta_2)\sigma_x^2 + \boldsymbol{\Sigma}_v$. We have two forms of the stochastic process such that (2.8) is the errors-in-variables representation while (2.9) is the VARMA representation. The former is a convenient form with trends and measurement errors and it may be difficult to recover (2.8) from the second form of (2.9), which may be popular in econometrics. If we multiply the vector $\boldsymbol{\beta}^{'} = (\beta_2, 1)$ to (2.8) or (2.9) from the left, we have the statistical relation

$$\boldsymbol{\beta}^{'} \mathbf{y}_i = u_i \ (= \boldsymbol{\beta}^{'} \mathbf{v}_i) \ , \tag{2.10}$$

which is a structural equation and u_i is a sequence of i.i.d. random variables with $\mathcal{E}(u_i) = 0, \mathcal{E}(u_i^2) = \boldsymbol{\beta}^{'} \boldsymbol{\Sigma}_v \boldsymbol{\beta}$.

Example 2 : We take the case when $\mathbf{x}_i = \boldsymbol{\mu}_i$, and $\boldsymbol{\mu}_i = \boldsymbol{\mu}_{i-1} + \mathbf{w}_i^{(x)}$, which is often called *spurious regression*. We also take the case when $\mathbf{s}_i = \mathbf{0}$ and $\mathbf{v}_i$ is a sequence of i.i.d. vectors. It can be written as

$$\mathbf{y}_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\mu}_i + \mathbf{v}_i \tag{2.11}$$

and the dimension of random walk is 2. Then $\boldsymbol{\beta}^{'} \mathbf{y}_i = \boldsymbol{\beta}^{'} \boldsymbol{\mu}_i + u_i$ and $u_i = \boldsymbol{\beta}^{'} \mathbf{v}_i$ for any $\boldsymbol{\beta} \neq \mathbf{0}$. In this case the non-stationary term of $\boldsymbol{\beta}^{'} \boldsymbol{\mu}_i$ is the trend term, which follows an $I(1)$ process.

Example 3 : Assume that the random vectors $\mathbf{s}_i = (s_{1i}, s_{2i})^{'}$ with $s_{1i} = \nu_i^{(s)} = \beta_2^{(s)} \mu_i^{(s)}$ and $s_{2i} = \mu_i^{(s)}$ satisfy $\mu_i^{(s)} = \mu_{i-s}^{(s)} + w_i^{(s)}$ $(s \geq 1 \ ; \ i = 1, \cdots, n)$ and $w_i^{(s)}$ are i.i.d. random variables with $\mathcal{E}(w_i^{(s)}) = 0$ and $\mathcal{E}(w_i^{(s)2}) = \sigma_s^2$. We take the case when $\mathbf{x}_i = \mathbf{0}$ and $\mathbf{v}_i$ is a sequence of i.i.d. vectors. Then we can write

$$\mathbf{y}_i = \begin{pmatrix} \beta_2^{(s)} \\ 1 \end{pmatrix} \mu_i^{(s)} + \mathbf{v}_i \ . \tag{2.12}$$

If we multiply the vector $\boldsymbol{\beta}_s^{'} = (1, -\beta_2^{(s)})$ to (2.11) from the left, we have the relation among seasonal components as

$$\boldsymbol{\beta}_s^{'}\mathbf{y}_i = u_i \ (= \boldsymbol{\beta}_s^{'}\mathbf{v}_i) \tag{2.13}$$

and $\mathbf{y}_i$ has the common seasonal component.

Example 4 : We consider the situation when $\mathbf{x}_i = \boldsymbol{\mu}_i$, $\boldsymbol{\mu}_i = \boldsymbol{\mu}_{i-1} + \mathbf{w}_i^{(x)}$ with $\boldsymbol{\Sigma}_x = \sigma_x^2 \mathbf{I}_2$ (which is proportional to the identity) as the non-stationary trends and $\mathbf{s}_i = (s_{1i}, s_{2i})^{'}$ with $s_{1i} = \nu_i^{(s)} = \beta_2^{(s)}\mu_i^{(s)}$, $s_{2i} = \mu_i^{(s)}$, $\mu_i^{(s)} = \mu_{i-s}^{(s)} + w_i^{(s)}$ ($w_i^{(s)}$ are i.i.d. random variables) and $\boldsymbol{\Sigma}_s \geq 0$ (non-negative definite) as the non-stationary seasonals. In this case the non-stationary trends do not have any common trend, but there is a common non-stationary seasonal. The standard regression of one non-stationary variable on another non-stationary variable may not give a meaningful information on the underlying relationships among trends and seasonals.

3 The non-stationary common factor case without seasonality

We first consider the non-stationary time series without seasonality because the presence of seasonality may make some complication into our analysis. We shall introduce our main idea for this case, and then extend it to the non-stationary time series with stochastic seasonality.

Let $p \geq 2$ and $\mathbf{s}_i = \mathbf{0}$ and assume that $\mathbf{v}_i$ is a sequence of i.i.d. measurement error vectors in this section. We consider the multivariate time series model having the representation

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i = \boldsymbol{\Pi}\boldsymbol{\mu}_i + \mathbf{v}_i \ , \tag{3.1}$$

where $\mathbf{w}_i^{(x)} = \Delta\mathbf{x}_i$, $\mathcal{E}(\mathbf{w}_i^{(x)}) = \mathbf{0}$, and $\mathcal{E}(\mathbf{w}_i^{(x)}\mathbf{w}_i^{(x)'}) = \boldsymbol{\Sigma}_x$. We assume that the rank of non-zero $p \times q_x$ matrix $\boldsymbol{\Pi}$ is q_x $(1 \leq q_x \leq p)$ and $\boldsymbol{\mu}_i$ are $q_x \times 1$ vectors. We denote $\mathcal{E}(\boldsymbol{\mu}_i) = \mathbf{0}$ and $\mathcal{E}[(\Delta\boldsymbol{\mu}_i)(\Delta\boldsymbol{\mu}_i^{'})] = \boldsymbol{\Sigma}_\mu$, which is a $q_x \times q_x$ non-singular matrix. Since the rank of $\boldsymbol{\Pi}$ is q_x, there exists a non-zero $r_x \times p$ (non-zero) matrix $\mathbf{B}^{'}$ such that $\mathbf{B}^{'}\boldsymbol{\Pi} = \mathbf{O}$ and $\mathbf{B}^{'}\mathbf{y}_i = \mathbf{u}_i$ $(= \mathbf{B}^{'}\mathbf{v}_i)$, which are the set of r_x structural equations when $0 < r_x = p - q_x < p$. They are often called the co-integrated relations in the non-stationary time series analysis.

We consider the situation when $\Delta \mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are mutually independent and each of the component vectors are independently, identically, and normally distributed as $N_p(\mathbf{0}, \mathbf{\Sigma}_x)$ and $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$, respectively. We use an $n \times p$ matrix $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ and consider the distribution of $np \times 1$ random vector $(\mathbf{y}_1^{'}, \cdots, \mathbf{y}_n^{'})^{'}$. Given the initial condition $\mathbf{y}_0$, we have

$$\text{vec}(\mathbf{Y}_n) \sim N_{n \times p}\left(\mathbf{1}_n \cdot \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes \mathbf{\Sigma}_x\right) , \tag{3.2}$$

where $\mathbf{1}_n^{'} = (1, \cdots, 1)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix}_{n \times n} . \tag{3.3}$$

Then, given the initial condition $\mathbf{y}_0$, the conditional maximum likelihood (ML) estimator can be defined as the solution of maximizing the conditional log-likelihood function [2] except a constant as

$$\begin{aligned} L_n^* &= \log |\mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes \mathbf{\Sigma}_x|^{-1/2} \\ &\quad -\frac{1}{2}[vec(\mathbf{Y}_n - \bar{\mathbf{Y}}_0)^{'}]^{'}[\mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes \mathbf{\Sigma}_x]^{-1}[vec(\mathbf{Y}_n - \bar{\mathbf{Y}}_0)^{'}] \quad , \end{aligned}$$

where

$$\bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0^{'} . \tag{3.4}$$

We use the transformation $\mathbf{K}_n^*$ that from $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = \mathbf{K}_n^* \left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) \ , \ \mathbf{K}_n^* = \mathbf{P}_n \mathbf{C}_n^{-1} , \tag{3.5}$$

where

$$\mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{n \times n} , \tag{3.6}$$

[2]It may be possible to use the unconditional likelihood function with an assumption on the initial condition, which makes some complication but may have a better finite sample property.

and

$$(3.7) \qquad \mathbf{P}_n = (p_{jk}^{(n)}) \ , \ p_{jk}^{(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos\left[\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})\right] \ .$$

By using the spectral decomposition $\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1} = \mathbf{P}_n\mathbf{D}_n\mathbf{P}_n^{'}$ and $\mathbf{D}_n$ is a diagonal matrix with the k-th element

$$d_k = 2[1-\cos(\pi(\frac{2k-1}{2n+1}))] \ (k=1,\cdots,n) \ .$$

Then the conditional log-likelihood function given the initial condition is proportional to

$$(3.8) \qquad L_n^{(SI)} = \sum_{k=1}^{n} \log|a_{kn}^{*}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x|^{-1/2} - \frac{1}{2}\sum_{k=1}^{n} \mathbf{z}_k^{'}[a_{kn}^{*}\mathbf{\Sigma}_v + \mathbf{\Sigma}_x]^{-1}\mathbf{z}_k \ ,$$

where

$$(3.9) \qquad a_{kn}^{*} \ (= d_k) \ = 4\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right] \ (k=1,\cdots,n) \ .$$

We have used the transformation $\mathbf{K}_n^{*}$ to the non-stationary time series $\mathbf{y}_i$ $(i = 1,\cdots,n)$ to the sequence of independent random vectors $\mathbf{z}_k$ $(k=1,\cdots,n)$, which follows $N_p(\mathbf{0}, \mathbf{\Sigma}_x + a_{kn}^{*}\mathbf{\Sigma}_v)$, and the coefficients a_{kn}^{*} is a dense sample of $4\sin^2(x)$ in $(0,\pi/2)$. [3]

Since we are dealing with an errors-in-variables model, there is an issue whether we can identify the structural equation of our interest. When $\mathbf{x}_i$ are i.i.d. random vectors, for instance, the coefficient parameters are not identified when we have the general variance-covariances for hidden variables and measurement errors without some further restrictions. In the classical homogeneous case, where the observed random vectors $\{\mathbf{y}_i\}$ are independent, there is no way to identify the covariance matrix of the hidden variables for instance. (See Anderson (1984) for the details of the classical errors-in-variables models.)

For the present case, we consider the conditional likelihood function when $p \geq 2$ and $q_x = 1$. We take a $p \times 1$(non-zero) vector $\mathbf{b}$ and apply the matrix formulae that for a $p \times p$ positive definite $\mathbf{A}$

$$|\mathbf{A} + \boldsymbol{b}\boldsymbol{b}^{'}| = |\mathbf{A}|[1 + \boldsymbol{b}^{'}\mathbf{A}^{-1}\boldsymbol{b}]$$

[3]We have used the notation $\mathbf{K}_n^{*}$ and a_{kn}^{*}, which are different from $\mathbf{K}$ and a_{kn} in Kunitomo and Sato (2013) and $\mathbf{K}_n = \sqrt{n}\mathbf{K}_n$, $a_{kn} = na_{kn}^{*}$.

and

$$[\mathbf{A}+\boldsymbol{b}\boldsymbol{b}^{'}]^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\boldsymbol{b}[1+\boldsymbol{b}^{'}\mathbf{A}^{-1}\boldsymbol{b}]^{-1}\boldsymbol{b}^{'}\mathbf{A}^{-1}$$

for $\mathbf{A}=a_{kn}^{*}\boldsymbol{\Sigma}_v$ $(k=1,\cdots,n)$, $\boldsymbol{\Sigma}_x=\mathbf{b}\mathbf{b}^{'}$, $\mathbf{b}=\sigma_{\mu}\boldsymbol{\pi}=\boldsymbol{\pi}^{*}$, ($\boldsymbol{\pi}$ is the same as $\boldsymbol{\Pi}$ except a vector) $\sigma_{\mu}^{2}=\mathcal{E}[(\Delta\boldsymbol{\mu}_i)^2]$, and $\mathbf{b}_{*}=\boldsymbol{\Sigma}_v^{-1}\mathbf{b}$.
Then $L_n^{(SI)}$ is proportional to (-1/2) times

$$\begin{aligned} L_{1n} &= \sum_{k=1}^{n}\left[\log|a_{kn}^{*}\boldsymbol{\Sigma}_v|+\log(1+a_{kn}^{*-1}\pi^{*'}\boldsymbol{\Sigma}_v^{-1}\pi^{*})+a_{kn}^{*-1}\mathbf{z}_k^{'}\boldsymbol{\Sigma}_v^{-1}\mathbf{z}_k-\frac{a_{kn}^{*-1}(\mathbf{z}_k^{'}\boldsymbol{\Sigma}_v^{-1}\pi^{*})^2}{a_{kn}^{*}+\pi^{*'}\boldsymbol{\Sigma}_v^{-1}\pi^{*}}\right] \\ &= \sum_{k=1}^{n}\log|a_{kn}^{*}\boldsymbol{\Sigma}_v|+\sum_{k=1}^{n}a_{kn}^{*-1}\mathbf{z}_k^{'}\boldsymbol{\Sigma}_v^{-1}\mathbf{z}_k+\sum_{k=1}^{n}\left[\log(1+a_{kn}^{*-1}c)-\frac{a_{kn}^{*-1}(\mathbf{z}_k^{'}\mathbf{b}_{*})^2}{a_{kn}^{*}+c}\right]\ , \end{aligned}$$

where we take $c=\pi^{*'}\boldsymbol{\Sigma}_v^{-1}\pi^{*}$ as a parametrization.
Then it may be a natural to consider the maximum likelihood (ML) estimation for the present errors-in-variables model. One of interesting aspects of the present problem is the fact that it is not a trivial task to maximize the (conditional) likelihood function. The detailed investigation of this problem requires many discussions and it will be given by Kunitomo, Awaya and Kurisu (2017) in a systematic way and here we give an illustration of Example 1 in Section 2.2. We set the true parameter values in Example 1 as $\sigma_{\mu}^{2}=0.4, \beta_2=1.0$ and

$$\boldsymbol{\Sigma}_v=\begin{pmatrix}0.45 & 0.23\\ 0.23 & 0.4\end{pmatrix}\ ,\ \boldsymbol{\Sigma}_x=\sigma_{\mu}^{2}\boldsymbol{\pi}\boldsymbol{\pi}^{'}\ ,\ \boldsymbol{\pi}=\begin{pmatrix}1\\ -\beta_2\end{pmatrix}.$$

Then we generate a set of simulated observations as a typical realization and we have drawn the Gaussian log-likelihood function with respect to β_2 in Figure 3.1 when the number of replications is $1,000$, given the true values for other parameters. We have found that the Gaussian log-likelihood function could have some peculiar form in some cases as illustrated by Figure 3.1. This may be one of important consequences in the non-stationary errors-in-variables models.

One may think that as an estimator of $\boldsymbol{\Sigma}_x$, we could use

$$\mathbf{S}_n=\frac{1}{n}\sum_{k=1}^{n}\mathbf{z}_k\mathbf{z}_k^{'}\ . \tag{3.10}$$

Because

$$\mathcal{E}[\mathbf{S}_n]=\boldsymbol{\Sigma}_x+(\frac{1}{n}\sum_{k=1}^{n}a_{kn}^{*})\boldsymbol{\Sigma}_v\ , \tag{3.11}$$

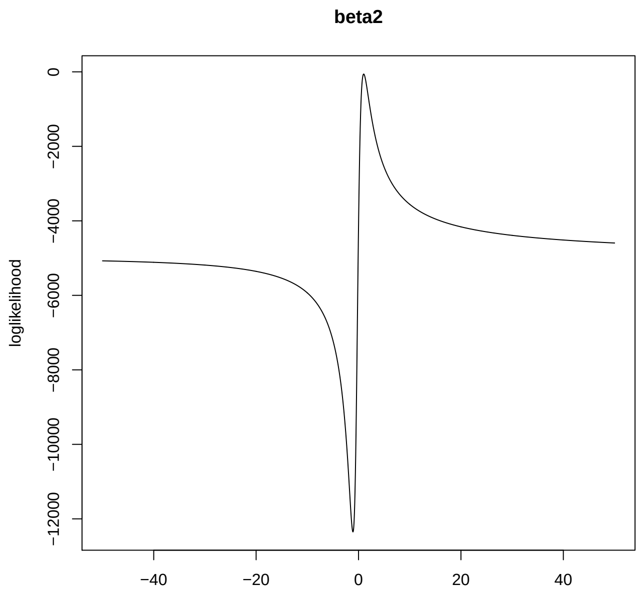


Figure 3.1 : Gaussian Log-Likelihood Function of β_2 ($n = 1,000$)

then $\mathbf{S}_n$ is not a consistent estimator of $\mathbf{\Sigma}_x$, and it is straight-forward to show that $(1/n)\sum_{k=1}^{n} a_{kn}^{*} \to 2$ as $n \to \infty$.

It is straight-forward to extend the above likelihood analysis to cases for more general q_x $(1 \leq q_x \leq p)$ and we have the corresponding results. It may not be obvious to find a general way to construct the consistent estimator of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ as well as the coefficients in the non-stationary errors-in-variable model.

4 Macro SIML estimation

Although we have considered the likelihood function in the errors-in-variables models under Gaussianity, we need a simple robust procedure, such that the assumptions of Gaussianity and the specifications of components are not crucial for the resulting estimation results.

We notice that $a_{kn}^{*} \to 0$ as $n \to \infty$ for a fixed k. When k is small, a_{kn}^{*} is small and we can expect that $k = k_n$ depending n is still small when n is large. However, $(1/m_n)\sum_{k=1}^{m_n} a_{kn}^{*}$ is not small if m_n is near to n, which suggests the condition $m_n/n \to 0$ as $n \to \infty$. The separating information maximum likelihood (SIML) estimator of $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ can be defined by

$$\hat{\mathbf{\Sigma}}_{x,SIML} = \frac{1}{m_n}\sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'} \ . \tag{4.1}$$

It is because

$$\mathcal{E}[\hat{\mathbf{\Sigma}}_{x,SIML}] = \mathbf{\Sigma}_x + [\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^{*}]\mathbf{\Sigma}_v \tag{4.2}$$

and the second term is $o(1)$ when $m_n/n \to 0$.

This estimator of the variance-covariance chooses the information in the frequency domain, which corresponds to the trend part from the time series observations. By the similar reason, we expect that it is possible to extract the information of seasonality, which we shall discussed in Section 5. For $\hat{\mathbf{\Sigma}}_x$, the number of terms m_n should be dependent on n. Then we need the order requirement that $m_n = O(n^{\alpha})$ and $0 < \alpha < 1$.

As the same reasoning as (4.2), we can utilize the conditions

$$\mathcal{E}[\mathbf{z}_k \mathbf{z}_k^{'}] = \mathbf{\Sigma}_x + o(1) \ \text{ for } \ k = 1, \cdots, m_n \tag{4.3}$$

and

$$(4.4) \qquad \mathcal{E}[a_{kn}^{*-1}\mathbf{z}_k\mathbf{z}_k^{'}] = \mathbf{\Sigma}_v + \frac{1}{4}\mathbf{\Sigma}_x + o(1) \text{ for } k = n+1-m_n,\cdots,n\ .$$

Then it is possible to construct consistent estimators of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_v$ by utilizing these relations.

Asymptotic properties of SIML

For the estimation of the variance-covariance matrix $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$, we have the next result and the proof will be given in Appendix A.

Theorem 4.1 : We assume (2.1)-(2.4) with $\mathbf{s}_i = \mathbf{0}$ and $\mathbf{x}_i = \mathbf{\Pi}\boldsymbol{\mu}_i$. The rank of non-zero $p \times q$ matrix $\mathbf{\Pi}$ is q_x ($1 \leq q_x \leq p$) and $\boldsymbol{\mu}_i$ are $q_x \times 1$ vectors with $\mathcal{E}(\boldsymbol{\mu}_i) = \mathbf{0}$ and $\mathcal{E}[(\Delta\boldsymbol{\mu}_i)(\Delta\boldsymbol{\mu}_i^{'})] = \mathbf{\Sigma}_\mu$, which is a $q_x \times q_x$ non-singular matrix. We also assume that $\mathbf{w}_i^{(x)} = (w_{ji}^{(x)})$ $\mathbf{e}_i = (e_{ji})$ are a sequence of independent random variables with $\mathcal{E}[w_{ig}^{(x)4}] < \infty$ and $\mathcal{E}[e_{ig}^4] < \infty$ $(i,j = 1,\cdots,n; g,h = 1,\cdots,p)$. We further assume that there exists ρ such that $0 \leq \rho < 1$ and $\|\mathbf{C}_j\| = O(\rho^j)$ in (2.4).
Then (i) For $m_n = [n^\alpha]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

$$(4.5) \qquad \hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}\ .$$

(ii) For $m_n = [n^\alpha]$ and $0 < \alpha < 0.8$, as $n \longrightarrow \infty$

$$(4.6) \qquad \sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{\mathcal{L}} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right)\ .$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g,h,k,l = 1,\cdots,p)$.

For estimating the variance-covariance matrix $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$,the number of terms m_n should be dependent on n because we need the resulting desirable asymptotic properties. Then we need the order requirement that $m_n = O(n^\alpha)$ $(0 < \alpha < 0.8)$. Because the properties of the SIML estimation method depend on the choice of m_n, which is dependent on n, we have investigated the asymptotic effects as well as the small sample effects with several choices of m_n. There is a trade-off between the bias and the asymptotic variance. For the macro-SIML, we can obtain an optimal choice of m_n.

Theorem 4.2 : In the setting of Theorem 4.1, an optimal choice of $m_n = [n^\alpha]$ $(0 <$

$\alpha < 1$) with respect to the asymptotic mean squared error when n is large is given by $\alpha^* = 0.8$.

It may be natural to use the sample quantities

$$\hat{\mathbf{\Sigma}}_x = (\frac{1}{m_n}\sum_{k=1}^{m_n} z_{ik}z_{jk}) \tag{4.7}$$

in order to make statistical inference on $\mathbf{\Sigma}_x$. For instance, the estimation of the Pearson's correlation coefficients among the trend variables is a typical case, which is given by

$$\hat{\rho}_{ij} = \frac{\sum_{k=1}^{m_n} z_{ik}z_{jk}}{\sqrt{\sum_{k=1}^{m_n} z_{ik}^2}\sqrt{\sum_{k=1}^{m_n} z_{jk}^2}} \ . \tag{4.8}$$

Furthermore, we consider the estimation of the structural relationships in the non-stationary time series process satisfying (2.5). Here we notice that the present statistical problem could be regarded as the estimation of structural relationships with the covariance matrix $\mathbf{\Sigma}_x(\boldsymbol{\theta})$ with $\boldsymbol{\theta}$ being the vector of parameters. In standard statistical multivariate analysis, Anderson (1984, 2004) has discussed statistical models of estimating structural relationships among a set of variables based on n independent observations.

We consider the estimation of the parameter vector $\boldsymbol{\beta}$ in the structural equation

$$\boldsymbol{\beta}^{'}\mathbf{y}_i = u_i \ , \tag{4.9}$$

where u_i is defined by $u_i = \boldsymbol{\beta}^{'}\mathbf{v}_i$) and $\mathbf{v}_i$ is given by (2.4). It is a simple case when $p \geq 2$ and $q_x = 1$. It may be natural to consider the characteristic equation

$$\left[\hat{\mathbf{\Sigma}}_x - \lambda\mathbf{\Sigma}_v\right]\hat{\boldsymbol{\beta}} = \mathbf{0} \ . \tag{4.10}$$

where $\hat{\mathbf{\Sigma}}_x$ is given by (4.7) and λ is the (scalar) characteristic root. Here we need to use a consistent estimator $\hat{\mathbf{\Sigma}}_v$ for $\mathbf{\Sigma}_v$. When we take the smallest eigenvalue λ_1 in (4.10) and $\hat{\mathbf{\Sigma}}_{v,SIML}$ in (4.7), we have the $\hat{\boldsymbol{\beta}}_{SIML}$, which is called the SIML estimator of $\boldsymbol{\beta}$.

Theorem 4.3 : In the setting Theorem 4.1 with its assumptions, we further assume

$q_x = p - 1$. Let $\hat{\boldsymbol{\beta}}$ be the characteristic vector with the corresponding minimum characteristic root of (4.10), which is the SIML estimator of $\boldsymbol{\beta}$. We further assume that we have a consistent estimator $\hat{\boldsymbol{\Sigma}}_v = \boldsymbol{\Sigma}_v + O_p(m_n^{-1/2})$.
Then for $m_n = [n^\alpha]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \xrightarrow{p} \mathbf{0} \; . \tag{4.11}$$

It is possible to derive the limiting distribution of $\hat{\boldsymbol{\beta}}_2$, but we need lenthy arguments and we have omitted them. Under a set of regularity conditions, we also find that the smallest eigenvalue λ_1 of (4.10),

$$\lambda_1 \longrightarrow 0 \text{ (in probability)} \tag{4.12}$$

as $n \to \infty$ because the rank of $\boldsymbol{\Sigma}_x$ is $p - 1$.
Then we define the SILS (Separating Information Least Squares) method by solving

$$\hat{\boldsymbol{\Sigma}}_x \hat{\boldsymbol{\beta}}_{SILS} = \mathbf{0} \; . \tag{4.13}$$

When $p = 2, q_x = 1$, $\boldsymbol{\beta} = (1, -\beta_2)'$, $\hat{\boldsymbol{\beta}}_{*,SIML} = (1, -\hat{\beta}_2)'$ and $\boldsymbol{\pi} = (\beta_2, 1)'$, then the SILS estimation becomes

$$\hat{\beta}_2 = \frac{\sum_{k=1}^{m_n} z_{1k} z_{2k}}{\sum_{k=1}^{m_n} z_{2k}^2} \; , \tag{4.14}$$

which is the regression coefficient of the first transformed variable on the second transformed variable in $\mathbf{z}_k$ $(= (z_{1k}, z_{2k})')$ $(k = 1, \cdots, m_n)$.

To construct a consistent estimator of $\boldsymbol{\Sigma}_v$, one might use (3.10). However, we notice the fact that

$$\mathbf{S}_n \xrightarrow{p} \boldsymbol{\Sigma}_x + 2\boldsymbol{\Sigma}_v \; . \tag{4.15}$$

Then we can construct a consistent estimator of $\boldsymbol{\Sigma}_v$ by using (4.1), (4.2), and the fact $\hat{\boldsymbol{\Sigma}}_{SIML,x} \xrightarrow{p} \boldsymbol{\Sigma}_x$.

Although we have developed the SIML estimation of a structural relationship in (4.9) when $q_x = 1$, it is straight-forward to extend the SIML procedure when we have several structural relationships among trend variables at the same time. The

SIML estimation can be defined by the smaller q_x ($\leq p$) roots and the corresponding q_x ($\leq p$) vectors of the characteristic equation. It may correspond to the standard situation in the statistical multivariate analysis except the fact that the classical multivariate analysis was based on the case when the observations are realizations of independent random variables without seasonality as well as non-stationarity in time series.

5 Discussions on Seasonality

We consider the estimation problem of seasonal factors and consider the general case when we have $\mathbf{y}_i = \mathbf{x}_i + \mathbf{s}_i + \mathbf{v}_i$ $(i = 1, \cdots, n)$, where $\mathbf{x}_i$ is a sequence of trend components, $\mathbf{s}_i$ is a sequence of seasonal components and $\mathbf{v}_i$ is a sequence of i.i.d. measurement error components. We transform the observed data using the difference operator $\Delta = 1 - \mathcal{L}$ $(\mathcal{L}\mathbf{y}_i = \mathbf{y}_{i-1})$ and $\mathbf{K}_n^*$ in (3.5). Then we can utilize the transformation

$$\mathbf{B}_n^{(3)} = (b_{jk}^{(3)}) = \mathbf{P}_n \mathbf{C}_n^{-2} \mathbf{C}_n^{(s)} \ , \tag{5.1}$$

where $\mathbf{C}_n^{(s)} = \mathbf{C}_N \otimes \mathbf{I}_s$,

$$\mathbf{C}_N^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{N \times N} , \tag{5.2}$$

and we have assumed that N, s (≥ 2) and $n = Ns$ are positive integers.
Then Lemma A-3 in Appendix gives

$$\sum_{j=1}^{n} b_{kj}^{(3)} b_{k^{'},j}^{(3)} = 4\delta(k, k^{'}) \frac{\sin^4 \left[\frac{\pi}{2} \frac{2k-1}{2n+1}\right]}{\sin^2 \left[\frac{\pi}{2} \frac{2k-1}{2n+1} s\right]} + O(\frac{1}{n}) \ . \tag{5.3}$$

By ignoring the correlations of $O(n^{-1})$, the criterion function in the general case, which extends the conditional log-likelihood function in Section 3, can be defined as

$$L_n^{(SI)} = \sum_{k=1}^{n} \log |a_{kn}^* \mathbf{\Sigma}_v + a_{kn}^{(s)} \mathbf{\Sigma}_s + \mathbf{\Sigma}_x|^{-1/2} - \frac{1}{2} \sum_{k=1}^{n} \mathbf{z}_k^{'} [a_{kn}^* \mathbf{\Sigma}_v + a_{kn}^{(s)} \mathbf{\Sigma}_s + \mathbf{\Sigma}_x]^{-1} \mathbf{z}_k \ , \tag{5.4}$$

where a_{kn}^* is given by (3.22) and

$$a_{kn}^{(s)} = 4 \frac{\sin^4 \left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right]}{\sin^2 \left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}s\right)\right]} \quad (k = 1, \cdots, n) \ . \tag{5.5}$$

For the estimation of the trend variance-covariance matrix we have the next result, which is a direct extension of Theorem 4.1.

Theorem 5.1 : In the setting of (2.1) with N, s, n $(= Ns)$ (positive integers), we assume the moment conditions on the seasonal components as $\mathcal{E}[w_{ig}^{(s)4}] < \infty$ in addition to the conditions of Theorem 4.1.
Let $\hat{\mathbf{\Sigma}}_x$ be given by (4.1).
Then (i) For $m_n = [n^{\alpha}]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O} \ . \tag{5.6}$$

(ii) For $m_n = [n^{\alpha}]$ and $0 < \alpha < 0.8$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{\mathcal{L}} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right) \ . \tag{5.7}$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g, h, k, l = 1, \cdots, p)$.

For the estimation of the seasonal variance-covariance matrix $\mathbf{\Sigma}_s = (\sigma_{gh}^{(s)})$ and $\hat{\mathbf{\Sigma}}_s = (\hat{\sigma}_{gh}^{(s)})$, we use

$$\hat{\mathbf{\Sigma}}_{s,SIML} = \frac{1}{m_n} \sum_{k \in \mathrm{I}_n^{(s)}} a_{kn}^{(s)-1} \mathbf{z}_k \mathbf{z}_k^{'} \ , \tag{5.8}$$

where s is the seasonal integer, $[x]$ is the largest integer being equal to or less than x and $\mathrm{I}_n^{(s)}$ is the set of integers such that $\mathrm{I}_{1n}^{(s)} = \{[2n/s]+1, \cdots, [2n/s]+m_n]\}$ with $m_n = [n^{\alpha}]$ $(0 < \alpha < 1)$.
Alternatively, $\mathrm{I}_{1n}^{(s)}$ can be replaced by a symmetric region
$\mathrm{I}_{2n}^{(s)} = \{[2n/s] - [m_n/2], \cdots, [2n/s], \cdots, [2n/s] + [m_n/2]]\}$.
In this formulation $[2n/s]$ corresponds to the seasonal frequency in the frequency domain of the observed time series. For the quarterly and monthly data, we take $s = 4$ and $s = 12$, respectively.

When we have the trend, seasonal, and stationary components, we have the relation

$$\mathcal{E}[\mathbf{z}_k \mathbf{z}_k^{'}] = \mathbf{\Sigma}_x + a_{kn}^{(s)}\mathbf{\Sigma}_s + a_{kn}^*\mathbf{\Sigma}_v \ . \tag{5.9}$$

Hence we have

$$\mathcal{E}[a_{kn}^{(s)-1}\mathbf{z}_i\mathbf{z}_i^{'}] = \mathbf{\Sigma}_s + a_{kn}^{(s)-1}\mathbf{\Sigma}_x + \frac{a_{kn}^{*}}{a_{kn}^{(s)}}\mathbf{\Sigma}_v \ . \tag{5.10}$$

Therefore, we have the next result.

Theorem 5.2 : In the setting of (2.1) we assume the moment conditions on the seasonal components as $\mathcal{E}[w_{ig}^{(s)4}] < \infty$ in addition to the conditions of Theorem 4.1. Let $\hat{\mathbf{\Sigma}}_s$ be given by (5.8) with $\mathrm{I}_{1n}^{(s)}$ or $\mathrm{I}_{2n}^{(s)}$.
Then (i) for $m_n = [n^{\alpha}]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_s - \mathbf{\Sigma}_s \xrightarrow{p} \mathbf{O} \ . \tag{5.11}$$

(ii) For $m_n = [n^{\alpha}]$ and $0 < \alpha < 0.8$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(s)} - \sigma_{gh}^{(s)}\right] \xrightarrow{\mathcal{L}} N\left(0, \sigma_{gg}^{(s)}\sigma_{hh}^{(s)} + \left[\sigma_{gh}^{(s)}\right]^2\right) \ . \tag{5.12}$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(s)} - \sigma_{gh}^{(s)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(s)} - \sigma_{kl}^{(s)}]$ is given by $\sigma_{gk}^{(s)}\sigma_{hl}^{(s)} + \sigma_{gl}^{(s)}\sigma_{hk}^{(s)}$ $(g, h, k, l = 1, \cdots, p)$.

Then it is possible to estimate the structural relationships of seasonal factors as we have discussed in Section 4. Also it is possible to construct a consistent estimator of $\mathbf{\Sigma}_v$ by utilizing the relation

$$\mathcal{E}[(\frac{1}{m})\sum_{k\in \mathrm{I}_{n/2}^{(s)}} a_{kn}^{*-1}\mathbf{z}_i\mathbf{z}_i^{'}] = \mathbf{\Sigma}_v + [\sum_{k\in \mathrm{I}_{n/2}^{(s)}} \frac{1}{a_{kn}^{*}}]\mathbf{\Sigma}_x + [\sum_{k\in \mathrm{I}_{n/2}^{(s)}} \frac{a_{kn}^{(s)}}{a_{kn}^{*}}]\mathbf{\Sigma}_s \ . \tag{5.13}$$

Alternatively, it has been a common practice to use the seasonal difference of original time series since Box and Jenkins (1970) if we observe clear seasonal fluctuations. When we transform the observed data by using the seasonal difference operator $\Delta_s = 1 - \mathcal{L}^s$ ($\mathcal{L}^s\mathbf{y}_i = \mathbf{y}_{i-s}$) and $\mathbf{P}_n$, we have

$$\Delta_s\mathbf{y}_i = (1 + \mathcal{L} + \cdots + \mathcal{L}^{s-1})\Delta\mathbf{x}_i + (1 - \mathcal{L}^s)\mathbf{s}_i + (1 - \mathcal{L}^s)\mathbf{v}_i \ . \tag{5.14}$$

Then there can be alternative possibilities of transformation of $\mathbf{Y}_n$; however, we may use $\mathbf{Z}_n^{(s)}$ $(= (\mathbf{z}_k^{(s)'}))$ by

$$\mathbf{Z}_n^{(s)} = \mathbf{P}_n\mathbf{C}_n^{(s)-1}\left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) \ , \tag{5.15}$$

where $\mathbf{C}_n^{(s)-1} = \mathbf{C}_N^{-1} \otimes \mathbf{I}_s$ and we have assumed that N, s and $n = Ns$ are positive integers.
When we use the transformation matrix

$$\mathbf{B}_n^{(1)} = (b_{jk}^{(1)}) = \mathbf{P}_n \mathbf{C}_n^{(s)-1} \ , \tag{5.16}$$

Lemma A-1 in Appendix gives

$$\sum_{j=1}^{n} b_{kj}^{(1)} b_{k',j}^{(1)} = \delta(k, k^{'}) 4 \sin^2 \left[\frac{\pi}{2} \frac{2k-1}{2n+1} s \right] + O(\frac{1}{n}) \ . \tag{5.17}$$

Then for the estimation of the seasonal covariance matrix $\mathbf{\Sigma}_s = (\sigma_{gh}^{(s)})$ and $\hat{\mathbf{\Sigma}}_s = (\hat{\sigma}_{gh}^{(s)})$, we may use

$$\hat{\mathbf{\Sigma}}_{s.BJ} = \frac{1}{m_n} \sum_{k \in I_n^{(s)}} \mathbf{z}_k^{(s)} \mathbf{z}_k^{(s)'} \ , \tag{5.18}$$

where s is the seasonal integer, $[x]$ is the largest integer being equal to or less than x and $I_n^{(s)}$ is the set of integers such that $I_n^{(s)} = \{[2n/s]+1, \cdots, [2n/s]+m_n]\}$ with $m_n = [n^{\alpha}]$ $(0 < \alpha < 1)$.
Then it is possible to obtain the similar results and

$$\hat{\mathbf{\Sigma}}_{s.BJ} - \mathbf{\Sigma}_s \xrightarrow{p} \mathbf{O} \ . \tag{5.19}$$

When we use (4.25) for the seasonally transformed data $\Delta_s \mathbf{y}_i$ $(i = 1, \cdots, n)$ in Theorem 5.2, however, its probability limit is given by

$$\hat{\mathbf{\Sigma}}_x \xrightarrow{p} s\mathbf{\Sigma}_x + \mathbf{\Sigma}_s \tag{5.20}$$

because the transformed trend component is given by

$$\Delta_s \mathbf{x}_i = (1 + \mathcal{L} + \cdots + \mathcal{L}^{s-1}) \mathbf{w}_i^{(x)} \ . \tag{5.21}$$

The bias can be significant when $s > 1$.

6 Simulations and an empirical example

In order to examine the finite sample properties of the procedure we have discussed in the previous sections, we have done several simulations. The data length is 80 in the basic case because our setting may be a reasonable approximation to many macroeconomic time series. (For the present GDP in Japan the various estimates of components are calculated from 1994 by the Cabinet office.) The number of simulations is 3,000, $\alpha = 0.6$, and $m_n = [n^\alpha]$ in each case. We have set three cases with the non-stationary trend and seasonality, whose typical simulation paths are given as Figures 6-1 to 6.3. We have done a number of simulations including the traditional linear seasonal models, and we report some results which may provide a reasonable description of economic quarterly data ($s = 4$). Since we deal with non-stationary seasonality, we need to control the parameter values carefully including the initial conditions. Figure 6-1 does not have any seasonality while Figures 6-2 and 6-3 have non-linear seasonality and represent rather extreme cases in our simulations.

In these simulations we first generated the initial uniform random variables $s_{j,-3}, \cdots, s_{j,0}$, the sequence of i.i.d. random variable $sv_{j,i}$ for $j = 1,2; i = 0,\cdots,n$. Then, we set $\mathbf{s}_i = (s_{1i}, s_{2i})'$ such that $sw_{j,i} = sw_{j,i-1} + sv_{j,i}$, $s_{j,i} = s_{j,i}^{(0)} \times sw_{j,i}$ and $s_{j,i}^{(0)} = s_{j,i-4}^{(0)}$ $(n \geq i \geq 4)$. We have summarized the four simulation results in Tables 6.1-6.4. In our tables $cor = 0.9$ means the true correlation coefficient among trend components and cor is the SIML estimate, where vol-1 is the correlation estimate based on the first differenced data and vol-4 is the correlation estimate based on the seasonal differenced data with $s = 4$.

When we have the basic model with trend and noise components and without the seasonal and cycle components, the optimal choice of $m_n = [n^\alpha]$ in an asymptotic sense would be $\alpha = 0.8$; however, it seems that the choice of $\alpha = 0.6$ would be appropriate to obtain robust results when we have finite samples with seasonality as well as non-stationary trends when $n = 80$. We have a tentative impression that $n = 80$ is a situation of small sample and we need a further investigation on the effects of small sample size.

Also we have investigated the estimation of the correlation coefficient of the seasonal components and given Table 6-4 when the seasonals were generated by $\mathbf{s}_i = (s_{1i}, s_{2i})'$ and $\mathbf{w}_i^{(s)} = (w_{1i}^{(s)}, w_{2i}^{(s)})'$ such that $s_{ji} = -s_{j,i-1} - s_{j,i-2} - s_{j,i-3} + w_{ji}^{(s)}$ $(i = 1,\cdots,n; j = 1,2)$ given the initial random variables, and we also have trend components and noise components (Simulation 4). The number of data was 400 and we took $\alpha = 0.4$ and we have given a typical sample path as Figure 6-4.

We have found that, even with the extreme cases given in our figures, the macro SIML method gives reasonable estimates, whereas in more standard cases we have more favorable results using the SIML estimation.

Table 6-1 : Simulation-1
($n = 80, \alpha = 0.6,$ nsim=3,000)

cor= 0.9	corr	vol-4	vol-1
mean	0.852	0.733	0.491
SD	0.088	0.076	0.095
cor= 0.0	corr	vol-4	vol-1
mean	0.007	0.003	0.001
SD	0.278	0.168	0.119

Table 6-2 : Simulation-2
($n = 80, \alpha = 0.6,$ nsim=3,000)

cor= 0.9	corr	vol-4	vol-1
mean	0.805	0.663	0.133
SD	0.118	0.088	0.295
cor= 0.0	corr	vol-4	vol-1
mean	-0.007	2.59E-03	0.005
SD	0.278	1.62E-01	0.287

Table 6-3 : Simulation-3
($n = 80, \alpha = 0.6,$ nsim=3,000)

cor= 0.9	corr	vol-4	vol-1
mean	0.672	0.344	0.034
SD	0.196	0.185	0.191
cor= 0.0	corr	vol-4	vol-1
mean	0.002	0.002	0.002
SD	0.284	0.149	0.184

Table 6-4 : Simulation-4
($n = 400, \alpha = 0.40,$ nsim=1,000)

cor= 0.8	corr	vol-4	vol-1
mean	0.7475	0.3358	0.7084
SD	0.1463	0.0699	0.2405

Finally, we report an empirical estimate of Japanese (real) GDP and fixed investment represented in Figure 1-1 as a typical example. We have used quarterly data which were taken from the official estimates from the Japanese Cabinet Office. When we take the first differences and the estimate of the correlation coefficient of the GDP-trend and investment-trend is 0.726176 while we take the seasonal difference and the estimate of the correlation coefficient of the GDP-trend and investment-trend is -0.12159. On the other hand, the SIML estimate of the correlation coefficient of the GDP-trend and investment-trend is 0.614224 (0.069623) while the SIML estimate of the correlation coefficient of the GDP-seasonal and investment-seasonal is 0.169324 (0.108598). We have used the symmetric region $\mathrm{I}_{2n}^{*}(s)$ and the parenthesis is the estimate of standard deviation calculated by the standard asymptotic formula in statistical multivariate analysis $(1-\hat{\rho}^2)/\sqrt{[m_n]}$. These estimates give some information on the statistical relationship between quarterly GDP and quarterly fixed-investment in Japan.

7 Concluding Remarks

In this study, we propose a new statistical method for estimating the statistical relationships in the non-stationary time series with trends, seasonality and noises. Instead of using seasonally adjusted data published by the official statistics agencies, we are proposing to use the separating information maximum likelihood (SIML) estimation, which can be regarded as a modification of the classical maximum likelihood (ML) method in some sense. We have pointed out that in the the standard approach of time series econometrics it is not easy to handle the measurement errors with non-stationary trends and seasonality as illustrated in Section 2.2 and instead we have used the additive decomposition of components. We have shown that the SIML estimator has reasonable asymptotic properties; that is, it is consistent and it has asymptotic normality when the sample size is large under reasonable conditions. The SIML estimator has reasonable finite sample properties and asymptotic robustness properties. We have also suggested a number of possible applications in macroeconomic non-stationary time series since many important macro time series exhibit clear trends and seasonality.

There are several possible extensions and related topics. First, it is interesting to incorporate the non-stationary components and stationary components in the multivariate time series decompositions. There has not been any computer program, which is free and public in this case, as DECOMP at ISM. Second, it may be straightforward to extend the cases when we have double unit roots in the trend variables. Third, as we indicated in Section 3, there is an interesting question on the merits and demerits of the ML method and the SIML method. Some results on this issue will be also reported in Kunitomo, Awaya, and Kurisu (2017) in details.

Finally, there is an important problem to determine the number of non-stationary trends and seasonal factors q_x. If we denote the numbers of seasonal components and stationary components as q_s and q_c, respectively, we also have the same problem. An obvious way is to use an information criterion as AIC under the Gaussian assumptions and the ML estimation (Akaike (1973)). Since there may be some doubts on the validity of the Gaussian assumptions and the ML estimation in practice as we have discussed in the previous sections, this problem is currently under investigation.

References

[1] Akaike, H. (1973), "Information Theory and an extension of the maximum likelihood principle," 2nd International Symposium on Information Theory, B.N. Petrov and F. Csaki edited, Academiai Kiado, Budapest, 267-281.

[2] Anderson, T.W. (1984), "Estimating Linear Statistical Relationships," *Annals of Statistics*, 12, 1-45.

[3] Anderson, T.W. (2003), *An Introduction to Statistical Multivariate Analysis*, 3rd Edition, John-Wiley.

[4] Box, G.E.P. and G. Jenkins (1970), *Time Series Analysis : Forecasting and Control*, Holden-Day, San Francisco.

[5] Engle, R. and C.W.J. Granger (1987), "Co-integration and Error Correction," *Econometrica*, Vol.55, 251-276.

[6] Findley, D., B.C.Monsell, W.R. Bell, M.C. Otto and B.C. Chen (1998), "New Capabilities and Methods of the X-12-ARIMA Seasonal Adjustment Program," *Journal of Business and Economic Statistics*, 16, 127-176(with Discussions).

[7] Hayashi, F. (2000), *Econometrics*, Princeton University Press.

[8] Fuller, W. (1987), *Measurement Error Models*, John-Wiley.

[9] Johansen, S. (1995), *Likelihood Based Inference in Cointegrated Vector Autoregressive Models*, Oxford UP.

[10] Kitagawa, G., and Gersch, W. (1984), "A smoothness priors-state space modeling of time series with trend and seasonality," *Journal of the American Statistical Association*, 79(386), 378-389.

[11] Kitagawa, G. (2010), *Introduction to Time Series Modeling*, CRA Press.

[12] Kunitomo, N. and S. Sato (2013), "Separating Information Maximum Likelihood Estimation of Realized Volatility and Covariance with Micro-Market Noise," *North American Journal of Economics and Finance*, Elsevier, 26, 282-309.

[13] Kunitomo, N., N. Awaya and D. Kurisu (2017), "Some Properties of Estimation Methods for Structural Relationships in Non-stationary Errors-in Variables Models," in preparation.

[14] Nerlove, M., D.M. Grether, and J. L. Varvalho (1995), *Analysis of Economics Time Series*, Academic Press.

APPENDIX : Mathematical Derivations

In this Appendix, we give some details of the proofs in Sections 4 and 5. Some of the proofs are are based on the extensions of the results by Kunitomo and Sato (2013) and thus there are similar features. However, there are important differences to which we shall mention explicitly at several places including several new lemmas.

Proof of Theorem 4.1 :
(**Step 1**) : Let $\mathbf{z}_k^{(x)} = (z_{kj}^{(x)})$ and $Z_k^{(v)} = (z_{kj}^{(v)})$ $(k = 1, \cdots, n)$ be the k-th row vector elements of $n \times p$ matrices

$$\text{(A.1)} \qquad \mathbf{Z}_n^{(x)} = \mathbf{K}_n^*(\mathbf{X}_n - \bar{\mathbf{Y}}_0) \ , \ \mathbf{Z}_n^{(v)} = \mathbf{K}_n^*\mathbf{V}_n \ , \ \mathbf{K}_n^* = \mathbf{P}_n\mathbf{C}_n^{-1} \ ,$$

respectively, where we denote $\mathbf{X}_n = (\mathbf{x}_k^{'}) = (x_{kg})$, $\mathbf{V}_n = (\mathbf{v}_k^{'}) = (v_{kg})$, $\mathbf{Z}_n = (\mathbf{z}_k^{'})$ $(= (z_{kg}))$ are $n \times p$ matrices with $z_{kg} = z_{kg}^{(x)} + z_{kg}^{(v)}$. We write $z_{kg}, z_{kg}^{(x)}, z_{kg}^{(v)}$ as the $g-$th component of $\mathbf{z}_k, \mathbf{z}_k^{(x)}, \mathbf{z}_k^{(v)}$ $(k = 1, \cdots, n; g = 1, \cdots, p)$.
We use the decomposition of $z_{kg}^{(f)}$ $(f = x, v)$ for investigating the asymptotic distribution of $\sqrt{m_n}[\hat{\Sigma}_x - \Sigma_x] = (\sqrt{m_n}(\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)})_{gh})$ for $g, h = 1, \cdots, p$. We use the decomposition

$$\text{(A.2)} \qquad \sqrt{m_n}\left[\hat{\Sigma}_x - \Sigma_x\right]$$
$$= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k\mathbf{z}_k^{'} - \Sigma_x\right]$$
$$= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'} - \Sigma_x\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}[\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'}]$$
$$+\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'} - \mathcal{E}[\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'}]\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{z}_k^{(x)}\mathbf{z}_k^{(v)'} + \mathbf{z}_k^{(v)}\mathbf{z}_k^{(x)'}\right] \ .$$

Then we will investigate the conditions that three terms except the first one of (A.2) are $o_p(1)$. When these conditions are satisfied, we could estimate the variance and covariance of the underlying processes consistently as if there were no noise terms because other terms can be ignored asymptotically as $n \to \infty$.
Let $\mathbf{b}_k = (b_{kj}) = \mathbf{e}_k^{'}\mathbf{P}_n\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0, \cdots, 1, 0, \cdots)$ be an $n \times 1$ vector. (We note that $b_{kj} = b_{kj}^{(1)}$ with $s = 1$ in Lemma A-1 below.) We write $z_{kg}^{(v)} = \sum_{j=1}^{n} b_{kj}v_{jg}$ for the noise part and use the relation

$$\text{(A.3)} \qquad (\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1}\mathbf{P}_n^{'})_{k,k'} = \delta(k, k^{'})4\sin^2[\frac{\pi}{2n+1}(k - \frac{1}{2})]$$

and $\sum_{j=1}^{n} b_{kj}b_{k'j} = \delta(k,k')a_{kn}^{*}$. We have

$$\mathbf{\Sigma}_v = (\sum_{j=-\infty}^{\infty} \mathbf{C}_j)\mathbf{\Sigma}_e(\sum_{j=-\infty}^{\infty} \mathbf{C}_j^{'}) ,$$

under the assumption that $\|\mathbf{C}_j| = O(\rho^j)$ $(0 \le \rho < 1)$ and then we can find K_1 (a constant) such that

$$\text{(A.4)} \qquad \mathcal{E}[(z_{kg}^{(v)})]^2 = \mathcal{E}[\sum_{i=1}^{n} b_{ki}v_{ig}\sum_{j=1}^{n} b_{kj}v_{jg}] \le K_1 \times a_{kn}^{*} .$$

It is because

$$\mathcal{E}[(z_{kg}^{(v)})]^2 = \sum_{i,j=1}^{n} b_{ki}b_{kj}\sigma_{gg}^{(v)}(i-j) ,$$

where $\sigma_{gg}^{(v)}(i-j)$ is the $(i-j)-$th auto-covariance of v_{ig} and v_{jg}. We denote $b_{ki}=0$ for $i<0$ and $i>n$ and then

$$\mathcal{E}[(z_{kg}^{(v)})]^2 = \sum_{l=-(n-1)}^{n-1} (\sum_{j=1}^{n} b_{kj}b_{k,j+l}\sigma_{gg}^{(v)}(l) \le [\sum_{j=1}^{n} b_{kj}^2]] \sum_{l=-\infty}^{\infty} |\sigma_{gg}^{(v)}(l)| .$$

Because $\|\mathbf{C}_j\| = O(\rho^j)$, $\sum_{l=-\infty}^{\infty} |\sigma_{gg}^{(v)}(l)|$ is bounded. Also from (3.9) it is straightforward to find that

$$\frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn}^{*} = \frac{1}{m_n}2\sum_{k=1}^{m_n}\left[1-\cos(\pi\frac{2k-1}{2n+1})\right] = O(\frac{m_n^2}{n^2}) ,$$

by using the relation

$$\sum_{k=1}^{m} 2\cos(\pi\frac{2k-1}{2n+1}) = \sum_{k=1}^{m}[e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})} + e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})}] = \frac{\sin(\frac{2\pi}{2n+1}m)}{\sin(\frac{\pi}{2n+1})}$$

and then the second term of (A.2) becomes

$$\text{(A.5)} \qquad \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}[z_{kg}^{(v)}]^2 \le K_1\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n} a_{kn}^{*} = O(\frac{m_n^{5/2}}{n^2}) ,$$

which is $o(1)$ if we set α such that $0<\alpha<0.8$.
(The arguments here are similar to the derivations in Kunitomo and Sato (2008, 2013), but there is a major difference on the conditions because there is no $\sqrt{n}$

factor in $\mathbf{P}_n$ and we use a^*_{kn} while they have used $a_{kn} = na^*_{kn}$.)
For the fourth term of (A.2),

$$
\begin{aligned}
\mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n} z_{kg}^{(x)} z_{kg}^{(v)}\right]^2 &= \frac{1}{m_n}\sum_{k,k'=1}^{m_n} \mathcal{E}\left[z_{kg}^{(x)} z_{k',g}^{(x)} z_{kg}^{(v)} z_{k',g}^{(v)}\right] \\
&= O(\frac{m_n^2}{n^2}) \ .
\end{aligned}
$$

In the above evaluation we have used the evaluation that if we set $s_{jk} = \cos[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})]$ $(j,k=1,2,\cdots,n)$, then we have the relation

$$
|\sum_{j=1}^{n} s_{jk}s_{j,k'}| \leq [\sum_{j=1}^{n} s_{jk}^2] = \frac{n}{2} + \frac{1}{4} \ \text{ for any } k \geq 1 \ .
$$

(See Lemma 3 of Kunitomo and Sato (2013) for instance.) For the third term of (A.2), we need to consider the variance of

$$
(z_{kg}^{(v)})^2 - \mathcal{E}[(z_{kg}^{(v)})^2] = \sum_{j,j'=1}^{n} b_{kj}b_{k,j'}\left[v_{jg}v_{j',g} - \mathcal{E}(v_{jg}v_{j',g})\right] \ .
$$

Then by using the assumption we made, after lengthy evaluations we can find a positive constant K_2 such that

$$
\begin{aligned}
& \mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}((z_{kg}^{(v)})^2 - \mathcal{E}[(z_{kg}^{(v)})^2])\right]^2 \\
= \ & \frac{1}{m_n}\sum_{k_1,k_2=1}^{m_n}\mathcal{E}\left[\sum_{j_1,j_2,j_3,j_4=1}^{n} b_{k_1,j_1}b_{k_1,j_2}(v_{j_1,g}v_{j_2,g} - \mathcal{E}(v_{j_1,g}v_{j_2,g}))\right. \\
& \qquad\qquad \times b_{k_2,j_3}b_{k_2,j_4}(v_{j_3,g}v_{j_4,g} - \mathcal{E}(v_{j_3,g}v_{j_4,g}))] \\
\leq \ & K_2\frac{1}{m_n}[\sum_{k=1}^{m_n} a^*_{kn}]^2 \\
= \ & O(\frac{1}{m_n} \times (\frac{m_n^3}{n^2})^2) \ ,
\end{aligned}
$$

which is $O(m_n^5/n^4)$. Here we just give an illustration of our derivations when $p=1$.

We need to evaluate

$$
\begin{aligned}
& \frac{1}{m}\sum_{k_1,k_2=1}^{m}\sum_{j_1,j_2,j_3,j_4} b_{k_1,j_1}b_{k_1,j_2}b_{k_2,j_3}b_{k_2,j_4}\mathcal{E}\{[v_{j_1,g}v_{j_2,g}-\mathcal{E}(v_{j_1,g}v_{j_2,g})][v_{j_3,g}v_{j_4,g}-\mathcal{E}(v_{j_3,g}v_{j_4,g})]\} \\
= & \frac{1}{m}\sum_{k_1,k_2=1}^{m}\sum_{j_1,j_2,j_3,j_4} b_{k_1,j_1}b_{k_1,j_2}b_{k_2,j_3}b_{k_2,j_4} \\
& \times \sum_{l_1,l_2,l_3,l_4=-\infty}^{\infty} c_{l_1}c_{l_2}c_{l_3}c_{l_4}\mathcal{E}\{[e_{j_1-l_1}e_{j_2-l_2}-\mathcal{E}(e_{j_1-l_1}e_{j_2-l_2})][e_{j_3-l_3}e_{j_4-l_4}-\mathcal{E}(e_{j_3-l_3}e_{j_4-l_4})]\}\ .
\end{aligned}
$$

Then we need to evaluate the corresponding terms for four cases when (i) $j_1-l_1=j_2-l_2=j_3-l_3=j_4-l_4$, (ii) $j_1-l_1=j_2-l_2\neq j_3-l_3=j_4-l_4$, (iii) $j_1-l_1=j_3-l_3\neq j_2-l_2=j_4-l_4$, (iv) $j_1-l_1=j_4-l_4\neq j_2-l_2=j_4-l_4$. For an instance, in Case (i) the corresponding terms are less than

$$
\begin{aligned}
& K_{21}(\frac{1}{m})\sum_{k_1,k_2=1}^{m}[\sum_{j_1=1}^{n} b_{k_1,j_1}^2]^{1/2}[\sum_{j_2=1}^{n} b_{k_1,j_2}^2]^{1/2}[\sum_{j_3=1}^{n} b_{k_2,j_3}^2]^{1/2}[\sum_{j_4=1}^{n} b_{k_2,j_4}^2]^{1/2} \\
& \times\sum_h[\sum_{j_1=1}^{n} c_{j_1-h}^2]^{1/2}[\sum_{j_2=1}^{n} c_{j_2-h}^2]^{1/2}[\sum_{j_3=1}^{n} c_{j_3-h}^2]^{1/2}[\sum_{j_4=1}^{n} c_{j_4-h}^2]^{1/2}\ ,
\end{aligned}
$$

where K_{21} is a positive constant. Because of the assumption $\|\mathbf{C}_j\|=O(\rho^j)$ with $0\leq\rho<1$ the last sum converges to a positive constant.
Hence the third term of (A.2) is negligible if we set α such that $0<\alpha<0.8$.

(**Step 2**) The second step is to give the asymptotic variance of the first term of (A.62), that is,

$$
\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)\prime}-\mathbf{\Sigma}_x\right] \tag{A.6}
$$

because it is of the order $O_p(1)$. We can write

$$
\begin{aligned}
& \frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)\prime} \\
= & \frac{1}{m_n}(\frac{2}{n+\frac{1}{2}})\sum_{k=1}^{m_n}[\sum_{i=1}^{n}\mathbf{r}_i\cos[\pi(\frac{2k-1}{2n+1})(i-\frac{1}{2})]\sum_{j=1}^{n}\mathbf{r}_j^{'}\cos[\pi(\frac{2k-1}{2n+1})(j-\frac{1}{2})]] \\
= & \sum_{i=1}^{n}c_{ii}^{*}\mathbf{r}_i\mathbf{r}_i^{'}+\sum_{i\neq j}c_{ij}^{*}\mathbf{r}_i\mathbf{r}_j\ ,
\end{aligned}
$$

where $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ and

$$\begin{aligned} c_{ii}^* &= (\frac{2}{2n+1})\left[1+\frac{1}{m}\frac{\sin 2\pi m(\frac{i-1/2}{2n+1})}{\sin(\pi\frac{i-1/2}{2n+1})}\right] , \\ c_{ij}^* &= \frac{1}{2m}(\frac{2}{2n+1})\left[\frac{\sin 2\pi m(\frac{i+j-1}{2n+1})}{\sin(\pi\frac{i+j-1}{2n+1})}+\frac{\sin 2\pi m(\frac{j-i}{2n+1})}{\sin(\pi\frac{j-i}{2n+1})}\right] \quad (i \neq j) . \end{aligned}$$

(We have used the notations c_{ii}^* and c_{ij}^* here instead of c_{ii} and c_{ij} in Kunitomo and Sato (2013), where $c_{ii} = nc_{ii}^*$ and $c_{ij} = nc_{ij}^*$ for $i,j = 1,\cdots,n$.) Then it is possble to show that

$$\text{(A.7)} \qquad \frac{\sqrt{m_n}}{n}\sum_{i=1}^{n}\left[\mathbf{r}_i\mathbf{r}_i^{'} - \mathbf{\Sigma}_x + (nc_{ii}^* - 1)\mathbf{r}_i\mathbf{r}_i^{'}\right] = o_p(1) .$$

Then we re-write (A.7) as

$$\text{(A.8)} \qquad \frac{\sqrt{m_n}}{n}\sum_{i=1}^{n}\left[nc_{ii}^*\ \mathbf{r}_i\mathbf{r}_i^{'} - \mathbf{\Sigma}_x\right] + \frac{\sqrt{m_n}}{n}\sum_{i\neq j}^{n}\left[nc_{ij}^*\ \mathbf{r}_i\mathbf{r}_j^{'}\right] .$$

After some albegra, we can evaluate the asymptotic variance of its second term. The variance of the limiting distribution of the (g,g)-the element of (A.8) is the limit of

$$\text{(A.9)} \qquad V_n(g,g) = 2\sum_{i,j=1}^{n}\frac{m_n}{n^2}[nc_{ij}^*]^2[\sigma_{gg}^{(x)}]^2 .$$

For $i,j = 1,\cdots,n$, we use the relation

$$c_{ij}^* = \frac{2}{m_n(n+\frac{1}{2})}\sum_{k=1}^{m}\cos\left[\frac{2\pi}{2n+1}(i-\frac{1}{2})(k-\frac{1}{2})\right]\cos\left[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})\right]$$

and as the result of lengthy but straightforward evaluations of trigonometric relations, we find that

$$\text{(A.10)} \qquad \sum_{i,j=1}^{n}[nc_{ij}^*]^2 = \frac{4}{m_n}\left[\frac{n}{2}+\frac{1}{4}\right]^2 .$$

Then as $n \to \infty$

$$\text{(A.11)} \qquad V_n(g,g) \longrightarrow V(g,g) = 2\left[\sigma_{gg}^{(x)}\right]^2 .$$

(**Step 3**) Finally, we need to give the proof of the asymptotic normality. Define the sequence of $\sigma-$fields $\mathcal{F}_{n,i}$ generated by the set of random variables $\{\mathbf{x}_j, \mathbf{v}_j; 1 \leq j \leq i \leq n\}$, for $(g,g)-$the element we shall use a sequence of random variables

$$U_n(g,g) = \sum_{j=2}^{n} [2 \sum_{i=1}^{j-1} \sqrt{m_n} c_{ij}^{*} r_{gi}] r_{gj} \ , \tag{A.12}$$

which is a discrete martingale and then we can apply the martingale central limit theorem. (In the present case the conditional variances r_{gj} $(j = 1, \cdots, n)$ are constant while they can be stochastic in Kunitomo and Sato (2013), and it is a considerable simplification.) Since the trend differences $r_{gi} = x_{gi} - x_{g,i-1}$ $(g = 1, \cdots, p; i = 1, \cdots, n)$ are also (discrete) martingales, we set
$X_{nj}(g,g) = (2\sum_{i=1}^{j-1} \sqrt{m_n} c_{ij}^{*} r_{gi}) r_{gj}$ $(j = 2, \cdots, n)$
and $V_{gg.n}^{*}(g,g) = \sum_{j=2}^{n} \mathcal{E}[X_{nj}^2 | \mathcal{F}_{n,j-1}]$.
Then in order to prove

$$U_n(g,g) = \sum_{i=1}^{n} X_{ni}(g,g) \xrightarrow{\mathcal{L}} N(0, V(g,g)) \tag{A.13}$$

we need to show the conditions (i) $\sum_{i=1}^{n} \mathcal{E}[X_{ni}(g,g)^2 | \mathcal{F}_{n,i-1}] \xrightarrow{p} V(g,g)$ and (ii) $\sum_{i=1}^{n} \mathcal{E}[X_{ni}(g,g)^2 I(|X_{ni}(g,g)| > \epsilon) | \mathcal{F}_{n,i-1}] \xrightarrow{p} 0$ (for any $\epsilon > 0$).
In the present situation, it is straightforward to show that these conditions are satisfied. (They have been given essentially in the proof of Theorem 3 in Kunitomo and Sato (2013) with detailed algebra.)
For the covariance of the trend term $\sigma_{sf}^{(x)}$ $(s, f = 1, \cdots, p)$, the arguments are quite similar, which are omitted here. By applying the martingale CLT, we obtain the corresponding result.
(**Q.E.D.**)

Proof of Theorem 4.2 : By the proof of Theorem 4.1, we have found that the main order of the bias of the SIML estimator is $m_n^{-1} \sum_{k=1}^{m_n} a_{kn} = O(n^{2\alpha-2})$. Since the normalization of the SIML estimator is in the form of $\sqrt{m_n}[\hat{\sigma}_{gg}^{(x)} - \sigma_{gg}^{(x)}] = O_p(1)$, its variance is of the order $O(n^{-\alpha})$. Hence when n is large we can approximate the mean squared error of $\hat{\sigma}_{gg}^{(x)}$ $(g = 1, \cdots, p)$ as

$$g_n(\alpha) = c_{1g} \frac{1}{n^{\alpha}} + c_{2g} n^{4\alpha-4} \ , \tag{A.14}$$

where c_{1g} and c_{2g} are some constants. The first term and the second term correspond to the order of the variance and the squared bias, respectively. By minimizing $g_n(\alpha)$

with respect to α, we obtain an optimal choice of m_n.
(**Q.E.D.**)

Proof of Theorem 4.3 : We consider the sample characteristic equation

$$\left[\hat{\boldsymbol{\Sigma}}_x - \lambda_1 \boldsymbol{\Sigma}_v\right] \hat{\boldsymbol{\beta}} = \mathbf{0} \ , \tag{A.15}$$

when λ_1 is the smallest root of the corresponding characteristic equation. By Theorem 4.1 we have

$$\hat{\boldsymbol{\Sigma}}_x \xrightarrow{p} \boldsymbol{\Sigma}_x \tag{A.16}$$

and we use

$$\boldsymbol{\beta}^{'} \left[\hat{\boldsymbol{\Sigma}}_x - \lambda_1 \boldsymbol{\Sigma}_v\right] \hat{\boldsymbol{\beta}} = 0 \ . \tag{A.17}$$

Then we find $\lambda_1 \xrightarrow{p} 0$ because λ_1 is the minimum root of the characteristic equation and the rank of $\boldsymbol{\Sigma}_x$ is less than p. Since $\boldsymbol{\Sigma}_v$ is a nonsingular matrix, we have the consistency of the SIML estimator.
(**Q.E.D.**)

For the proofs of Theorem 5.1 and Theorem 5.2, we give some preliminary lemmas, which are keys in our arguments.

Lemma A-1 : Let

$$\mathbf{B}_n^{(1)} = (b_{jk}^{(1)}) = \mathbf{P}_n \mathbf{C}_n^{(s)-1} \tag{A.18}$$

in (5.17). Then we have

$$\sum_{j=1}^{n} b_{kj}^{(1)} b_{k',j}^{(1)} = \delta(k,k^{'}) 4 \sin^2 \left[\frac{\pi}{2}\frac{2k-1}{2n+1}s\right] + O(\frac{1}{n}) \ . \tag{A.19}$$

Lemma A-2 : Let

$$\mathbf{B}_n^{(2)} = (b_{jk}^{(2)}) = \mathbf{P}_n \mathbf{C}_n^{(s)-1} \mathbf{C}_n \ . \tag{A.20}$$

Then we have

$$\sum_{j=1}^{n-s} b_{kj}^{(2)} b_{k',j}^{(2)} = \delta(k,k^{'}) \frac{\sin^2\left[\frac{\pi}{2}\frac{2k-1}{2n+1}s\right]}{\sin^2\left[\frac{\pi}{2}\frac{2k-1}{2n+1}\right]} + O(\frac{1}{n}) \ . \tag{A.21}$$

Lemma A-3 : Let $n = Ns, N$ and s be positive integers and

(A.22) $$\mathbf{B}_n^{(3)} = (b_{jk}^{(3)}) = \mathbf{P}_n \mathbf{C}_n^{-2} \mathbf{C}_n^{(s)} \ .$$

Then we have

(A.23) $$\sum_{j=1}^{n-s} b_{kj}^{(3)} b_{k',j}^{(3)} = \delta(k,k') 4 \frac{\sin^4 \left[\frac{\pi}{2}\frac{2k-1}{2n+1}\right]}{\sin^2 \left[\frac{\pi}{2}\frac{2k-1}{2n+1}s\right]} + O(\frac{1}{n}) \ .$$

Proof of Lemma A-1 : The proof is the result of lengthy, but straightforward calculations of the trigonometric functions. We set

(A.24) $$b_{kj}^{(1)} = p_{kj} - p_{k,j+s} \ \ (1 \le j \le n-s) \ ,$$

which can be written as

(A.25) $$\begin{aligned} b_{kj}^{(1)} &= \frac{1}{\sqrt{2n+1}}\{[1 - e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}] e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})} \\ &\quad + [1 - e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}] e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})}\} \ . \end{aligned}$$

Then we evaluate each terms of

$$\begin{aligned} \sum_{j=1}^{n-s} b_{kj}^{(1)} b_{k'j}^{(1)} &= \frac{1}{2n+1}\sum_{j=1}^{n-s}\{[A_{1j}(k) + A_{2j}(k)][A_{1j}(k') + A_{2j}(k')]\} \\ &= \frac{1}{2n+1}\sum_{j=1}^{n-s}\{A_{1j}(k)A_{1j}(k') + A_{2j}(k)A_{2j}(k') \\ &\quad + A_{1j}(k)A_{2j}(k') + A_{2j}(k)A_{2j}(k')\} \ , \end{aligned} \tag{A.26}$$

where we denote

$$A_{1j}(k) = (1 - e^{i\theta_k^s}) e^{i\theta_{k,j}} \ , A_{2j}(k) = (1 - e^{-i\theta_k^s}) e^{-i\theta_{k,j}} \ ,$$

and

$$\theta_k^s = \frac{2\pi}{2n+1}(k - \frac{1}{2})s, \ \theta_{k,j} = \frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2}) \ .$$

There are four terms in the summation of (A.26). For instance, the first term of (A.26) is given by

$$\begin{aligned} \sum_{j=1}^{n-s} A_{1j}(k)A_{1j}(k') &= (1 - e^{i\theta_k^s})(1 - e^{i\theta_{k'}^s}) \frac{1 - e^{i\frac{2\pi}{2n+1}(k+k'-1)(n-s+1)}}{1 - e^{i\frac{2\pi}{2n+1}(k+k'-1)}} \\ &\quad \times e^{i\frac{2\pi}{2n+1}(k+k'-1)\frac{1}{2}} \end{aligned}$$

and the third term of (A.26) is

$$\sum_{j=1}^{n-s} A_{1j}(k)A_{2j}(k^{'}) = (1-e^{i\theta_k^s})(1-e^{-i\theta_{k'}^s})\frac{1-e^{i\frac{2\pi}{2n+1}(k-k^{'})(n-s+1)}}{1-e^{i\frac{2\pi}{2n+1}(k-k^{'})}} \times e^{i\frac{2\pi}{2n+1}(k+k^{'}-1)\frac{1}{2}}$$

when $k \neq k^{'}$. When $k = k^{'}$, the third term of (A.26) becomes

$$\begin{aligned}\sum_{j=1}^{n-s} A_{1j}(k)A_{2j}(k^{'}) &= (n-s)(1-e^{i\theta_k^s})(1-e^{-i\theta_k^s}) \\ &= (n-s)(-1)[e^{-i\theta_k^s/2}-e^{i\theta_k^s/2}]^2 \\ &= 4(n-s)\sin^2[\frac{\theta_k^s}{2}] \ .\end{aligned} \tag{A.27}$$

Then by using similar calculations of the second and fourth terms and by summarizing four terms of (A.86), we have the desired result.
(**Q.E.D.**)

Proof of Lemma A-2 : The derivation of Lemma A-2 is similar to that of Lemma A-1. For $k = 1, \cdots, n; j = 1, \cdots, n-s+1$, we set

$$b_{kj}^{(2)} = p_{kj} + \cdots + p_{k,j+s-1} \ , \tag{A.28}$$

which can be written as

$$\begin{aligned}b_{kj}^{(2)} &= \frac{1}{\sqrt{2n+1}}\{\frac{1-e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}}{1-e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})}}e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})} \\ &\quad +\frac{1-e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}}{1-e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})}}e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})}\} \ .\end{aligned} \tag{A.29}$$

Then the rest of derivation is similar to that of Lemma A-1.
(**Q.E.D.**)

Proof of Lemma A-3 : The derivation of Lemma A-3 is similar to those of Lemmas A-1 and A-2. For $k = 1, \cdots, n; j = 1, \cdots, n-s-1$, we set

$$\begin{aligned}b_{kj}^{(3)} &= -[(p_{kj}-p_{k,j+1})-(p_{k,j+1}-p_{k,j+2})]+\cdots \\ &\quad +[(p_{k,(N-1)s}-p_{k,(N-1)s+1})-(p_{k,(N-1)s+1}-p_{k,(N-1)s+2})],\end{aligned} \tag{A.30}$$

which can be written as

$$
\text{(A.31)} \qquad b_{kj}^{(3)} = \frac{-1}{\sqrt{2n+1}}\{\frac{(1-e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})})^2}{1-e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}}e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})} + \frac{(1-e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})})^2}{1-e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})s}}e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})}\} .
$$

Then the rest of derivation is similar to those of Lemmas A-1 and A-2.
(**Q.E.D.**)

Proof of Theorem 5.1 : The proof of Theorem 5.1 is similar to that of Theorem 4.1 except the fact that we have used a different transformation of seasonal effects. Let $n = sN$ and N is an integer. (In the general case when $n = sN + j$ $(1 \leq j < s)$ we need some arguments, but the effects of additional terms $n = sN + j$ $(1 \leq j < s)$ are small.)

We set $\mathbf{z}_k^{(x)} = (z_{kg}^{(x)})$,$\mathbf{z}_k^{(v)} = (z_{kg}^{(v)})$ and $\mathbf{z}_k^{(s)} = (z_{kg}^{(g)})$, $(k = 1, \cdots, n; g = 1, \cdots, p)$ be the k-th vector elements of $n \times p$ matrix such that

$$
\mathbf{Z}_n^{(x)} = \mathbf{K}_n^*(\mathbf{X}_n - \bar{\mathbf{Y}}_0) \ , \ \mathbf{Z}_n^{(v)} = \mathbf{K}_n^*\mathbf{V}_n \ , \ \mathbf{Z}_n^{(s)} = \mathbf{K}_n^*\mathbf{S}_n \ , \ \mathbf{K}_n^* = \mathbf{P}_n\mathbf{C}^{-1} \ ,
$$

where $\mathbf{S}_n = (\mathbf{s}_i^{'}) = (s_{ig})$, $\mathbf{V}_n = (\mathbf{v}_i^{'})$ $(= (v_{ig}))$ and $\mathbf{Z}_n = (\mathbf{z}_k^{'})$ $(= (z_{kg}))$ are $n \times p$ matrices with $z_{kg} = z_{kg}^{(x)} + z_{kg}^{(v)} + z_{kg}^{(s)}$. Then we can write

$$
\mathbf{Z}_n^{(s)} = \mathbf{B}_n^{(3)}[\mathbf{C}_n^{(s)-1}\mathbf{C}_n\mathbf{S}_n]
$$

and we use the fact that $(1-\mathcal{L})^{-1}(1-\mathcal{L}^s)\mathbf{s}_i = \mathbf{w}_i^{(s)}$ and $\mathbf{w}_i^{(s)}$ are the sequence of i.i.d. random variables for $i = s, s+1, \cdots, n$ in (2.3), where we have set $\mathbf{B}_n^{(3)}$ in (A.23).

We denote $\mathbf{z}_k^{(x)} = z_{kg}^{(x)}$, $\mathbf{z}_k^{(s)} = z_{kg}^{(s)}$, and $\mathbf{z}_k^{(v)} = z_{kg}^{(v)}$. Then we have several additional terms in the decomposition of $\mathbf{z}_k$ $(k = 1, \cdots, m_n)$ as

$$
\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}(\mathbf{z}_k^{(s)}\mathbf{z}_k^{(s)'}), \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{z}_k^{(s)}\mathbf{z}_k^{(s)'} - \mathcal{E}(\mathbf{z}_k^{(s)}\mathbf{z}_k^{(s)'})\right],
$$

and

$$
\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}(\mathbf{z}_k^{(x)}\mathbf{z}_k^{(s)'} + \mathbf{z}_k^{(s)}\mathbf{z}_k^{(x)'}), \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}(\mathbf{z}_k^{(v)}\mathbf{z}_k^{(s)'} + \mathbf{z}_k^{(s)}\mathbf{z}_k^{(v)'}) \ .
$$

We need to show that these terms are stochastically negligible. The resulting evaluations are rather straightforward, but quite tedious. We illustrate a typical argument such that for any (non-zero) constant $p \times 1$ vector $\mathbf{a}$ and $\mathbf{b}$ we have

$$
\begin{aligned}
\mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{a}^{'}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(s)'}\mathbf{b}\right]^2 &\leq \frac{1}{m_n}\mathcal{E}\left[(\sum_{k=1}^{m_n}\mathbf{a}^{'}\mathbf{z}_k^{(x)})^2(\sum_{k=1}^{m_n}\mathbf{z}_k^{(s)'}\mathbf{b})^2\right] \\
&\leq \frac{1}{m_n}\mathcal{E}\left[(\sum_{k=1}^{m_n}\mathbf{a}^{'}\mathbf{z}_k^{(x)})^2\right]\mathcal{E}\left[(\sum_{k=1}^{m_n}\mathbf{z}_k^{(s)'}\mathbf{b})^2\right]
\end{aligned}
$$

because of the independence assumption of $\mathbf{z}_k^{(x)}$ and $\mathbf{z}_k^{(s)}$ $(k = 1, \cdots, m_n)$. Then by using Lemma A-3 it is possible to see the fact that this term and other extra terms due to the seasonality are of the smaller order $o_p(1)$ than constants. Since the evaluation of each terms are quite similar to the proof of Theorem 4.1, we omit some details.
(**Q.E.D.**)

Proof of Theorem 5.2 : The proof of Theorem 5.2 is similar to those of Theorem 4.1 and Theorem 5.1. Let $n = sN$ and N is an integer. (In the general case when $n = sN + j$ $(1 \leq j < s)$ we need some arguments, but the effects of additional terms $n = sN + j$ $(1 \leq j < s)$ are small.)

Let $\mathbf{z}_k^{(x)} = (z_{kg}^{(x)}), Z_k^{(v)} = (z_{kg}^{(v)})$ and $\mathbf{z}_k^{(s)} = (z_{kg}^{(s)})$ $(k = 1, \cdots, n; g = 1, \cdots, p)$ be the k-th vector elements of $n \times p$ matrices such that

$$
\mathbf{Z}_n^{(x)} = \mathbf{K}_n^*(\mathbf{X}_n - \bar{\mathbf{Y}}_0)\ ,\ \mathbf{Z}_n^{(v)} = \mathbf{K}_n^*\mathbf{V}_n\ , \mathbf{Z}_n^{(s)} = \mathbf{K}_n^*\mathbf{S}_n\ ,
$$

respectively, and $\mathbf{X}_n = (\mathbf{x}_k^{'}) = (x_{kg})$, $\mathbf{V}_n = (\mathbf{v}_k^{'}) = (v_{kg})$, $\mathbf{S}_n = (\mathbf{s}_k^{'})$ $(= (s_{kg}))$ $\mathbf{Z}_n = (\mathbf{z}_k^{'})$ $(= (z_{kg}))$ are $n \times p$ matrices with $z_{kg} = z_{kg}^{(x)} + z_{kg}^{(v)} + z_{kg}^{(s)}$. (We have written z_{kg} as the $g-$th component of $\mathbf{z}_k$.) Then we can write

$$
\mathbf{Z}_n^{(x)} = \mathbf{B}_n^{(3)-1}(\mathbf{X}_n - \bar{\mathbf{Y}}_n)\ ,\ \mathbf{Z}_n^{(v)} = \mathbf{B}_n^{(3)-1}\mathbf{B}^{(1)}\mathbf{V}_n\ .
$$

Next, we extend the decomposition in the present case as

$$
\begin{aligned}
&\sqrt{m_n}\left[\hat{\Sigma}_s - \Sigma_s\right] \\
= &\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k\in I_n(s)} \mathbf{z}_k\mathbf{z}_k^{'} - \Sigma_s\right] \\
= &\sqrt{m_n}\left[\frac{1}{m_n}\sum_{k\in I_n(s)} \mathbf{z}_k^{(s)}\mathbf{z}_k^{(s)'} - \Sigma_s\right] \\
&+\frac{1}{\sqrt{m_n}}\left[\sum_{k\in I_n(s)} \mathcal{E}(\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'}) + \sum_{k\in I_n(s)} \mathcal{E}(\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'})\right] \\
&+\frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)}\left[[\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'} - \mathcal{E}(\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'})] + [\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'} - \mathcal{E}(\mathbf{z}_k^{(v)}\mathbf{z}_k^{(v)'}])\right] \\
&+\frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)}\left(\mathbf{z}_k^{(s)}\mathbf{z}_k^{(x)'} + \mathbf{z}_k^{(x)}\mathbf{z}_k^{(s)'}\right) + \frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)}\left(\mathbf{z}_k^{(s)}\mathbf{z}_k^{(v)'} + \mathbf{z}_k^{(v)}\mathbf{z}_k^{(s)'}\right) \\
&+\frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)}\left(\mathbf{z}_k^{(x)}\mathbf{z}_k^{(v)'} + \mathbf{z}_k^{(v)}\mathbf{z}_k^{(x)'}\right) .
\end{aligned}
$$

In order to evaluate many terms, we use the relations of Lemmas A-1, A-2 and A-3. For instance, we can find a positive constant K_3 such that

$$
\mathcal{E}[(z_{ks}^{(v)})]^2 \leq K_3 \times a_{kn}^{(s)} , \tag{A.32}
$$

where

$$
a_{kn}^{(s)} = 4\sin^2[\frac{\pi}{2n+1}(k-\frac{1}{2})s] .
$$

Also we find that

$$
\frac{1}{m_n}\sum_{k\in I_n(s)} a_{kn}^{(s)} = \frac{1}{m_n}2\sum_{k\in I_n(s)}\left[1-\cos(\pi\frac{2k-1}{2n+1})s\right] = O(\frac{m_n^2}{n^2})
$$

and then the second term of the decomposition becomes

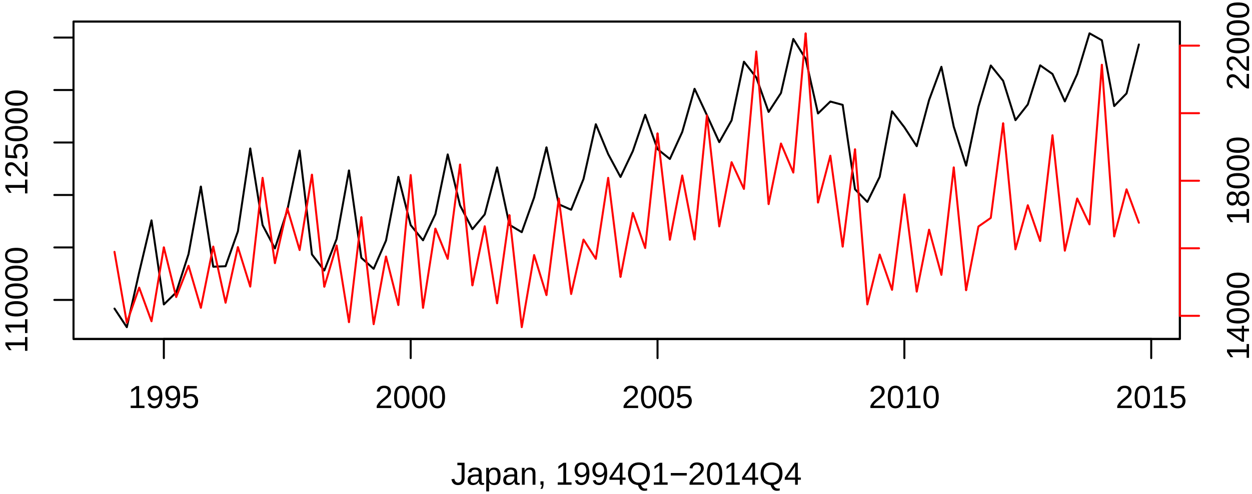
$$
\frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)} \mathcal{E}[z_{ks}^{(v)}]^2 \leq K_4\frac{1}{\sqrt{m_n}}\sum_{k\in I_n(s)} a_{kn}^{(s)} = O(\frac{m_n^{5/2}}{n^2}) , \tag{A.33}
$$

K_4 is a positive constant. This term is $o(1)$ if $0 < \alpha < 0.8$. The remaining arguments of the proof are quite similar to that of Theorem 4.1 and

$$
\begin{aligned}
\mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n}((z_{kg}^{(2)})^2-\mathcal{E}[(z_{kg}^{(2)})^2])\right]^2 &\le K_5\frac{1}{m_n}[\sum_{k=1}^{m_n}a_{kn}]^2 \\
&= O(\frac{1}{m_n}\times(\frac{m_n^3}{n^2})^2) ,
\end{aligned}
$$

where K_5 is a positive constant. Since the rest of arguments are quite similar to the proofs of Theorem 4.1 and Theorem 5.1, we omit some details.
(**Q.E.D.**)

Fig.1−1:Real GDP and Investment(red line)



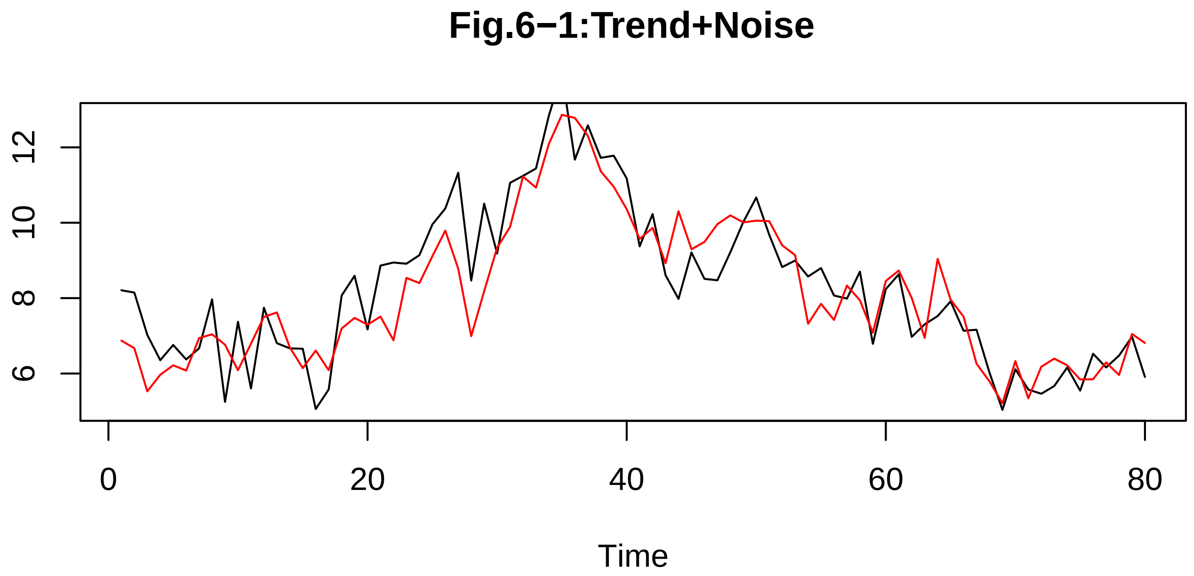


Fig.6−1:Trend+Noise
6
8
10
12
0
20
40
60
80
Time

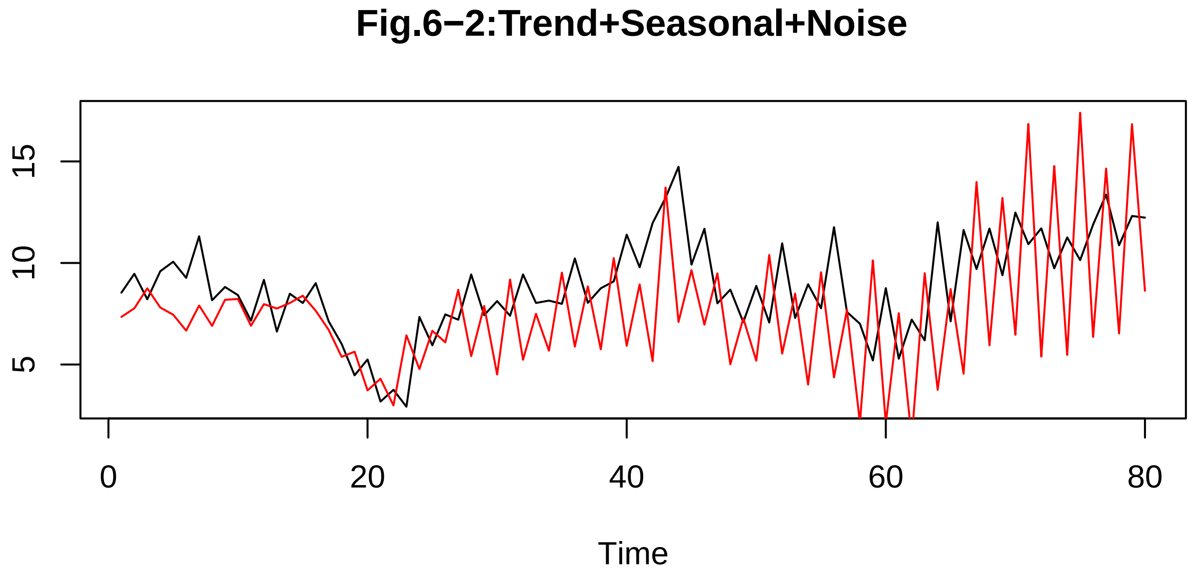


Fig.6−2:Trend+Seasonal+Noise
5
10
15
0
20
40
60
80
Time

Fig.6−3:Trend+Seasonal(irregular case)+Noise

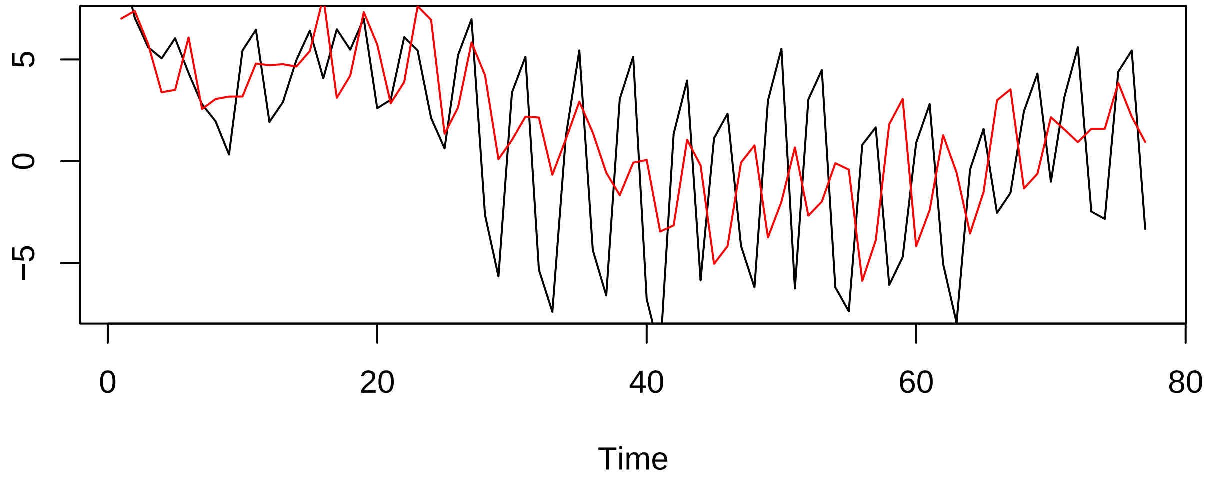


Fig.6−4:Trend+Seasonal+Noise(small)