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Naoto Kunitomo and Seisho Sato

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Local SIML Estimation of Some Brownian Functionals *

Naoto Kunitomo †
and
Seisho Sato ‡

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Abstract

To estimate Brownian functionals including the integrated volatility and co-volatility from high-frequency financial data, we introduce the local estimation of the integrated volatility and fourth-order variation of continuous-time diffusion process with micro-market noise. Although it is straight-forward to extend the realized volatility (RV) estimation to more general cases without micro-market noise, it may not be straight-forward to develop the estimation method of Brownian functionals in the presence of micro-market noise. To estimate higher-order Brownian functionals when we have micro-market noise, we develop the local SIML (LSIML) method, which is an extension of the separating information maximum likelihood (SIML) method proposed by Kunitomo, Sato and Kurisu (2018). The new method is simple and the LSIML estimator has some desirable asymptotic properties as well as reasonable finite sample properties.

Key Words

Integrated Volatility, Realized Volatility, Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), Higher-Order Brownian Functionals, Stable Convergence, Local SIML Estimation

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†The corresponding author, School of Political Science and Economics, Meiji University, Kanda-Surugadai 1-1 (Sarugakucho 3rd Building C-106), Chiyoda-ku, Tokyo 101-8301, naoto.kunitomo@gmail.com

‡Graduate School of Economics, University of Tokyo, Bunkyo-ku, Hongo7-3-1, Tokyo113-0033, seisho@e.u-tokyo.ac.jp

1. Introduction

In financial econometrics, several statistical methods have been proposed to estimate the integrated volatility and co-volatility from high-frequency data. The integrated volatility is one type of Brownian functionals and the realized volatility (RV) estimate has been often used when there does not exist any micro-market noise and the underlying diffusion process is directly observed. The asymptotic distribution of the RV estimator depends on the fourth-order integrated Brownian functional and then we need to estimate the fourth-order integrated moments to make statistical inference on the integrated volatility when the number of observations increases in a fixed interval. However, it has been known that the RV estimator is quite sensitive to the presence of micro-market noise in high-frequency financial data. Then several statistical methods have been proposed to estimate the integrated volatility and co-volatility. See Ait-Sahalia and Jacod (2014) for the detail of recent developments of financial econometrics.

When the micro-market noise cannot be ignored in high-frequency financial data, Knitomo, Sato and Kurisu (2018) have developed the separating information maximum likelihood (SIML) method for estimating the volatility and co-volatilities of security prices when the underlying processes are the class of diffusion processes. In this paper we extend the SIML method and develop the Local SIML (LSIML) estimation method for estimating higher-order Brownian functionals such as the fourth-order integrated moments, which is a new statistical method. The LSIML method was originally suggested in Chapter 8 of Kunitomo et al. (2018), but they did not give its detailed exposition. (To avoid the possible duplication of explanations on the SIML method, we will sometimes refer to the corresponding parts of Kunitomo, Sato and Kurisu (2018).) The main motivation for developing the LSIML method is to improve the SIML method and to estimate some Brownian functionals, which are general than the volatility and co-volatility. The fourth order integrated moments appear as the asymptotic variance of the limiting distribution of several estimation methods including the SIML estimation for instance. Because the main purpose of this paper it to propose the use of the LSIML method, we shall try to make our formulation not in the most general case, but concentrate on the simpler cases.

In this paper, we show that the local SIML method has some desirable asymptotic properties such as the consistency and asymptotic normality, and more importantly it may improve the asymptotic order of convergence. It also has reasonable finite sample properties, which are illustrated by several simulations. Since the LSIML method is a straightforward extension of the SIML estimation and it is quite simple, it will be useful in practical applications. Although there could be other methods for estimating higher-order Brownian functionals, the LSIML method has some merits such as its simplicity and desirable asymptotic properties.

In Section 2, we discuss the framework of estimation problem by using high-frequency financial data. In Section 3, we generalize the realized volatility and explain the method of local estimation in our study. Then in Section 4, we propose the local SIML method, which is a generalization of the SIML method originally developed by Kunitomo et. al (2018). In Section 5 we investigate the asymptotic properties of the local SIML method such as consistency as well as the asymptotic normality and in Section 6 we discuss the problem of choosing parameters needed in the LSIML estimation method. In Section 7 we give some finite sample properties of the LSIML estimation based on a set of Monte Carlo simulations. In Section 8, we discuss the possible generalizations of our results in more general setting and we give some concluding remarks in Section 9. Some mathematical details are given in the Appendix.

2. Estimation of Brownian Functionals

To see the essential feature of the local estimation method in this paper, we first consider the basic and simple time-varying cases in the univariate case when $p = 1$ (where p is the dimension). Let

$$Y(t_i^n) = X(t_i^n) + \epsilon_n v(t_i^n) \quad (i = 1, \cdots, n) \tag{2.1}$$

be the (one dimensional) observed (log-)price at t_i^n $(0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1)$ and $v(t_i^n)$ $(= v_i)$ be a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$ and $\mathbf{E}[v_i^2] = \sigma_v^2$ (> 0). We assume that

$$\epsilon_n = \frac{1}{n^\delta} , \tag{2.2}$$

where δ (≥ 0) is a constant. When $\delta = 0$, it is the micro-market noise model, while it is the high-frequency financial model without micro-market noise when $\delta = +\infty$. When $0 < \delta < +\infty$, it corresponds to the small-noise high-frequency model.

The underlying continuous-time Brownian martingale is given by

$$X(t) = X(0) + \int_0^t \sigma_s dB_s \ (0 \leq s \leq t \leq 1) , \tag{2.3}$$

which is independent of $v(t_i^n)$, σ_s is the (instantaneous) volatility function, which can be deterministic or stochastic time-varying (but it is bounded and continuous for the simplicity), and B_s is the standard Brownian motion.

Although it may be possible to apply the LSIML method to more general Itô semi-martingales, we first consider this situation because it gives the essential feature of the LSIML method in a simple way. (See Section 8 for its possible extensions.) We assume that when the volatility process is stochastic it has a representation of Ito's Brownian semi-martingale as

$$\sigma_t = \sigma_0 + \int_0^t \mu_s^\sigma ds + \int_0^t \omega_s^\sigma dB_s^\sigma \ (0 \leq s \leq t \leq 1) , \tag{2.4}$$

where μ_s^σ and ω_s^σ are the drift and diffusion coefficients (which are deterministic, bounded and continuous for the simplicity), and B_s^σ is another Brownian motion, which may be correlated with B_s. It has been known that many diffusion processes following the stochastic differential equations have such representation (see Ikeda and Watanabe (1989)).

When the micro-market noise is present in high-frequency financial data, some existing literatures such as Barndorff-Nielsen et al. (2008) for instance assume the independence of B_s and B_s^σ. This problem is related to the stochastic models for leverage effects in financial economics. We should note that on the LSIML estimation method in this paper at least, we do not need such assumption and the

Brownian motions can be correlated with the standard filtration.

The main problem of our interest is how to estimate Brownian functionals of the form

$$V(g,2r)=\int_0^1 g(s)\sigma_s^{2r}ds \tag{2.5}$$

for any positive integer r and a known function $g(s)$ from a set of observations of $Y(t_i^n)$ $(i=1,\cdots,n)$. We denote $V(2r)=V(g,2r)$ when $g(s)=1$ $(0\le s\le 1)$ for convenience.

There are important examples of this type of Brownian functionals. An obvious example is the integrated volatility that corresponds to the case when $r=1$.

Example 1 : When r=1, we have the integrated volatility, which is given by

$$V(2)=\int_0^1 \sigma_s^2 ds\ . \tag{2.6}$$

Example 2 : The asymptotic variance of the SIML estimator of integrated volatility $V(4)$ is given by

$$2V(4)=2\int_0^1 \sigma_s^4 ds\ . \tag{2.7}$$

It should be noted that the estimation of $V(4)$ with $r=2$ is a non-trivial task for which the SIML estimation cannot be used directly. Although we may have a known function $g(s)$ $(0\le s\le 1)$, it is straight-forward to treat this function in the estimation method we introduce in this paper. Zhang, L., Per A. Mykland, and Y. Ait-Sahalia (2005), Jacod, J., Y. L., Per A. Mykland, M. Podolskijc, and M. Vetter (2009), and Ait-Sahalia and Jacod (2014) discussed some estimation methods of higher-order Brownian functionals with different $g(s)$ functions, but it seems that they are more complicated than the method developed herein.

3. Local Estimation for the No-Micro-Market-noise Case

For simplicity, we take $t_j^n-t_{j-1}^n=1/n$ $(j=1,\cdots,n)$ and $t_0^n=0$. We divide $(0,1]$ into $b(n)$ sub-intervals and in every interval we allocate $c(n)^*$ observations. First, we consider the sequence $c^*(n)$ such that $c^*(n) \to \infty$ and we can take $b(n) \to \infty$ and $b(n) \sim n/c^*(n)$ as $n \to \infty$. A typical choice of observations in each interval would be $c^*(n) = [n^\gamma]$ $(0 < \gamma < 1)$, whereupon $b(n) \sim n^{1-\gamma}$. Because there are some extra observations (n may not be equal to $b(n)c^*(n)$) and $b(n)$ is a positive integer, we need to adjust the number of terms in each interval $c(n) = c^*(n) +$ (several terms). Although there can be finite sample effects, we will ignore the effects of extra terms in the following development because they are asymptotically negligible and hence we take $b(n)c(n) = n$.

When there does not have micro-market noise, we simply use the log-return process $r_j = y(t_j^n) - y(t_{j-1}^n)$ from the log-price process $y(t_j^n)$. We order the data r_j in each sub-intervals and denote $r_{k,(i)}$ $(k = 1, \cdots, c(n); i = 1, \cdots, b(n))$.
When $p = 1$, let the $2r-$th moment of $r_{k,(i)}$ in the i-th interval be

$$M^*_{2r,(i)} = \sum_{k=1}^{c(n)} [r_{k,(i)}]^{2r} \ . \tag{3.1}$$

Then we define the local realized moment (LRM) estimator of $V^*(2r)$ by

$$\hat{V}^*(2r) = \frac{n^{r-1}}{a_r} \sum_{i=1}^{b(n)} M^*_{2r,(i)} \tag{3.2}$$

where

$$a_r = \frac{2r!}{r!\ 2^r} \ . \tag{3.3}$$

When $r = 1$, it is the realized volatility (RV).
In this construction of the local realized moment (LRM) estimation, we need to normalize the sample moment due to the scale factor n^{r-1} and to use the local Gaussianity of underlying continuous martingales.

For the LRM estimator, we have the next result on the asymptotic properties, which could be obtained straight-forwardly by extending the standard arguments developed in the existing literature to the present case. (See Section 3.4 of Ait-Sahalia and Jacod (2014) on the standard arguments for instance.)

Proposition 1 : Assume that there is no micro-market noise, i.e. $\epsilon_n = 0$ with $p = 1$ and $r \geq 1$ in (2.1), (2.3) and (2.4). Also assume that $Y(t_i^n) = X(t_i^n)$ and

σ_s $(0 \leq s \leq 1)$ is bounded.

(i) As $n \longrightarrow \infty$

$$\hat{V}^*(2r) - V(2r) \xrightarrow{p} 0 \ . \tag{3.4}$$

(ii) As $n \longrightarrow \infty$

$$\sqrt{n}\left[\hat{V}^*(2r) - V(2r)\right] \xrightarrow{\mathcal{L}-s} N\left[0, W\right] \ , \tag{3.5}$$

where $\mathcal{L} - s$ means the stable convergence and

$$W = c_r^* \int_0^1 \left[\sigma_x(s)\right]^{4r} ds \ , \tag{3.6}$$

where c_r^* $(= a_{2r}/a_r^2 - 1)$ is a positive constant.

4. Local SIML Estimation

We consider the estimation problem of Brownian functionals when we have the micro-market noise as (2.1), (2.2) and (2.3). We utilize the localization of the estimation method in Section 3 and divide $(0, 1]$ into $b(n)$ sub-intervals and at every interval we allocate $c(n)$ observations. We consider the sequence $c^*(n)$ such that $c^*(n) \to \infty$ and we can take $b(n) \to \infty$ and $b(n) \sim n/c^*(n)$ as $n \to \infty$. We choose that the observations in each interval would be $c^*(n) = [n^\gamma]$ $(0 < \gamma < 1)$, whereupon $b(n) \sim n^{1-\gamma}$. Because there are some extra observations (n is not equal to $b(n)c^*(n)$) and $b(n)$ is a positive integer, we adjust the number of terms in each interval $c(n) = c^*(n) +$ (several terms) such that $n = b(n)c(n)$.

Then we apply the SIML method developed by Kunitomo et. al (2018) to each sub-intervals. To use the SIML transformation in each local interval, we set $m_c = [c(n)^\alpha]$ $(0 < \alpha < 0.5)$, in the i-th interval $(i = 1, \cdots, b(n))$ and the transformed data are denoted as $z_k(i)$ as the k-th data in the i-th interval $I_c(i)$ $(k = 1, \cdots, c(n); i = 1, \cdots, b(n))$. Now we explain the procedure for the general case when $p \geq 1$. Here we follow the notations in Chapter 3 of Kunitomo et al. (2018) for the $p-$dimensional stochastic process $\mathbf{y}(t_i^n)$ and in each sub-intervals we transform $c(n) \times p$ observation matrix $\mathbf{Y}_{c(n),(i)}$ to $c(n) \times p$ matrix $\mathbf{Z}_{n,(i)}$ $(= (\mathbf{z}^{'}_{k,(i)}))$ $(i = 1, \cdots, b(n))$ by

$$\mathbf{Z}_{c(n),(i)} = h_{c(n)}^{-1/2} \mathbf{P}_{c(n)} \mathbf{C}_{c(n)}^{-1} \left(\mathbf{Y}_{c(n),(i)} - \bar{\mathbf{Y}}_{0,(i)} \right) \tag{4.1}$$

where $h_{c(n)} = 1/c(n)$, $c(n) \times c(n)$ matrices

$$\mathbf{C}_{c(n)}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} , \tag{4.2}$$

$$\mathbf{P}_{c(n)} = (p_{jk}) \ , \ p_{jk} = \sqrt{\frac{2}{c(n) + \frac{1}{2}}} \cos \left[\frac{2\pi}{2c(n)+1} (k - \frac{1}{2})(j - \frac{1}{2}) \right] . \tag{4.3}$$

The initial conditions are given by the $p \times 1$ vector $\mathbf{y}_{0,(i)}$ and

$$\bar{\mathbf{Y}}_{0,(i)} = \mathbf{1}_{c(n)} \cdot \mathbf{y}^{'}_{0,(i)} . \tag{4.4}$$

Then we have the spectral decomposition

$$\mathbf{C}_{c(n)}^{-1} \mathbf{C}_{c(n)}^{'-1} = \mathbf{P}_{c(n)} \mathbf{D}_{c(n)} \mathbf{P}^{'}_{c(n)} , \tag{4.5}$$

where $\mathbf{D}_{c(n)}$ is a diagonal matrix with the k-th element $d_k = 2\left[1 - \cos(\pi(\frac{2k-1}{2c(n)+1}))\right]$ $(k = 1, \cdots, c(n))$. We define

$$a_{k,c(n)} = c(n) d_k = 4c(n) \sin^2 \left[\frac{\pi}{2} \left(\frac{2k-1}{2c(n)+1} \right) \right] \ (k = 1, \cdots, n) . \tag{4.6}$$

When $p = 1$ and for any positive integer r, let the $2r-$th moment in the i-th sub-interval be

$$M_{2r,(i)}^{n} = \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}]^{2r} . \tag{4.7}$$

Then we define the LSIML estimator of $V(2r)$ by

$$\hat{V}(2r) = \frac{b(n)^{r-1}}{a_r} \sum_{i=1}^{b(n)} M_{2r,(i)}^{n} \tag{4.8}$$

where

$$a_r = \frac{2r!}{r! \ 2^r} . \tag{4.9}$$

In particular, $a_1 = 1, a_2 = 3$ and $a_3 = 15$.

If we take $c(n) = n, b(n) = 1$ and $r = 1$, then we have the SIML estimator for integrated volatility.

In this construction of the LSIML estimator, we need to normalize (4.8) due to the fact that the scale factor $1/c(n)$ and we have the local Gaussianity for underlying continuous martingales.

5. Asymptotic Properties of Local SIML

We consider the case when σ_s is a time-varying bounded function when $p = 1$ in this section. First, we illustrate the derivations of asymptotic properties of the Local-SIML (LSIML) estimator for the cases of $r = 1$ and $r = 2$.

(i) The case of $r = 1$

First, we consider the asymptotic behavior of the quantity $(1/m_c)\sum_{k=1}^{m_c} z_{k,(i)}^2$. in the i-th interval $I_c(i) = ((i-1)\frac{c(n)}{n}, i\frac{c(n)}{n}]$ $(i = 1, \cdots, b(n))$, where we take $n = b(n)c(n)$. By using the analogous arguments as Chapter 5 of Kunitomo et al. (2018) (and (A.4) in the Appendix of this paper) to the local interval $I_c(i)$ $(i = 1, \cdots, b(n))$, we need to evaluate the stochastic part as

$$\sqrt{m_c}\sum_{k,l=1}^{c(n)}[c_{kl}r_{k,(i)}r_{l,(i)} - \delta(k,l)\int_{t_{k-1}^n}^{t_k^n}\sigma_s^2 ds](\frac{n}{c(n)}) = O_p(1)\ , \tag{5.1}$$

where $r_{k,(i)}$ are hidden returns in the interval $(t_{k-1}^n, t_k^n] \in I_c(i)$, $t_k^n - t_{k-1}^n = 1/n$, $c_{kl} = (2/m_c)\sum_{j=1}^{m_c} s_{kj}s_{lj}$, and $s_{jk} = \cos\left[\frac{2\pi}{2c(n)+1}(j-\frac{1}{2})(k-\frac{1}{2})\right]$.

By using Lemma A-3 given in the Appendix (see Chapter 5 of Kunitomo et al. (2018) also), $(1/m_c)\sum_{k=1}^{m_c} z_{k,(i)}^2 \sim O_p((\pi^2/3)m_c^2/c(n))$ because when $c(n)$ is large the bias term of $\sum_{i=1}^{b(n)}[1/m_c]\sum_{k=1}^{m_c} z_{k,(i)}^2$ in each interval is proportional to

$$\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)} = O(\frac{1}{m_c}\times\frac{m_c^3}{c(n)}) = O(\frac{m_c^2}{c(n)})\ . \tag{5.2}$$

The bias term can be written as asymptotically

$$\mathrm{AB}_n = b(n)\frac{\pi^2}{3}\frac{(m_c)^2}{c(n)}[\epsilon_n]^2\sigma_v^2 \tag{5.3}$$

Because the normalizing factor of (5.1) is $\sqrt{m_c}b(n)$ and then we find that

$$\mathbf{Var}\left[\frac{1}{m_c}\sum_{k=1}^{m_c} z_{k,(i)}^2 - \int_{s\in((i-1)\frac{c(n)}{n},i\frac{c(n)}{n}]} \sigma_s^2 ds\right] = O(\frac{1}{m_c b(n)^2}) \tag{5.4}$$

and we can take a positive constant K_1 if we assume that σ_s is a continuous function of time such that

$$\begin{aligned}\left|\int_{s\in((i-1)\frac{c(n)}{n},i\frac{c(n)}{n}]} \sigma_s^2 ds - \sigma_{(i-1)\frac{c(n)}{n}}^2[\frac{c(n)}{n}]\right| &\leq K_1\left|\int_{s\in((i-1)\frac{c(n)}{n},i\frac{c(n)}{n}]} [s-(i-1)\frac{c(n)}{n}]ds\right| \\ &= O_p\left((\frac{c(n)}{n})^2\right) ,\end{aligned}$$

which is $1/b(n)^2$. Then we have the relation that

$$\left[\sum_{i=1}^{b(n)} \frac{1}{m_c}\sum_{k=1}^{m_c} z_{k,(i)}^2\right] - \int_0^1 \sigma_s^2 ds \xrightarrow{p} 0 , \tag{5.5}$$

provided that $\max\{\frac{1}{b(n)},\frac{1}{m_c}\} \longrightarrow 0$ and

$$b(n)\frac{(m_c)^2}{c(n)}[\epsilon_n]^2 \longrightarrow 0 \tag{5.6}$$

as $n \to \infty$.

For the asymptotic normality of $\hat{V}(2)$ without any asymptotic bias term, we use the fact that the dominant factor of (5.1) is a martingale part (see (A.11) in the Appendix). A sufficient condition for the asymptotic normality (See Theorem 3.3 of Kunitomo et al. (2018)) would be

$$\sqrt{m_c b(n)}\, b(n)\frac{(m_c)^2}{c(n)}[\epsilon_n]^2 \longrightarrow 0 . \tag{5.7}$$

If we set $c(n) = n^{\gamma}$, $b(n) = n^{1-\gamma}$ and $m_c = [c(n)]^{\alpha}$, then

$$b(n)\frac{(m_c)^2}{c(n)}[\epsilon_n]^2 = n^{1-2\gamma+2\gamma\alpha-2\delta} , \tag{5.8}$$

and

$$\sqrt{m_c b(n)}\, b(n)\frac{(m_c)^2}{c(n)}[\epsilon_n]^2 = n^{\frac{1-\gamma}{2}+\frac{\alpha\gamma}{2}+1-2\gamma+2\gamma\alpha-2\delta} = n^{1-2\delta-2\gamma+\frac{5}{2}\alpha\gamma} . \tag{5.9}$$

By setting $\alpha_1^* = [2\gamma+2\delta-1]/[2\gamma]$, and $\alpha_2^* = [\frac{5}{2}\gamma+2\delta-\frac{3}{3}]/[\frac{5}{2}\gamma]$, we summarize the result on the asymptotic distribution of the local SIML estimation in the simplest case. It is indeed a corollary of Theorem 3 below, but we present the result for the case of $r = 1$, which may be useful to understand the more general cases. In the Appendix, we give some remarks on the derivations of Proposition 2 and the stable convergence used.

Proposition 2 : When $r = 1$ and $p = 1$ in (2.1), (2.2), (2.3) and (2.4). Also assume that $v(t_i^n)$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$, $\mathbf{E}[v_i^4] < +\infty$, σ_s $(0 \le s \le 1)$ is bounded, and $\alpha_1^* > 0$, $\alpha_2^* > 0$.
Then we have the following asymptotic properties of the LSIML estimator.
(i) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.5, \alpha_1^*\}$, as $n \longrightarrow \infty$

$$\hat{V}(2) - V(2) \xrightarrow{p} 0 \ . \tag{5.10}$$

(ii) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.4, \alpha_1^*\}$, as $n \longrightarrow \infty$

$$\sqrt{m_c b(n)} \left[\hat{V}(2) - V(2)\right] \xrightarrow{\mathcal{L}-s} N\left[0, W\right] \tag{5.11}$$

in the stable convergence sense, where

$$W = 2 \int_0^1 \left[\sigma_x(s)\right]^4 ds \ . \tag{5.12}$$

If we take $\delta = 0.0$ and $\gamma = 2/3$, then the above condition for consistency implies $0 < \alpha < 1/4$.

(ii) The case when $r = 2$

It may be straight-forward to extend the above arguments in (i) for the case of $r = 1$ to the general cases of $r \ge 1$. When $r = 2$, the evaluation of associated terms becomes more complicated because the main signal part of $(1/m_c)\sum_{k=1}^{m_c} z_k^4$ is given by

$$\sum_{j_1,j_2,j_3,j_4=1}^{c(n)} [(\frac{4}{\sqrt{m_c}} \sum_{k=1}^{m_c} s_{k,j_1} s_{k,j_2} s_{k,j_3} s_{k,j_4}] r_{j_1} r_{j_2} r_{j_3} r_{j_4} \ ,$$

where $s_{jk} = \cos\theta_{jk}$ and $\theta_{jk} = \frac{2\pi}{2c(n)+1}(j - \frac{1}{2})(k - \frac{1}{2})$ $(j = 1, \cdots, c(n); k = 1, \cdots, m_c)$.
There are many terms for its asymptotic variance such as

$$\sum_{i_1=i_2\neq i_3=i_4} \sum_{j_1=j_2\neq j_3=j_4} [(\frac{4}{m_c} \sum_{k=1}^{m_c} s_{k,i_1}^2 s_{k,i_3}^2][(\frac{4}{m_c} \sum_{k'=1}^{m_c} s_{k',j_1}^2 s_{k',j_3}^2] \mathbf{E}(r_{i_1}^2)\mathbf{E}(r_{i_3}^2)\mathbf{E}(r_{j_1}^2)\mathbf{E}(r_{j_3}^2) \ .$$

Then the main bias term comes from the main noise part and in the interval $I_c(i) = ((i-1)\frac{c(n)}{n}, i\frac{c(n)}{n}]$ $(i = 1, \cdots, b(n))$, it becomes

$$\frac{1}{m_c} \sum_{k=1}^{m_c} \sum_{j_1,j_2,j_3,j_4=1}^{c(n)} b_{k,j_1} b_{k,j_2} b_{k,j_3} b_{k,j_4} \mathbf{E}[v_{j_1} v_{j_2} v_{j_3} v_{j_4}] ,$$

where $b_{kj} = (h_{c(n)}^{-1/2} \mathbf{P}_{c(n)} \mathbf{C}_{c(n)}^{-1})_{kj}$ $(k, j = 1, \cdots, c(n))$.
Hence when $r = 2$, we have the typical bias term $[m_c^4/c(n)^2]$ in each sub-interval because

$$\frac{1}{m_c} \sum_{i_1=j_1 \neq i_2=j_2} b_{k,i_1}^2 b_{k,i_2}^2 = \frac{1}{m_c} \sum_{k=1}^{m_c} a_{k,c(n)}^2 = O(\frac{m_c^4}{c(n)^2}) , \tag{5.13}$$

where

$$a_{k,c(n)} = 4c(n) \sin^2 \left[\frac{\pi}{2} \frac{2k-1}{2c(n)+1} \right] . \tag{5.14}$$

Then the condition for consistency of the LSIML estimator becomes

$$b(n)[b(n) \frac{(m_c)^4}{c(n)^2}][\epsilon_n]^4 \longrightarrow 0 . \tag{5.15}$$

If we set $c(n) = n^\gamma$, $b(n) = n^{1-\gamma}$ and $m_c = [c(n)]^\alpha$, then

$$b(n)^2 [\frac{(m_c)^4}{c(n)^2}][\epsilon_n]^4 = n^{2(1-\gamma)-2\gamma+4\gamma\alpha-4\delta} = n^{2[1-2\gamma+2\gamma\alpha-2\delta]} . \tag{5.16}$$

The condition for the asymptotic normality without bias becomes

$$\sqrt{m_c b(n)} b(n)^2 [\frac{(m_c)^4}{c(n)^2}][\epsilon_n]^4 = n^{\frac{1-\gamma+\alpha\gamma}{2} + 2(1-2\gamma)+4\gamma\alpha-4\delta} . \tag{5.17}$$

(iii) The general case when $r \geq 1$

It is straight-forward to extend the results for $p = 1$ and $r = 2$ to more general cases. For instance, the condition for the bias term becomes

$$b(n)^{r-1} \times b(n)[\frac{(m_c)^{2r}}{c(n)^r}][\epsilon_n]^{2r} = n^{r(1-\gamma)-r\gamma+2r\gamma\alpha-2r\delta} \tag{5.18}$$

and

$$\sqrt{m_c b(n)} b(n)^r [\frac{(m_c)^{2r}}{c(n)^r}][\epsilon_n]^{2r} = n^{\frac{1-\gamma+\alpha\gamma}{2} + r(1-2\gamma)+2r\gamma\alpha-2r\delta} = n^{\frac{1}{2}-\frac{\gamma}{2}+r(1-2\gamma)-2r\delta+\alpha[(2r+\frac{1}{2})\gamma]} . \tag{5.19}$$

(See Lemma A-1 in Appendix.) We have a generalization of Propositions 1 and 2 when $r \geq 1$ and $p = 1$ as follows, which is the summary of the asymptotic properties of the local SIML estimation in this paper. (We give some remarks on the derivation and stable convergence in the Appendix.)

Theorem 3 : When $p = 1$ and $r \geq 1$ in (2.1), (2.2), (2.3) and (2.4), assume that $v(t_i^n)$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$, $\mathbf{E}[v_i^{4r}] < +\infty$ and σ_s $(0 \leq s \leq 1)$ is bounded. We set $\alpha_{1r}^* = [2\gamma + 2\delta - 1]/[2\gamma]$ and $\alpha_{2r}^* = [(4r+1)\gamma - (1+2r) + 4r\delta]/[(4r+1)\gamma]]$.

Then we have the following asymptotic properties of the LSIML estimator.

(i) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.5, \alpha_{1r}^*\}$ $((\alpha_{1r}^* > 0)$, as $n \longrightarrow \infty$

$$\hat{V}(2r) - V(2r) \xrightarrow{p} 0 \ . \tag{5.20}$$

(ii) For $m_c = [c(n)^\alpha]$ and $0 < \alpha < \min\{0.4, \alpha_{2r}^*\}$ $(\alpha_{2r}^* > 0)$, as $n \longrightarrow \infty$

$$\sqrt{m_c b(n)}\left[\hat{V}(2r) - V(2r)\right] \stackrel{\mathcal{L}-s}{\rightarrow} N\left[0, W\right] \ , \tag{5.21}$$

where

$$W = c_r^* \int_0^1 [\sigma_x(s)]^{4r} \, ds \ , \tag{5.22}$$

where c_r^* $(= a_{2r}/a_r^2 - 1)$ is a positive constant.

In particular, when $r = 1$, $c_1^* = 2$ and we have Proposition 2. When $r = 2$ $c_2^* = 105/3^2 - 1$. In the general case, $c_r^* = a_{2r}/a_r^2 - 1$.

It is because

$$a_{2r} = \frac{4r!}{2r!2^r} = \frac{4r!}{4r \cdot 2(2r-1)\cdots 2} = (4r-1)(4r-3)\cdots 1 \ .$$

It is interesting to find that the form of the asymptotic variance for the LSIML estimation is the same as the one for RV when there is no micro-market noise.

6. An Optimal Choice of α and γ

Because the properties of the LSIML estimation method depends crucially on the choice of $c(n)$ and m_c, which are dependent on n, we need to investigate the asymptotic effects as well as the small-sample effects of their choice.

As we have derived in the previous section, the asymptotic bias of the LSIML estimator is proportional to

$$\text{AB}_n \sim [b(n)^{r-1} \times b(n) \times \frac{m_c^{2r}}{c(n)^r}][\epsilon_n]^{2r} \tag{6.1}$$

and the asymptotic variance is proportional to

$$\text{AV}_n \sim \frac{1}{m_c b(n)} = \frac{1}{n}[c(n)]^{1-\alpha} \ \ . \tag{6.2}$$

Hence when n is large, we can approximate the mean squared error of the LSIML estimator as

$$g_n = c_1 \frac{1}{n}[c(n)]^{1-\alpha} + c_2[b(n)^r \times \frac{m_c^{2r}}{c(n)^r}]^2[\epsilon_n]^{4r} \ , \tag{6.3}$$

where c_{1g} and c_{2g} are some constants. If we set $c(n) = n^{\gamma}, b(n) = n^{1-\gamma}$ ($\gamma > 0$), we can rewrite

$$g_n = c_1 \frac{1}{n}[c(n)]^{1-\alpha} + c_2 \left[n^{2[(1-2\gamma)r+2r\alpha\gamma]}\right][\epsilon_n]^{4r} \ \ . \tag{6.4}$$

Then, by differentiation MSE with respect to α we have the condition that $n^{-1}c(n)^{1-\alpha}$ is proportional to $n^{2[(1-2\gamma)r+2r\alpha\gamma]} \times n^{-4r\delta}$. By rearranging the related terms, we have the next result.

Theorem 4 : When $p = 1$ and $r \geq 1$ in (2.1), (2.2), (2.3) and (2.4), assume that $v(t_i^n)$ is a sequence of i.i.d. random variables with $\mathbf{E}[v_i] = 0$ and $\mathbf{E}[v_i^{4r}] < +\infty$, and σ_s ($0 \leq s \leq 1$) is bounded. An optimal choice of $m_c = [c(n)^{\alpha}]$ and $c(n) = n^{\gamma}$ (with $\epsilon_n = n^{-\delta}$) to minimize MSE when n is large, is given by

$$-1 + \gamma(1-\alpha) = 2[(1-2\gamma)r + 2r\alpha\gamma] - 4r\delta \ , \tag{6.5}$$

which means the choice as

$$\alpha^* = \frac{(4r+1)\gamma + 4r\delta - 2r - 1}{(4r+1)\gamma} = 1 + \frac{4r\delta - 2r - 1}{(4r+1)\gamma} \ . \tag{6.6}$$

For instance, when $\delta = 0$, $\alpha^* = 1 - 3/[5\gamma]$ for $r = 1$ and $\alpha^* = 1 - 5/[9\gamma]$ for $r = 2$. When $\delta = 0$ and we take α^*, then the MSE is proportional to $n^{-1+\gamma^*(1-\alpha^*)}$), which is

$$\text{MSE} \sim n^{\frac{-2r}{4r+1}} \ \ . \tag{6.7}$$

When $r = 1$, we find that $2r/[4r+1] = 2/5$, which is the same as the asymptotic order of the SIML estimation. Moreover, when $r = 2$, we have $2r/[4r+1] = 4/9$.

7. Simulations

As an experimental exercise, we have done some simulation when $r = 1$ and $r = 2$, for the true parameters $V(2)$ and $V(4)$. We note that the variance of the SIML estimator of integrated volatility corresponds to $2\hat{V}(4)$. In our simulations we set $b(n) = [n^{1-\gamma}]$, $c(n) = [n^{\gamma}]$ and the number of replications is 3,000. Also we have investigated several cases in which the instantaneous volatility function σ_s^2 is given by

$$\sigma_s^2 = \sigma_0^2 \left[a_0 + a_1 s + a_2 s^2 \right], \tag{7.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_s > 0$ for $s \in [0, 1]$. This is a typical time-varying (but deterministic) case and the integrated volatility $V(2)$ is given by

$$V(2) = \int_0^1 \sigma_s^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{7.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

In the following tables, the true parameter values of V(2) and V(4) are $\int_0^1 \sigma_s^2 ds$ and $\int_0^1 \sigma_s^4 ds$, respectively. In Tables, the values of c_r^* are 2 and 10.66, respectively.

Table 1 : Estimation of integrated fourth-order functional
($a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_u^2 = 0.0005$, $b(n) = 5, c(n) = 521$; $\alpha = 0.4, \gamma = 0.795$)

n=2,605	$V(2) = 2.00$	$V(4) = 4.0$
mean	2.009	4.053
Var	0.134	2.837
AV	0.133	2.843

Table 2 : Estimation of integrated fourth-order functional
$(a_0 = 1.0, a_1 = 0.0, a_2 = 0.0; \sigma_u^2 = 0.0005,\ b(n) = 10, c(n) = 1,000;$
$\alpha = 0.33, \gamma = 0.75)$

n=10,000	$V(2) = 2.00$	$V(4) = 4.0$
mean	2.013	4.056
Var	0.092	1.973
AV	0.089	1.895

Table 3 : Estimation of integrated fourth-order functional
$(a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_u^2 = 0.0005,\ b(n) = 10, c(n) = 1,000;$
$\alpha = 0.33, \gamma = 0.75)$

n=10,000	$V(2) = 2.00$	$V(4) = 7.2$
mean	2.023	7.167
Var	0.160	15.093
AV	0.160	17.056

Table 4 : Estimation of integrated fourth-order functional
$(a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_u^2 = 0.0005,\ b(n) = 40, c(n) = 1,261;$
$\alpha = 0.45, \gamma = 0.66)$

n=50,440	$V(2) = 2.00$	$V(4) = 7.2$
mean	2.070	7.457
Var	0.016	1.650
AV	0.015	1.599

Table 5 : Estimation of integrated fourth-order functional
($a_0 = 6.0, a_1 = -24.0, a_2 = 24.0; \sigma_u^2 = 0.0005$, $b(n) = 18, c(n) = 5,622;$
$\alpha = 0.33, \gamma = 0.75$)

n=101,196	$V(2) = 2.00$	$V(4) = 7.2$
mean	2.022	7.273
Var	0.049	5.128
AV	0.047	5.016

In the above tables we first confirm that the LSIML method work well for the estimation of the integrated volatility. Although there may be some loss of estimation accuracy when the underlying true stochastic process is known, the LSIML method gives desirable finite and asymptotic properties. The most important result in our simulation is the estimation of 2V(4), which is the asymptotic variance of the SIML estimator of integrated volatility. As we see in Tables, the mean and SD (standard deviation) have reasonable values.

In order to investigate the asymptotic distribution of the LSIML estimator, we give some typical empirical distribution of a set of simulated data in Figure 7.1 ($r = 1, b(n) = 14, c(n) = 3371$) ad Figure 7.2 ($r = 2, b(n) = 76, c(n) = 677$). (We have taken $a_0 = 6.0, a_1 = -24.0, a_2 = 24.0$.) We can confirm that we have the asymptotic normality of the SIML estimator and the limiting normal distribution gives a reasonable approximation of the finite sample distribution. Also we found that when $r = 2$, we need more sample size to use the limiting normal distribution as a finite sample approximation in comparison with the case when $r = 1$.

From our simulations we found that the LSIML estimator of integrated volatility V(2) and V(4) perform quite well as we expected. The behaviors of the LSIML estimator for higher Brownian functionals as $r = 2$ are reasonable given the difficulties of the problem involved.

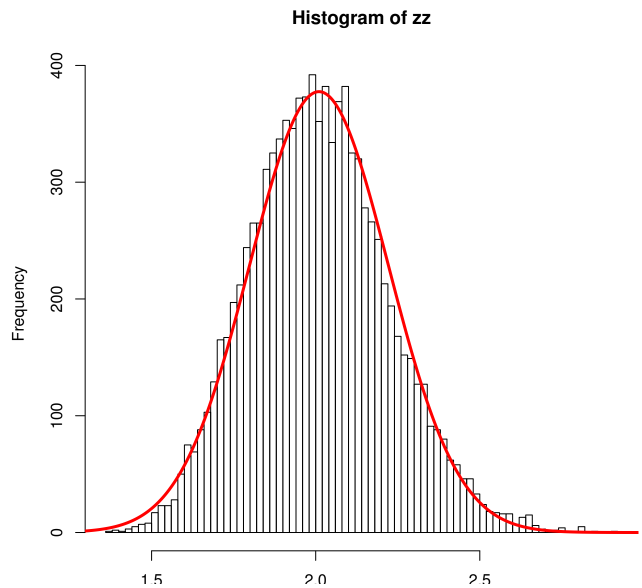

Figure 7.1: Normalized Histogram and Normalized Distribution ($r = 1$)

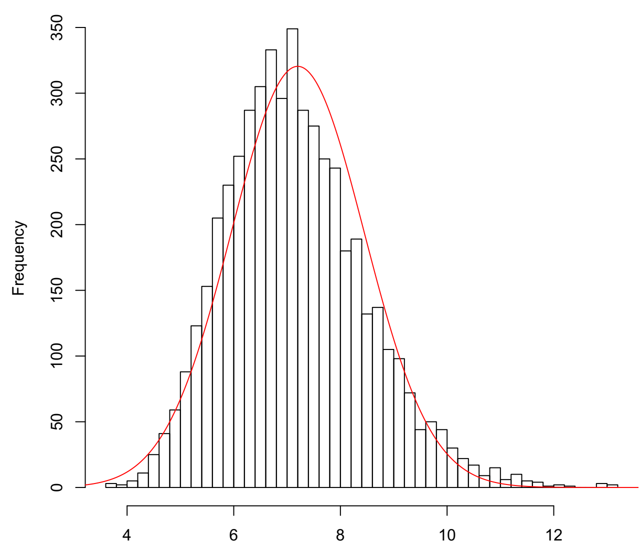

Figure 7.2: Normalized Histogram and Normalized Distribution ($r = 2$)

8. Possible Extensions

There are possible generalizations of our results in the previous sections. Let

$$\mathbf{Y}(t_i^n) = \mathbf{X}(t_i^n) + \epsilon_n \mathbf{v}(t_i^n) \ \ (i = 1, \cdots, n) \tag{8.1}$$

be the (p-dimensional) observed (log-)prices $\mathbf{Y}(t_i^n) = (Y_j(t_i^n))$ at t_i^n $(0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1)$ and $\mathbf{v}(t_i^n)$ $(= (v_j(t_i^n)))$ be a sequence of $(p \times 1)$ i.i.d. random vectors with $\mathbf{E}[\mathbf{v}(t_i^n)] = 0$ and $\mathbf{E}[\mathbf{v}(t_i^n)\mathbf{v}(t_i^n)^{'}] = \mathbf{\Sigma}_v$ (> 0).

As the underlying continuous-time process, it is straight-forward to consider the class of multi-dimensional diffusion processes. As the theory of continuous-time stochastic processes $\mathbf{X}(t_i^n) = (X_j(t_i^n))$, a more general form of the SDE for the p-dimensional continuous-time stochastic processes is given by

$$d\mathbf{X} = \boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{B}_t \ \ , \tag{8.2}$$

which has been called the diffusion-type continuous process, where $\boldsymbol{\mu}(s)$ is the $p \times 1$ drift vector, $\boldsymbol{\sigma}(s)$ is the $p \times q_1$ diffusion matrix, and $\mathbf{B}_t$ is the $q_1 \times 1$ Brownian motions. It also has the representation as

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \boldsymbol{\mu}(s) ds + \int_0^t \boldsymbol{\sigma}(s) d\mathbf{B}_s \ \ , \tag{8.3}$$

where the first term is an integration in the sense of Riemann while the second term is an Itô's stochastic integration with respect to the Brownian motion B_t ($q_1 \times 1$ vector).

A detailed theory of stochastic differential equation (SDE) and stochastic integration has been explained by Ikeda and Watanabe (1989). When the volatility parameters are stochastic, the volatility process $\boldsymbol{\sigma}(t) = (\sigma_{ij}(t))$ is stochastic it is a diffusion type process as

$$\sigma_{ij}(t) = \sigma_{ij}(0) + \int_0^t \mu_{ij}^{\sigma}(s) ds + \int_0^t \boldsymbol{\omega}_{ij}^{\sigma}(s) d\mathbf{B}_s^{\sigma} \ (0 \leq s \leq t \leq 1) \ , \tag{8.4}$$

where $\mu_{ij}(s)$ is the drift coefficient, $\boldsymbol{\omega}_{ij}^{\omega}(s)$ is $1 \times q_2$ diffusion coefficients and $\mathbf{B}_s^{\sigma}$ is another $q_2 \times 1$ Brownian motion vector, which may be correlated with $\mathbf{B}_s$.

An an example of the estimation problem, we take $p \times p$ variance-covariance (or the integrated volatility) matrix $\mathbf{\Sigma}_x = \int_0^1 \boldsymbol{\sigma}_s \boldsymbol{\sigma}_s' ds$, which is the same as $\mathbf{V}(2) = (V_{gh}(2))$ in our notation. In this case, the terms $(1/m_c)\sum_{k=1}^{m_c}[z_{k,(i)}]^2$ and the asymptotic variance $2\int_0^1 [\sigma_x(s)]^4\, ds$ in Section 5 are replaced by

$$\hat{V}(g,h;2) = \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{gk,(i)} z_{hk,(i)}] \tag{8.5}$$

and

$$\int_0^1 \left[\sigma_{gg}^{(x)}(s)\sigma_{hh}^{(x)}(s) + (\sigma_{gh}^{(x)}(s))^2\right] ds \ , \tag{8.6}$$

where

$$\mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_x(s) ds = \begin{pmatrix} \sigma_{gg}^{(x)} & \sigma_{gh}^{(x)} \\ \sigma_{gh}^{(x)} & \sigma_{hh}^{(x)} \end{pmatrix} .$$

The most important fact is that both the SIML method and the local-SIML method are simple and it is straightforward to use them when the dimension p of underlying processes is large. This aspect is quite different from other methods proposed in the past. Recently, Kunitomo (2018) has considered a statistical procedure to detect the number of factors of the hidden covariation r_x when it is substantially less than the dimension p, for instance. We expect that under a set of regularity conditions, we have the results on the asymptotic properties of the local SIML estimator in more general settings.

9. Concluding Remarks

In this paper we have developed the Local SIML (LSIML) method for estimating higher-order Brownian functionals, which is a new statistical method. We extend the separating information maximum likelihood (SIML) method, which was proposed by Kunitomo, Sato and Kurisu (2018). The main motivation of LSIML is to estimate higher order Brownian functionals including the integrated volatility and co-volatility when we have micro-market noise by using high-frequency financial data. We have shown that the local SIML method has desirable asymptotic properties such as the consistency and asymptotic normality, and it also has reasonable finite sample properties, which are illustrated by several simulations. Although there could be other methods for estimating higher-order Brownian functionals, the LSIML method is simple and it has desirable asymptotic properties. Hence it should be useful for practical situations.

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APPENDIX : Mathematical Derivations of Theorems

In this Appendix, we give some details of the derivations of the results in Section 5. Since we have used the stable convergence in Proposition 2 and Theorem 3 and it may not be standard in statistics literature, we will give a discussion how we can apply the basic arguments to our situation at the end of this Appendix.

We first give two lemmas.

Lemma A-1 : Let r be any positive integer. Then

(A.1) $$\frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)}^r \sim (\frac{\pi^{2r}}{2r+1})\frac{m_c^{2r}}{c(n)^r}$$

as $c(n), m_c \to \infty$ and $m_c/c(n) \to 0$.

Proof of Lemma A-1 : Since $m_c/c(n) \to 0$ as $n \to \infty$ and $\sin x \sim x$ when x is small, we can evaluate

$$\begin{aligned} \frac{1}{m_c}\sum_{k=1}^{m_c} a_{k,c(n)}^r &= [\pi]^{2r}\frac{m_c^{2r}}{c(n)^r}\left[\frac{1}{m_c}\sum_{k=1}^{m_c}(\frac{k}{m_c})^{2r} + o(1)\right] \\ &= \frac{\pi^{2r}}{2r+1}\left[\frac{m_c^{2r}}{c(n)^r} + o(1)\right] \end{aligned}$$

because

$$\frac{1}{m_c}\sum_{k=1}^{m_c}(\frac{k}{m_c})^{2r} - \int_0^1 x^{2r}dx = o(1) \ .$$

(**QED**)

Lemma A-2 : Let

(A.2) $\quad b_{kj} = \sqrt{c(n)}[p_{kj} - p_{k,j+1}] = \dfrac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{kj} - \dfrac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{k,j+1}$

for $k = 1, \cdots, c(n); j = 1, \cdots, c(n) - 1$ and

$$b_{k,c(n)} = \frac{2\sqrt{c(n)}}{\sqrt{2c(n)+1}}\cos\theta_{j,c(n)} \ ,$$

where $\theta_{kj} = \frac{2\pi}{2c(n)+1}(k - 1/2)(j - 1/2)$.

Then

$$\sum_{j=1}^{c(n)}[b_{kj}]^2 = [1 + O(\frac{1}{c(n)})]a_{k,c(n)} \ ,$$

and

$$\sum_{j=1}^{c(n)}[b_{kj}]^4 = [\frac{3}{2c(n)} + o(\frac{1}{c(n)})][a_{k,c(n)}]^2 \ ,$$

Proof of Lemma A-2 :

$$\begin{aligned}
\frac{2c(n)+1}{c(n)}\sum_{j=1}^{c(n)-1}[b_{kj}]^2 \quad &= \quad \sum_{j=1}^{c(n)-1}[(1-e^{i\theta_k})e^{i\theta_{kj}}]^2 + \sum_{j=1}^{c(n)-1}[(1-e^{-i\theta_k})e^{-i\theta_{kj}}]^2 \\
&\quad +2(c(n)-1)(1-e^{i\theta_k})(1-e^{-i\theta_k}) \ ,
\end{aligned}$$

where $\theta_k = [2\pi/(2c(n)+1](k - 1/2)$ $(k = 1, \cdots, c(n))$.

Then we use the relation

$$\begin{aligned}
\sum_{j=1}^{c(n)-1}[e^{i\theta_{kj}}]^2 \quad &= \quad e^{i\theta_k}\frac{1 - e^{i(4\pi/(2c(n)+1))(k-1/2)n}}{1 - e^{2i\theta_k}} \\
&= \quad e^{-i\theta_{kj}}\frac{1 + e^{i\theta_k}}{1 - e^{2i\theta_k}}
\end{aligned}$$

because we have

$$e^{i(4\pi/(2c(n)+1)(k-1/2)n} = e^{i(\pi/(2c(n)+1)(2k-1)(2n+1-1)} = -e^{-i\theta_k}$$

and $e^{i\theta_{kn}} + e^{-i\theta_{kn}} = e^{i\pi(k-1/2)}[e^{-i\theta_k} + e^{i\theta_n}] = 2\sin\theta_k = 4\sin(\theta_k/2)\cos(\theta_k/2)$.
Hence by arranging each terms and use the relation

$$(1-e^{i\theta_k})(1-e^{-i\theta_k}) = (e^{-i\frac{\theta_k}{2}} - e^{i\frac{\theta_k}{2}})(e^{i\frac{\theta_k}{2}} - e^{-i\frac{\theta_k}{2}}) \ ,$$

we have the result. By using the similar but tedious arguments for the fourth-powers, after some calculations we find that

$$\begin{aligned}\frac{[2c(n)+1]^2}{c(n)^2}\sum_{j=1}^{c(n)-1}[b_{kj}]^4 &= \sum_{j=1}^{c(n)-1}[b_{kj}]^4[e^{i\theta_{kj}}(1-e^{i\theta_k}) + e^{-i\theta_{kj}}(1-e^{-i\theta_k})]^4 \\ &= [6c(n)+O(1)] \times 4^2 \sin^4\frac{\theta_k}{2} \ .\end{aligned}$$

(**Q.E.D.**)

An Outline of Derivations of Proposition 2 and Theorem 3 :
We illustrate our derivations of the results in Section 5 when $p=1$ and $r=2$. (The derivations of the case of $r=1$ in Proposition 2 can be quite similar, but much simpler than the present case.) Although it is possible to extend the derivations, the notations become rather complicated and the essential arguments are the same. (Basically, we extend the arguments in Chapter 5 of Kunitomo et. al (2018).)

We first consider the case when $\sigma_s^2 = \sigma_x^2$ $(0 \le s \le 1)$ (a constant volatility). Then we apply the method for this basic case to the time-varying volatility case and the stochastic case as in Chapter 5 of Kunitomo et. al (2018).
With the transformation (4.1) in the set $I_c(i) = ((i-1)\frac{c(n)}{n}, i\frac{c(n)}{n}]$, we set $z_{k,(i)} = z_{k,(i)}^{(1)} + z_{k,(i)}^{(2)}$, where $z_{k,(i)}^{(1)}$ and $z_{k,(i)}^{(2)}$ correspond to the $k-$th elements of $\mathbf{Z}_{c(n),(i)}^{(1)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}(\mathbf{X}_{c(n),(i)} - \bar{\mathbf{y}}_{0,(i)})$ and $\mathbf{Z}_{c(n),(i)}^{(2)} = h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}\mathbf{V}_{c(n),(i)}$ ($\mathbf{V}_{c(n),(i)}$ is the noise vector in $I_c(i)$ $(i=1,\cdots,b(n))$), respectively. Then we have $\mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(1)}] = \mathbf{0}$, $\mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(2)}] = \mathbf{0}$ and

$$\text{(A.3)} \qquad \mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(1)}\mathbf{Z}_{c(n),(i)}^{(1)'}] = \sigma_x \mathbf{I}_{c(n)} \ , \mathbf{E}[\mathbf{Z}_{c(n),(i)}^{(1)}\mathbf{Z}_{c(n),(i)}^{(1)'}] = \sigma_v^2 h_{c(n)}^{-1}\mathbf{D}_{c(n)} \ ,$$

where $\mathbf{D}_{c(n)} = \mathrm{diag}(d_k)$ $(k \in I_c(i))$.
In $I_c(i)$ we write $z_{k,(i)}^{(2)} = \sum_{j=1}^{c(n)} b_{kj} v_j(i)$, $v_j(i)$ are noise terms in $I_c(i)$ $(i=1,\cdots,b(n))$

and b_{kj} are the corresponding coefficients of $h_{c(n)}^{-1/2}\mathbf{P}_{c(n)}\mathbf{C}_{c(n)}^{-1}$. Also we have

$$\text{(A.4)} \qquad z_{k,(i)}^{(1)} = \sqrt{\frac{4c(n)}{2c(n)+1}\sum_{j=1}^{c(n)} s_{kj} r_{j,(i)}} \ ,$$

$s_{kj} = \cos\theta_{kj}$ and $\theta_{kj} = (2\pi/(2m_c+1))(k-1/2)(j-1/2)$ $(j,k=1,\cdots,c(n))$. Then we can represent $(1/m_c)\sum_{k=1}^{m_c}[z_{k,(i)}^{(1)}]^2 = \sum_{j,j'=1}^{c(n)} c_{jj'}^{*} r_{j,(i)} r_{j',(i)}$, and $c_{j,j'}^{*} = [2c(n)/(2c(n)+1)]c_{j,j'}$.

Lemma A-3 : When $p=1$ and $r=1$, we have the asymptotic bias term as (5.3).

Proof of Lemma A-3 : By using the transformation (4.1), the dominant term of the variance of the noise terms in each interval $I_c(i)$ is given by $(1/m_c)\sum_{k=1}^{m_c} a_{k,c(n)}$. By using Lemma A-1, we can evaluate the dominant asymptotic bias as $b(n) \times (1/m_c)\sum_{k=1}^{m_c} a_{k,c(n)}$, which is given by (5.2). (These arguments are essentially the same as those in Lemma 5.3 and its proof in Chapter 5 of Kunitomo et al. (2018).) (**Q.E.D.**)

When $r \geq 2$, the evaluation of each term becomes more complicated, but the essential procedure is the same as the case of $r=1$. We illustrate the typical derivation when $r=2$ in the following. In our derivations of the results we make an extensive use of the decomposition of the local sample moments as

$$\begin{aligned}
\text{(A.5)} \qquad & \frac{1}{m_n}\sum_{k=1}^{m_n}\left[z_{k,(i)}^4 - \mathbf{E}(z_{k,(i)}^4)\right] \\
= \ & \frac{1}{m_c}\sum_{k=1}^{m_c}\Big\{[(z_{k,(i)}^{(1)})^4 - \mathbf{E}((z_{k,(i)}^{(1)})^4)] \\
& +[\mathbf{E}(z_{k,(i)}^{(2)4})] + 6[\mathbf{E}(z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2})] + [z_{k,(i)}^{(2)4} - \mathbf{E}(z_{k,(i)}^{(2)4})] \\
& +4[z_{k,(i)}^{(1)} z_{k,(i)}^{(2)3} + z_{k,(i)}^{(1)3} z_{k,(i)}^{(2)}] + 6[z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2} - \mathbf{E}(z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2})]\Big\} \ .
\end{aligned}$$

By using Lemma A.2 and the boundedness of moments, we evaluate $\mathbf{E}((z_{k,(i)}^{(1)})^4)$ as

$$\begin{aligned}
\text{(A.6)}\mathbf{E}[\sum_{j=1}^{c(n)} b_{kj} v(t_j^{c(n)})]^4 &= \mathbf{E}[\sum_{j=1}^{c(n)} b_{kj}^4 v_j^4 + \sum_{j\neq j'} b_{kj}^2 b_{kj'}^2 v_j^2 v_{j'}^2] \\
&= \sum_{j=1}^{c(n)} b_{kj}^4[\mathbf{E}(v_j^4) - (\mathbf{E}(v_j^2))^2) + \sum_{j,j'=1}^{c(n)} b_{kj}^2 b_{kj'}^2 \mathbf{E}(v_j^2)\mathbf{E}(v_{j'}^2)]
\end{aligned}$$

$$= \quad [a_{k,c(n)}^2 + O(\frac{1}{c(n)})]M_4 \ ,$$

where $M_4 = \mathbf{E}[v(t_1^{c(n)})]^4$.

By using the similar arguments, it is straight-forward to show that the asymptotic bias term is negligible for $r = 2$ when

$$b(n) \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} \mathbf{E}[z_{k,(i)}^{(2)4}] \longrightarrow 0 \ .$$

Because of Lemma A-1 with $r = 2$, we have the condition (i) in Proposition 2.

For the consistency of the LSIML estimator, we need a set of sufficient conditions such that

$$b(n) \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(2)4} - \mathbf{E}(z_{k,(i)}^{(2)4}] \xrightarrow{p} 0 \ ,$$

$$b(n) \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)} z_{k,(i)}^{(2)3} + z_{k,(i)}^{(1)} z_{k,(i)}^{(2)}] \xrightarrow{p} 0 \ ,$$

and

$$b(n) \sum_{i=1}^{b(n)} \frac{1}{m_c} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2} - \mathbf{E}(z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2})] \xrightarrow{p} 0 \ .$$

The evaluations of each terms are straight-forward, but it is tedious to evaluate the corresponding orders. For instance, we find that

$$\text{(A.7)} \quad b(n) \sum_{i=1}^{b(n)} \mathbf{E}[z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2}] = O(b(n)^2 \frac{c(n)}{n} \times \frac{m_c^2}{c(n)} \times \epsilon_n^2) = O(n^{1-2\gamma+2\gamma\alpha-2\delta})$$

because of Lemma A-1 and $\mathbf{E}[r_j^2] = O(1/n)$.

Hence under the conditions in Proposition 2 are sufficient for that each term except the leading term in the decomposition (A.5) are negligible and hence they are sufficient for the consistency. The above arguments can be used when the diffusion coefficient σ_s $(0 \leq s \leq 1)$ is time varying and stochastic, we have the result in Proposition 2.

We also consider the conditions that each term except the first term in the decomposition (A.5) of the normalized LSIML estimator are asymptotically negligible when $n \longrightarrow \infty$

$$\sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_c} \mathbf{E}[z_{k,(i)}^{(2)4}] \longrightarrow 0 \ ,$$

$$\sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_c} [z_{k,(i)}^{(2)4} - \mathbf{E}(z_{k,(i)}^{(2)4}] \xrightarrow{p} 0 \ ,$$

$$\sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)} z_{k,(i)}^{(2)3} + z_{k,(i)}^{(1)3} z_{k,(i)}^{(2)}] \xrightarrow{p} 0 \ ,$$

and

$$\sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_c} [z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2} - \mathbf{E}(z_{k,(i)}^{(1)2} z_{k,(i)}^{(2)2}] \xrightarrow{p} 0 \ .$$

We use a set of sufficient condition that the above convergence results hold when $n \to \infty$ in Proposition 2 and Theorem 3 when $r = 2$. Then in order to show the asymptotic normality of the local SIML estimator, we write the main term of the above decomposition as

$$\begin{aligned} & \sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_n} \left[z_{k,(i)}^{(1)4} - \mathbf{E}(z_{k,(i)}^{(1)4}) \right] \\ = \ & \sqrt{\frac{b(n)^3}{m_c}} \sum_{i=1}^{b(n)} \sum_{k=1}^{m_n} \sum_{j_1,j_2,j_3,j_4=1}^{c(n)} s_{k,j_1} s_{k,j_2} s_{k,j_3} s_{k,j_4} [r_{j_1}(i) r_{j_2}(i) r_{j_3}(i) r_{j_4}(i) \\ & \quad - \mathbf{E}[r_{j_1}(i) r_{j_2}(i) r_{j_3}(i) r_{j_4}(i)]] \ . \end{aligned}$$

The conditions in Proposition 2 are sufficient for that each term except the leading term in the decomposition (A.5) are negligible. By applying the CLT for discretized stochastic processes and using the stable-convergence in the present case, we have the asymptotic normality.

For the asymptotic normality, we need to apply the martingale CLT for stochastic processes. (See the remarks on stable convergence below.) In each interval $I_c(i)$, $i = 1, \cdots, c(n)$ we can decompose $\sum_{k=1}^{c(n)} [z_{k,(i)}^4 - \mathbf{E}(z_{k,(i)}^4)]$ as the martingale part M_i^n and the remaining parts. Because the stochastic order of the remaining parts is small than the martingale part, they can be stochastically ignored as we used in Chapter 5 of Kunitomo et al. (2018)) for the case $r = 1$. Then we can use the martingale CLT to the sum of matringales $S_n = \sum_{i=1}^{b(n)} M_i^n$ and we have the asymptotic normality by taking the appropriate normalizing factor to the martingale part.

In the general case of stochastic volatility, we need the stable-convergence because the limiting terms as $\int_0^1 \sigma_s^{2r} ds$ are stochastic. Since we are considering higher order Brownian motions of the form (2.5) under (2.4) and the dominant terms are martingale differences, it is possible to show the stable-convergence. (See some discussion on the stable convergence below.)

On Stable Convergence in Proposition 2 and Theorem 3 :
We give an outline of the underlying arguments of stable convergences in Proposition 2 and Theorem 3. We consider the case when μ_s^σ and ω_s^σ in (2.3) and (2.4) are bounded and continuous with $p = r = 1$ and $b(n) = 1$. Then we denote $c(n) = n$ and $m_c = m_n$ as in Kunitomo et al. (2018). ($m_n \to \infty$ and $m_n = O(n^\alpha)$ $(0 < \alpha < .4)$.)

By using Itô's formula, we can represent

$$\sigma_t^4 = \sigma_0^4 + \int_0^t \mu_s^{\sigma *} ds + \int_0^t \omega_s^{\sigma *} dB_s^\sigma \ (0 \leq s \leq t \leq 1) \ , \tag{A.8}$$

where $\mu_s^{\sigma *}$ and $\omega_s^{\sigma *}$ are the drift and diffusion coefficients and B_s^σ is Brownian motion, which may be correlated with B_s.
For $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ we write

$$V(4) = \sigma_0^4 + \sum_{j=1}^{n} [\int_{t_{j-1}^n}^{t_j^n} (\int_0^t \mu_s^{\sigma *} ds) dt + \int_{t_{j-1}^n}^{t_j^n} (\int_0^t \omega_s^{\sigma *} dB_s^\sigma) dt] \ , \tag{A.9}$$

and where the last term of $V(4)$ becomes the sum of

$$V_i^n = \int_{t_{i-1}^n}^{t_i^n} (\int_s^1 dt) \omega_s^{\sigma *} dB_s^\sigma \quad (i = 1, \cdots, n). \tag{A.10}$$

By using the standard arguments, we can show that the effects of drift terms are negligible as $n \to \infty$. By using the arguments in Chapter 5 of Kunitomo et al. (2018), the leading martingale term of the SIML estimator is

$$U_n = \sum_{j=2}^{n} U_j^n \ , \tag{A.11}$$

where $U_j^n = [\sum_{i=1}^{j-1} 2\sqrt{m_n} c_{ij} r_i] r_j$, $c_{ij} = (2/m_n) \sum_{k=1}^{m_n} s_{ki} s_{kj}$ and $s_{ij} = \cos\left[\frac{2\pi}{2c(n)+1}(i - \frac{1}{2})(j - \frac{1}{2})\right]$ $(i, j = 1, \cdots, n)$.
Then we can evaluate the conditional expectations as

$$W_j^n = \mathbf{E}[U_j^n V_j^n | \mathcal{F}_{j-1,n}] = [\sum_{i=1}^{j-1} 2\sqrt{m} c_{ij} r_i] \int_{t_{j-1}^n}^{t_j^n} \sigma_s (1 - s) \omega_s^{\sigma *} ds \ , \tag{A.12}$$

where $\mathcal{F}_{j-1,n}$ is the $\sigma-$field generated at t_{j-1}^n $(j=1,\cdots,n)$. We notice that for any j $(j=1,\cdots,n)$ $\int_{t_{j-1}^n}^{t_j^n}\sigma_s(1-s)\omega_s^{\sigma *}ds=O_p(1/n)$, which can be approximated as $[\sigma(t_{j-1}^n)(1-t_{j-1}^n)\omega^{\sigma *}(t_{j-1}^n)][B(t_j^n)-B(t_{j-1}^n)]$ with the error order being $O(1/n^2)$. By using (A.5) with $t=t_{j-1}^n$ for each j, $\sigma(t_{j-1}^n)$ can be further represented as the sum of drift terms and Brownian motion parts given $\mathcal{F}_{i-1,n}$ for $t_{j-1}^n>t_{i-1}^n$ $(j=1,\cdots,n)$. By re-writing the sum of conditional expectations as

(A.13) $$\sum_{j=2}^{n}W_j^n=\sum_{i=1}^{n-1}[\sum_{j=i+1}^{n}\sqrt{m_n}c_{ij}\int_{t_{j-1}^n}^{t_j^n}\sigma_s(1-s)\omega_s^{\sigma *}ds]r_i\ ,$$

it is possible to show that as $n\longrightarrow\infty$

(A.14) $$\sum_{j=2}^{n}W_j^n\xrightarrow{p}0\ .$$

In order to show this convergence, we use several facts that the function $\sigma_s(1-s)\omega_s^{\sigma *}$ is bounded and continuous, σ_s is a Brownian semi-martingale with (2.4) for any s, and $r_j^n=\int_{t_{j-1}^n}^{t_j^n}\sigma_s dB_s$ can be approximated by $r_j^{*n}=\sigma(t_{j-1}^n)(B(t_j^n)-B(t_{j-1}^n))$ with errors order being $O(1/n^2)$. We also have the representation for $i\neq j$

(A.15) $$c_{ij}=\frac{1}{2m_n}[\frac{\sin\frac{\pi}{2n+1}(i+j-1)m_n}{\sin\frac{\pi}{2n+1}(i+j-1)}+\frac{\sin\frac{\pi}{2n+1}(i-j)m_n}{\sin\frac{\pi}{2n+1}(i-j)}]$$

(see Section 3.2 and Lemma 5.2 of Kunitomo et al. (2018)). By using Lemma 5.1 (for $l>0$ $\sum_{k=1}^{m}2\cos(2\pi/(2n+1))(k-1/2)l=\sin l(2\pi m/(2n+1)/\sin l(\pi/(2n+1))$, we find that as $n\rightarrow+\infty$

(A.16) $$\frac{1}{n}\sum_{j=1}^{n}\frac{\sin\frac{\pi}{2n+1}mj}{\sin\frac{\pi}{2n+1}j}\sim O(1)\ .$$

We have the last relation because for any positive integer N

(A.17) $$\int_{-\pi}^{\pi}\frac{\sin(N+\frac{1}{2}\alpha)}{2\pi\sin\frac{1}{2}\alpha}d\alpha=1$$

(see (3.2.7) of Brillinger (1980) on this relation and Chapter 5 of Kunitomo et al. (2018) on several properties of c_{ij} $(i,j=1,\cdots,n)$ and other quantities appeared in the SIML estimation method.)

By using the lengthy arguments, it is possible to show that the martingale U_n and the martingale part of $V(4)$ are asymptotically uncorrelated.

Finally, by using the convergence of each terms and applying Theorem 2.2.15 of Jacod and Protter (2012) to the martingale parts, we have the stable convergence for a sequence of random variables. (The derivation of the CLT for the main term in the normalized SIML estimator, U_n, has been given in Chapter 5 of Kunitomo et al. (2018).) Since the normalized SIML estimator and $V(4)$ $(= \int_0^1 \sigma_s^4 ds)$ (and higher order Brownian functionals) are uncorrelated in the asymptotic sense, we have the stable convergence of the martingale U_n to the limiting normal random variable given $\int_0^1 [\sigma_x(s)]^4 ds$ such that as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{V}(2) - V(2)\right] \xrightarrow{\mathcal{L}-s} N\left[0, W\right] \ , \tag{A.18}$$

where

$$W = 2\int_0^1 \left[\sigma_x(s)\right]^4 ds \ . \tag{A.19}$$

It is tedious, but straight-forward to extend the above arguments to more general cases. (See Jacod ad Protter (2012), and Hausler and Luschgy (2015) for the details of stable convergence.)

When $r = 2$ and $p = 1$ the essential derivations are parallel to the case $r = 1$, but with substantial notational complications. For instance, instead of (A.11) we need to evaluate

$$U_j^n = (\frac{4}{\sqrt{m}}) \sum_{0<i_1<i_2<i_3<j} (\sum_{k=1}^{m} s_{kj}s_{k,i_1}s_{k,i_2}s_{k,i_3})r_{i_1}r_{i_2}r_{i_3}]r_j \ , \tag{A.20}$$

By using similar arguments on trigonometric functions (addition theorem) and (A.16), it is possible to show (A.14) and then we have the stable convergence as consequence in the present case. (An important fact is that there are only n terms when $i = j$ $(i, j = 1, \cdots, n)$, while there are $O(n^2)$ terms when $i \neq j$ $(i, j = 1, \cdots)$ in (A.15), which are dominant asymptotically. In (A.20), there are $O(n^3)$ terms when $j - i_1 = i_2 - i_3$ $(i_1, i_2, i_3, j = 1, \cdots, n)$ for instance while there are $O(n^4)$ other terms in (A.15), which are dominant asymptotically.) In more general cases, it is tedious to write the corresponding notations, but the essential arguments are the same.