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Cyclical Components Synthesization Approach to Construct a Coincident Index of Business Cycles

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Abstract

The accurate assessment of business conditions is a long-standing problem in macroeconomics. To construct a coincident index of growth cycles from a given set of indicators, we propose a new approach, the cyclical components synthesization (CCS) approach. We refer to the coincident index of growth cycles as the index of business cycles (IBC) of coincident economic indications. The IBC based on the CCS approach has the following properties: (1) its mean is globally stationary; (2) it is constructed as a common factor in the stationary parts of the selected economic indicators; and (3) its variations are as large as possible and, therefore, it contains a larger amount of information on business fluctuations. We examine the performance of the proposed IBC by comparing it with the composite index and the Nikkei Business Index. The results show that the CCS approach has a number of advantages over existing methods.

Keywords: Business cycle index; Bayesian modeling; State space model; Kalman filter; Composite index; Nikkei Business Index

1 Introduction

Following the seminal work of Burns and Mitchell (1946), the measurement of business cycles has been recognized as an important issue in macroeconomic studies. Furthermore, favorable or unfavorable business conditions are of great interest to many, as the assessment of business conditions has a significant influence on government economic policies. However, conventional indexes may not accurately capture business cycles. Therefore, the question that arises is: what is a better way to produce useful and reliable indexes? This is a long-standing problem and was one of the central questions addressed by Stock and Watson (1989). In this paper, we aim to present an alternative approach to constructing a business cycle index using coincident economic indicators. Specifically, we attempt to measure growth cycles and apply the proposed method to Japanese data. As a monthly coincident index of growth cycles in Japan is, to our knowledge, a new development, it may be of broad interest to macroeconomists.

Conventionally, business conditions are assessed using summary measures for the state of macroeconomic activity in Japan and the United States (US). The composite index (CI) and diffusion index (DI) are representative summary measures. Although both have the advantage of manageability, they have faced criticism because they are not based on a strict statistical model. Given this, since the 1980s, there have been many studies of statistical methods for business cycle analysis. Pioneering works in this area are by Stock and Watson (1989, 1991), who developed a statistical method to construct an index of business cycles (IBC) based on a state space model. Stock and Watson (1989, 1991) define the business cycle as a co-movement of macroeconomic variables, and the Stock–Watson index is constructed by extracting the common factor hidden in multiple macroeconomic time series data. Thus, their proposed model is commonly termed the dynamic factor model, and it was first applied to analyze the US business cycle. Ohkusa (1992) and Fukuda and Onodera (2001) have applied the Stock–Watson dynamic factor modeling approach to analyze Japanese business cycles.

The dynamic factor modeling approach has since been extended by Kanoh and Saito (1994), Mariano and Murasawa (2003, 2010), Watanabe (2003), and Urasawa (2014). Kanoh and Saito (1994) extended the dynamic factor model to include qualitative data from the Short-Term Economic Survey of Enterprises (abbreviated to *Tankan*), a statistical survey conducted by the Bank of Japan. Focusing on the companies' judgments about business conditions in *Tankan*, Kanoh and Saito (1994) considered that if such judgments reflected an overall assessment of actual business conditions, then a business index that statistically extracts the actual state of the economy from such judgments would be a more appropriate measure of the overall state of the economy than conventional indexes. They concluded that the peaks and troughs projected by their proposed index have systematic relationships with the business cycles identified by experts from the Japanese government. Mariano and Murasawa (2003) pointed out that the CI and the Stock–Watson coincident index have two shortcomings: first, they ignore information contained in quarterly indicators, such as quarterly real gross domestic product (GDP); and second, they lack economic interpretation. Therefore, Mariano and Murasawa (2003) extended the Stock–Watson coincident index by applying a maximum likelihood factor analysis to a mixed-frequency series of quarterly real GDP and monthly coincident business cycle indicators. The resulting index is related to latent monthly real GDP. Furthermore, Mariano and Murasawa (2010) estimated Gaussian vector autoregression (VAR) and factor models for latent monthly real GDP and other coincident indicators using observable mixed-frequency series. For the maximum likelihood estimation of a VAR model, the expectation-maximization algorithm helps to identify a good starting value for a quasi-Newton method. Mariano and Murasawa (2010) concluded that the smoothed estimate of latent monthly real GDP is a natural extension of the Stock–Watson coincident index. To obtain early estimates of Japan's quarterly GDP growth in real time, Urasawa (2014) estimated a dynamic factor model using mixed-frequency data on GDP, industrial production, employment, private consumption, and exports. The results of a real-time forecasting exercise suggested that the model performs well.

Another prominent approach that differs from dynamic factor modeling is the regime-switching modeling approach developed by Hamilton (1989). Whereas dynamic factor modeling is associated with a CI-type index, the regime-switching modeling is associated with a DI-type index. Kim and Nelson (1998) developed a method that combined the above two approaches. Watanabe (2003) applied the approach of Kim and Nelson (1998) to the Japanese economy, and estimated a dynamic Markov switching factor model using macroeconomic data.

Here, we note a data treatment problem regarding the Stock–Watson dynamic factor modeling approach and its extensions. Specifically, many earlier studies used differencing in time series data to obtain stationariness in cases where nonstationary data (e.g., the mean) was used for convenience. This may result in a loss of significant information. In this paper, we propose an alternative approach to construct a coincident IBC via the decomposition of time series data into several possible components. In contrast to the Stock–Watson index, we consider an IBC that has the following properties: (1) it is globally stationary in mean; (2) it is constructed using the common factor of all the relevant coincident indicators; and (3) it has variations that are as large as possible and, therefore, it contains a larger amount of information on business cycle fluctuations. The IBC based on our proposed approach has a higher correlation with the cyclical component of GDP than the CI or the Nikkei Business Index (NBI). That is, the results show that our index performs better than either of these indexes.

The remainder of this paper is organized as follows. In Section 2, we present a brief review of some existing approaches. In Section 3, we explain the framework of our new approach for constructing a business cycle index. Then, we describe the parameter estimation procedure in Section 4. In Section 5, we compare the performance of the IBC based on the newly-proposed approach with the CI of coincident indicators and the NBI using Japanese economic data. In Section 6, we discuss our new approach. We present our conclusions in Section 7.

2 Brief review of existing approaches and evaluation

2.1 Diffusion index, composite index, and the MTV model

There is no uniform set of official indicators used to determine business conditions among developed countries. For example, the CI has a central role in measuring business cycles in the US, the United Kingdom, and Italy, whereas the growth rates of GDP are emphasized in the measurement of business cycles in Canada.

In Japan, business conditions are typically measured using business cycle indicators such as the CI and DI, which are compiled by the Economic and Social Research Institute (ESRI) of the Japanese Cabinet Office. Since April 2008, the ESRI has placed greater emphasis on the CI than on the DI in assessing business conditions in Japan.

According to the Cabinet Office, the indicators for indexes of business conditions are re-examined after each business cycle and changed if it is expected that the performance of the indexes will be improved. The DI and CI complied by Japan's ESRI consist of three indexes: a leading index (which is constructed based on 11 indicators), a coincident index (based on 10 indicators), and a lagging index (based on nine indicators). The DI represents the ratio of a series that increased in value during the most recent 3-month period. The coincident DI is considered an important index of peaks and troughs: its 50% line is frequently used to identify the turning point of a business cycle. However, official reference dates are determined by the ESRI on the basis of much broader information, including other economic indicators and professional opinions. The CI uses the same data series as the DI. However, unlike the DI, which only considers the direction of changes in indicators, the CI takes into account the degree of change. Therefore, the CI is considered useful in measuring the speed and magnitude of a business cycle. Originally, the CI and DI were developed by the National Bureau of Economic Research (NBER) in the US and the business cycle indicators of the Japanese Cabinet Office are based on the NBER calculation method. Although these business cycle indexes are widely used, it is recognized that they suffer from various defects. In particular, they are not derived from any sound statistical methods (Kanoh and Saito, 1994).

Kariya (1988, 1993) proposed a multivariate time series variance component model known as the MTV model. The basic concept of this model is that it uses principal component analysis with time series data on macroeconomic variables. When we apply the MTV model to coincident indicators, a principal component of these indicators can be regarded as an IBC. Note that the MTV model may lead to different results depending on the component on which it focuses. Therefore, a definite conclusion cannot be obtained unless there is some objective criterion justifying a focus on the obtained principal component.

2.2 Approach of Stock and Watson

Stock and Watson (1989, 1991) proposed a more reliable index of business cycles that measures the state of the economy using a time series model. In this approach, the signal for business fluctuations is generally expressed by a co-movement of selected macroeconomic variables. Thus, Stock and Watson suggested decomposing macroeconomic variables into common factors and unique (idiosyncratic) factors. The common factor is considered to be derived by a single unobservable dynamic variable, which is called the state of the economy.

Specifically, a macroeconomic time series x_{in}, for one of m indicators, is expressed as:

$$\begin{aligned} \Delta x_{in} &= \mu_i + \gamma_i \Delta c_n + u_{in}, &(1)\\ \phi(L)\Delta c_n &= \eta_n, &(2)\\ d_i(L)u_{in} &= \epsilon_{in} \quad (i = 1, 2, \ldots, m), &(3) \end{aligned}$$

where μ_i is an unknown parameter, γ_i is the unknown factor loading for the ith indicator, Δc_n and u_{in} are the common and unique factors of Δx_{in}, respectively, with $i = 1, 2, \ldots, m$, and $\phi(L)$ and $d_i(L)$ are polynomials with appropriate orders. Here, L denotes a lag operator, such as $L^j \Delta c_n = \Delta c_{n-j}$, with Δ as the difference operator defined by $\Delta = 1 - L$. Moreover, η_n and ϵ_{in} are usually assumed to be Gaussian white noise sequences, which are independent of each other.

Thus, the models in Eqs (2) and (3) are autoregressive (AR) models for Δc_n and u_{in}. To obtain good properties for estimates, it is assumed that Δc_n and u_{in} are stationary in mean; therefore, Δx_{in} is also stationary in mean. For most practical situations, time series data for the indicators show distinct trends and, therefore, it is necessary to take the first difference of the time series x_{in} so that Δx_{in} is stationary. Note that the difference Δc_n for factor c_n ensures the correspondence between x_{in} and c_n. When the time series x_{in} is considered stationary, Δx_{in} in Eq. (1) can be replaced with x_{in}, and Δc_n in Eqs (1) and (2) can be replaced with c_n.

The above dynamic factor models can be expressed by a state space representation. Thus, the Kalman filter algorithm can be applied to estimate the parameters and the common factor Δc_n and, hence, c_n. The estimate for c_n is called the Stock–Watson index (SWI).

2.3 Evaluation

As mentioned in Fukuda (1994) and Komaki (2001), the approaches used to construct business cycle indexes can be classified in terms of whether they consider deterministic or stochastic trends and the cyclical components of time series data. Specifically, DI, CI, and MTV modeling approaches are based on a deterministic perspective. In contrast, dynamic factor modeling is based on a stochastic perspective. An advantage shared by the DI and CI is their simplicity of construction. However, because they are not based on a clear statistical assumption, there can be difficulties in using them for prediction. The MTV modeling approach is based on a model involving principal component analysis. However, in general, the data structure is not clearly defined. Furthermore, principal component analysis often results in interpretation difficulties.

Conversely, the Stock–Watson dynamic factor modeling approach is based on a state space model and the Kalman filter. Therefore, this approach is based on a set of statistical models expressing the structure of the data. However, to satisfy the assumption of stationarity for the data used, many earlier methods take an ad hoc approach to the treatment of data, such as the use of differences for the time series. Another problem with the Stock–Watson approach is the difficulty of determining whether to use differencing.

Moreover, because of the treatment of the data, it may be difficult to interpret the SWI results from an economic perspective. Consider here a case, as in Eq. (1), where the difference in the data is used. Let the polynomial $\phi(L)$ in Eq. (2) be given by:

$$\phi(L) = \alpha_0 + \sum_{j=1}^{p} \alpha_j L^j$$

with $\alpha_0 \neq 0$. Then, from Eq. (2), we can see that the estimated SWI is a result of the autoregressive integrated moving average model given by:

$$c_n = c_{n-1} - \frac{1}{\alpha_0} \sum_{j=1}^{p} \alpha_j L^j (c_n - c_{n-1}) + \frac{1}{\alpha_0} \eta_n. \tag{4}$$

Thus, the behavior of the estimated SWI is very complicated because it is a weighted random walk process. Therefore, we agree with Mariano and Murasawa (2003) that the results of the SWI cannot be interpreted from an economic perspective. Moreover, a further difficulty of the Stock–Watson approach is that both the stationary and nonstationary indicators are included in the coincident indicators.

It is widely accepted that GDP is an inclusive measure of an economy's performance and, therefore, that it can be used as a real business cycle index. However, using GDP in this way does pose a number of problems, largely due to a lack of statistical sufficiency and the conflict between the need for immediate reports and the fact that GDP statistics are made quarterly, not monthly, and there is a long lag prior to their publication. It should be noted that Mariano and Murasawa (2003, 2010) added GDP together with coincident indicators to construct a business cycle index. Although this is an excellent concept, the problem inherent in the Stock–Watson approach cannot be completely resolved because it remains based on dynamic factor modeling.

3 Proposed approach

3.1 Model construction

Let $y_i(n)$ denote a monthly time series, which is one of m coincident indicators with $i = 1, 2, \ldots, m$. Generally, it is considered that each indicator (or transformed indicator) comprises nonstationary and stationary parts. Furthermore, the nonstationary part contains a trend component (which expresses the long-term tendency) and a seasonal component (which expresses a patterned variation that appears repeatedly every year). In contrast, the stationary part is considered to have a cyclical component (which is caused by business cycles) and an irregular component. In Kitagawa (2010) and Kitagawa and Gersh (1984), it is assumed that the cyclical component can be expressed by a stationary AR model, also called the AR component.

Let $g(y_i(n))$ be a one-to-one transformation of the time series $y_i(n)$ for the i-th indicator. Based on the above consideration, the model for $g(y_i(n))$ can be written as:

$$g(y_i(n)) = t_i(n) + s_i(n) + a_i(n) + \varepsilon_i(n) \quad (i = 1, 2 \ldots, m), \tag{5}$$

where $t_i(n)$, $s_i(n)$, and $a_i(n)$ denote the trend, seasonal, and cyclical components, respectively, and $\varepsilon_i(n)$ is the irregular component, also called the observation error. Note that the function g is determined so that $g(y_i(n))$ can be reasonably expressed in an additive multi-component form, as expressed in Eq. (5). As an example, a logarithmic function is frequently applied to g.

In this paper, we consider a case in which the time series $y_i(n)$ has been seasonally adjusted. Therefore, the seasonal component $s_i(n)$ does not necessarily need to be taken into account. Thus, the model in Eq. (5) is rewritten as follows:

$$g(y_i(n)) = t_i(n) + a_i(n) + \varepsilon_i(n) \quad (i = 1, 2 \ldots, m), \tag{6}$$

where the observation error $\varepsilon_i(n)$ is a random variable distributed with $\varepsilon_i(n) \sim \mathrm{N}(0, \sigma_i^2)$. To obtain meaningful estimates for every component, we employ a Bayesian approach and treat $t_i(n)$ and $a_i(n)$ as random variables. Thus, we need to introduce prior models based on the assumptions about the structure of each component.

As is typical for economic analysis, we assume that the trend component varies smoothly over time and that the cyclical component is globally stationary. Therefore, we introduce a second-order stochastic difference equation as follows:

$$t_i(n) = 2t_i(n-1) - t_i(n-2) + \omega_i(n) \quad (i = 1, 2, \ldots, m), \tag{7}$$

where $\omega_i(n)$ is a random error distributed with $\omega_i(n) \sim \mathrm{N}(0, \tau_i^2)$. Furthermore, we use an AR model for $a_i(n)$ as follows:

$$a_i(n) = \sum_{j=1}^{q_i} \beta_j^{(i)} a_i(n-j) + \psi_i(n) \quad (i = 1, 2, \ldots, m), \tag{8}$$

where q_i is the model order, $\{\beta_j^{(i)}; j = 1, 2, \ldots, q_i\}$ are the AR coefficients, and $\psi_i(n)$ is a Gaussian white noise distributed with $\psi_i(n) \sim \mathrm{N}(0, \eta_i^2)$.

The model in Eq. (6) is an observation model for the time series $g(y_i(n))$, and the models in Eqs (7) and (8) can be considered as a set of prior models for the trend $t_i(n)$ and cyclical $a_i(n)$ components, respectively. Obviously, Eqs (6)–(8) take the form of a Bayesian linear Gaussian model for the trend and cyclical components. For the case where the model for each value of i is individually managed, Kitagawa (2010) developed a maximum likelihood method to estimate the parameters σ_i^2, τ_i^2, η_i^2, and $\{\beta_j^{(i)}; j = 1, 2, \ldots, q_i\}$ based on the Kalman filter algorithm.

As mentioned in Section 1, Stock and Watson (1989, 1991) defined the SWI as co-movements in a set of macroeconomic variables. Therefore, it is constructed by extracting a common factor that is hidden in multiple macroeconomic time series data.

In contrast to the Stock–Watson approach, we consider that economic cycles can be organized using the basic classification of short- and long-term cycles. Short-term cycles generally mean business cycles or business fluctuations. Thus, to simplify the analysis of business fluctuations, it is necessary to separate business cycles from other economic cycles in the longer term.

Therefore, we expect that an index for business fluctuations would have the following properties:

1. It is globally stationary in mean.

2. It is based on a common factor from the stationary parts of all of the economic indicators used.

3. It has variations that are as large as possible and, therefore, it contains a larger amount of information on business fluctuations.

Thus, to construct an IBC using a common factor extracted from a set of cyclical components in a time series for coincident indicators, we take the common factor of the cyclical components $\{a_i(n); i = 1, 2 \ldots, m\}$. This can be easily achieved by applying the method of principal component analysis to the data for $\{a_i(n); i = 1, 2 \ldots, m\}$.

To derive an index that is free from the scale of the data, we construct the IBC based on the results of normalized cyclical components. That is, we normalize the cyclical components by:

$$a_i^*(n) = \frac{a_i(n)}{\mathrm{SD}\{a_i(n)\}} \quad (i = 1, 2, \ldots, m). \tag{9}$$

where $\mathrm{SD}\{a_i(n)\}$ is the standard deviation of the time series $\{a_i(n)\}$. Then, we construct the IBC using the first principal component of the normalized cyclical components $\{a_i^*(n); i = 1, 2, \ldots, m\}$ as follows:

$$b(n) = \sum_{i=1}^{m} w_i a_i^*(n), \tag{10}$$

where $\{w_i; i = 1, 2, \ldots, m\}$ is the principal component loadings for the first principal component, in which $\sum_{i=1}^{m} w_i^2 = 1$.

Hereinafter, the index $b(n)$ defined in Eq. (10) is referred to as the *IBC*, and the method to construct the IBC is called the *cyclical components synthesization* (CCS) approach. In a procedure to estimate the parameters, the cyclical component can be estimated as a stationary AR process with mean zero, so that the IBC becomes a stationary process with mean zero. Thus, as with the CI, the IBC can measure the speed and magnitude of business fluctuations. Moreover, the zero value of the IBC can be regarded as the midpoint between states of prosperity and depression.

3.2 Modeling and parameter estimation

For the models in Eqs (6)–(8), we make the following assumptions:

1. $\varepsilon_i(n)$, $\omega_i(n)$, and $\psi_i(n)$ are independent of one another for all values of i, n.

2. $\varepsilon_i(n_1)$ and $\varepsilon_\ell(n_2)$ are independent of each other when $i \neq \ell$ or $n_1 \neq n_2$.

3. $\omega_i(n_1)$ and $\omega_\ell(n_2)$ are independent of each other when $i \neq \ell$ or $n_1 \neq n_2$.

4. $\psi_i(n_1)$ and $\psi_\ell(n_2)$ are independent when $n_1 \neq n_2$ but may be dependent when $i \neq \ell$.

5. The parameters σ_i^2, τ_i^2, η_i^2, and $\{\beta_j^{(i)}; j = 1, 2, \ldots, q_i\}$ are constants over time.

Under the above assumptions, we can see that except for $t_i(n)$ and $a_i(n)$, we have $q_i + 3$ parameters for each indicator. When the values of these parameters are given, we can obtain the estimates for $t_i(n)$ and $a_i(n)$. Hereafter, for each value of i, based on Bayesian modeling, we refer to $t_i(n)$ and $a_i(n)$ as stochastic parameters. The other parameters, that is, σ_i^2, τ_i^2, η_i^2, and $\{\beta_j^{(i)}; j = 1, 2, \ldots, q_i\}$, are called the hyperparameters.

The fourth assumption is based on the consideration that among the cyclical components $a_i(n)$ with $i = 1, 2, \ldots, m$, there is a common factor to be estimated. Thus, in principle, $t_i(n)$ and $a_i(n)$ should be estimated simultaneously for all time points and all values of $i = 1, 2, \ldots, m$. That is, when the sample size for the time series $y_i(n)$ is N, it is necessary to estimate $\{t_i(n); n = 1, 2, \ldots, N; i = 1, 2, \ldots, m\}$, and $\{a_i(n); n = 1, 2, \ldots, N; i = 1, 2, \ldots, m\}$ simultaneously based on the estimates for the $\sum_i^m q_i + 3m$ hyperparameters. This is very difficult to do in practice.

To mitigate the difficulties in estimating hyperparameters, we develop a new estimation technique combining the maximum likelihood method and principal component analysis. A key quantity is the ratio of variances for error in the cyclical component to that in the trend component, that is, for $i = 1, 2, \ldots, m$, the proportion:

$$\lambda_i = \frac{\tau_i^2}{\eta_i^2} \tag{11}$$

is a key parameter for the present problem. Introducing this parameter, we can control the relative variations in the trend and cyclical components and, therefore, maintain the balance between the variances of errors in the models for the two components.

As mentioned in the preceding subsection, the first objective of this study is to obtain the IBC in Eq. (10) with a larger variance, and the second is to ensure the goodness of fit of the models to the data. Thus, we have two criteria for the modeling and parameter estimation: the first criterion is the variance of the constructed IBC and the second is the likelihood of the parameters. In the present paper, we first estimate the ratio λ_i by maximizing the contribution rate (CR) of the IBC, which is constructed as a principal component. Then, we estimate the other hyperparameters by maximizing the likelihood based on the constructed model.

Moreover, as will be shown in the following section, a set of initial estimates for the cyclical components $a_i(n)$ is necessary in an algorithm for parameter estimation, even though these initial estimates are tentative. As the initial estimates for $a_i(n)(i = 1, 2, \ldots, m)$, we will use the differences between $y_i(n)$ and its moving average with term $2L + 1$:

$$\widehat{a}_i(n)^{(0)} = y_i(n) - \frac{1}{2L+1} \sum_{\ell=-L}^{L} y_i(n+\ell) \tag{12}$$

for $n = L + 1, L + 2, \ldots, N - L$; otherwise, we will use $\widehat{a}_i(n)^{(0)} = 0$.

3.3 Standard score of a business cycle

The scale of the IBC is not essential for its application because it may change according to the data used. Thus, it is necessary to determine a standard to compare different IBC scales. To ensure easy understanding and application, we utilize the concept of a standard score (e.g., as used to evaluate students) and define the standard score of a business cycle (SSBC) as:

$$\mathrm{SSBC}(n) = 50 + \frac{10}{\mathrm{SD}\{b(n)\}} b(n)$$

with $\mathrm{SD}\{b(n)\}$ being the standard deviation of $b(n)$. Because the mean of $b(n)$ is zero, we can see that the mean and the standard deviation of $\mathrm{SSBC}(n)$ are 50 and 10, respectively. That is, the level of the SSBC is 50, representing the midpoint between the states of prosperity and depression.

To avoid conceptual confusion, we mainly focus on the IBC below.

4 Method for parameter estimation

4.1 Estimating the trend and cyclical components

To simplify the estimation problem, we tentatively assume that the parameters can be estimated individually for the model of each indicator with $i = 1, 2, \ldots, m$. To express the model in a state space representation, we define the state vector based on related quantities in the trend and cyclical components as:

$$\boldsymbol{x}_n^{(i)} = (t_i(n), t_i(n-1), a_i(n), a_i(n-1), \cdots, a_i(n-q_i+1))^{\mathrm{T}},$$

and we use the following matrices:

$$\boldsymbol{F}^{(i)} = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \beta_1^{(i)} & \beta_2^{(i)} & \cdots & \cdots & \beta_{q_i}^{(i)} \\ \vdots & \vdots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}^{\mathtt{T}}.$$

Moreover, to reduce the number of parameters, which will be estimated using numeric methods, we follow Kitagawa (2010) and combine:

$$\widetilde{\varepsilon}_i(n) = \frac{\varepsilon_i(n)}{\sigma_i}, \quad \widetilde{\omega}_i(n) = \frac{\omega_i(n)}{\sigma_i}, \quad \widetilde{\psi}_i(n) = \frac{\psi_i(n)}{\sigma_i},$$

with:

$$\widetilde{\tau}_i^2 = \frac{\tau_i^2}{\sigma_i^2}, \quad \widetilde{\eta}_i^2 = \frac{\eta_i^2}{\sigma_i^2}. \tag{13}$$

Then, the models in Eqs (6)–(8) can be expressed by the following state space representation:

$$\boldsymbol{x}_n^{(i)} = \boldsymbol{F}^{(i)}\boldsymbol{x}_{n-1}^{(i)} + \boldsymbol{G}\boldsymbol{v}_n^{(i)}, \tag{14}$$

$$y_i(n) = \boldsymbol{H}\boldsymbol{x}_n^{(i)} + \varepsilon_i(n), \tag{15}$$

where $\boldsymbol{v}_n^{(i)} = (\widetilde{\omega}_i(n), \widetilde{\psi}_i(n))^{\mathtt{T}}$. In the state space model comprising Eqs (14) and (15), the time-varying stochastic parameters $t_i(n)$ and $a_i(n)$ are included in the state vector $\boldsymbol{x}_n^{(i)}$. Therefore, their estimates can be obtained from the estimate of $\boldsymbol{x}_n^{(i)}$, under the assumption that the AR model order q_i and the hyperparameters, $\widetilde{\tau}_i^2$, $\widetilde{\eta}_i^2$, $\boldsymbol{\beta}^{(i)} = (\beta_1^{(i)}, \beta_2^{(i)}, and \ldots, \beta_{q_i}^{(i)})^{\mathtt{T}}$, are given.

Let $\boldsymbol{x}_0^{(i)}$ denote the state at the initial time point and let $\mathrm{Y}_k^{(i)} = \{y_i(1), y_i(2), \ldots, y_i(k)\}$ denote a set of observations for $y_i(n)$ up to the time point k. Assume that $\boldsymbol{x}_0^{(i)} \sim \mathrm{N}(\boldsymbol{x}_{0|0}^{(i)}, \boldsymbol{C}_{0|0}^{(i)})$. It is well known that the density function $f(\boldsymbol{x}_n^{(i)}|\mathrm{Y}_k^{(i)})$ for the state $\boldsymbol{x}_n^{(i)}$, conditional on $\mathrm{Y}_k^{(i)}$, is Gaussian and, therefore, it is only necessary to obtain the mean $\boldsymbol{x}_{n|k}^{(i)}$ and the covariance matrix $\boldsymbol{C}_{n|k}^{(i)}$ of $\boldsymbol{x}_n^{(i)}$ with respect to $f(\boldsymbol{x}_n^{(i)}|\mathrm{Y}_k^{(i)})$.

When we are given the values of q_i, $\widetilde{\tau}_i^2$, $\widetilde{\eta}_i^2$, and $\boldsymbol{\beta}^{(i)}$, the distribution $\mathrm{N}(\boldsymbol{x}_{0|0}^{(i)}, \boldsymbol{C}_{0|0}^{(i)})$ for $\boldsymbol{x}_0^{(i)}$, and a set of observations for $y_i(n)$ up to the time point N, then the estimates for the state $\boldsymbol{x}_n^{(i)}$ can be obtained using the well-known Kalman filter (for $n = 1, 2, \ldots, N$) and fixed-interval smoothing (for $n = N-1, N-2, \ldots, 1$) recursively as follows (see Kitagawa, 2010):

[**Kalman filter**]

$$\begin{aligned}
\boldsymbol{x}^{(i)}_{n|n-1} &= \boldsymbol{F}^{(i)}\boldsymbol{x}^{(i)}_{n-1|n-1},\\
\boldsymbol{C}^{(i)}_{n|n-1} &= \boldsymbol{F}^{(i)}\boldsymbol{C}^{(i)}_{n-1|n-1}(\boldsymbol{F}^{(i)})^{\mathtt{T}} + \boldsymbol{G}\boldsymbol{Q}_i\boldsymbol{G}^{\mathtt{T}}.\\
\boldsymbol{L}^{(i)}_{n} &= \boldsymbol{C}^{(i)}_{n|n-1}\boldsymbol{H}^{\mathtt{T}}(\boldsymbol{H}\boldsymbol{C}^{(i)}_{n|n-1}\boldsymbol{H}^{\mathtt{T}} + \boldsymbol{R}_i)^{-1},\\
\boldsymbol{x}^{(i)}_{n|n} &= \boldsymbol{x}^{(i)}_{n|n-1} + \boldsymbol{L}^{(i)}_{n}(y_n - \boldsymbol{H}\boldsymbol{x}^{(i)}_{n|n-1}),\\
\boldsymbol{C}^{(i)}_{n|n} &= (\boldsymbol{I} - \boldsymbol{L}^{(i)}_{n}\boldsymbol{H})\boldsymbol{C}^{(i)}_{n|n-1}.
\end{aligned}$$

[**Fixed-interval smoothing**]

$$\begin{aligned}
\boldsymbol{P}^{(i)}_{n} &= \boldsymbol{C}^{(i)}_{n|n}(\boldsymbol{F}^{(i)})^{\mathtt{T}}(\boldsymbol{C}^{(i)}_{n+1|n})^{-1},\\
\boldsymbol{x}^{(i)}_{n|N} &= \boldsymbol{x}^{(i)}_{n|n} + \boldsymbol{P}^{(i)}_{n}(\boldsymbol{x}^{(i)}_{n+1|N} - \boldsymbol{x}^{(i)}_{n+1|n}),\\
\boldsymbol{C}^{(i)}_{n|N} &= \boldsymbol{C}^{(i)}_{n|n} + \boldsymbol{P}^{(i)}_{n}(\boldsymbol{C}^{(i)}_{n+1|N} - \boldsymbol{C}^{(i)}_{n+1|n})(\boldsymbol{P}^{(i)}_{n})^{\mathtt{T}}.
\end{aligned}$$

Here, $\boldsymbol{R}_i = 1$, $\boldsymbol{Q}_i = \mathrm{diag}(\widetilde{\tau}_i^2, \widetilde{\eta}_i^2)$, and $\boldsymbol{I}$ denotes an identity matrix.

Then, the posterior distribution of $\boldsymbol{x}^{(i)}_n$ can be defined using $\boldsymbol{x}^{(i)}_{n|N}$ and $\boldsymbol{C}^{(i)}_{n|N}$. Subsequently, the estimates for the time-varying stochastic parameters $t_i(n)$ and $a_i(n)$ can be obtained because the state space model described by Eqs (14) and (15) incorporates $t_i(n)$ and $a_i(n)$ in the state vector $\boldsymbol{x}^{(i)}_n$. Hereinafter, the estimates of $t_i(n)$ and $a_i(n)$ are denoted by $\widehat{t}_i(n)$ and $\widehat{a}_i(n)$, respectively.

4.2 Estimating the hyperparameters

For $i = 1, 2, \ldots, m$, Eqs (11) and (13) lead to:

$$\lambda_i = \frac{\widetilde{\tau}_i^2}{\widetilde{\eta}_i^2}.$$

Thus, we have:

$$\widetilde{\tau}_i^2(\lambda_i) = \lambda_i\widetilde{\eta}_i^2,$$

that is, $\widetilde{\tau}_i^2$ corresponds with λ_i one-to-one for the given value of $\widetilde{\eta}_i^2$. Therefore, we regard λ_i as a hyperparameter instead of $\widetilde{\tau}_i^2$. When the values of q_i, σ_i^2, and λ_i, together with the time series data $Y^{(i)}_N = \{y_i(1), y_i(2), \ldots, y_i(N)\}$, are given, a likelihood function for the hyperparameters $\widetilde{\eta}_i^2$ and $\boldsymbol{\beta}^{(i)}$, which is derived from the algorithm of the Kalman filter, is as follows:

$$f^{(i)}(Y^{(i)}_N|\sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}) = \prod_{n=1}^{N} f^{(i)}_n(y_i(n)|Y^{(i)}_{n-1}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}),$$

where $f_n^{(i)}(y_i(n)|Y_{n-1}^{(i)}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)})$ is the conditional density function of $y_i(n)$ given the past history $Y_{n-1}^{(i)} = \{y_i(n-1), y_i(n-2), \ldots\}$ and λ_i. Assume that $Y_0^{(i)} = \{y_i(0), y_i(-1), \ldots\}$ is an empty set. Then, we have:

$$f_1^{(i)}(y_i(1)|Y_0^{(i)}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}) = f_1^{(i)}(y_i(1)|\sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}).$$

By taking the logarithm of $f^{(i)}(Y_N^{(i)}|\sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)})$, the log-likelihood is obtained as:

$$\begin{aligned} LL_i(\widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}|\sigma_i^2, \lambda_i) &= \log f^{(i)}(Y_N^{(i)}|\sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}) \\ &= \sum_{n=1}^{N} \log f_n^{(i)}(y_i(n)|Y_{n-1}^{(i)}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}). \end{aligned} \tag{16}$$

As given by Kitagawa (2010), when using the Kalman filter, the conditional density $f_n^{(i)}(y_i(n)|Y_{n-1}^{(i)}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)})$ has a normal density given by:

$$f_n^{(i)}(y_i(n)|Y_{n-1}^{(i)}, \sigma_i^2, \widetilde{\tau}_i^2(\lambda_i); \widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}) = \frac{1}{\sqrt{2\pi\sigma_i^2 w_{n|n-1}^{(i)}}} \exp\left\{ - \frac{(y_i(n) - \widehat{y}_{n|n-1}^{(i)})^2}{2\sigma_i^2 w_{n|n-1}^{(i)}} \right\}, \tag{17}$$

where $\widehat{y}_{n|n-1}^{(i)}$ is the one-step ahead prediction for y_n and $w_{n|n-1}^{(i)}$ is the variance of the predictive error. They are given, respectively, by:

$$\widehat{y}_{n|n-1}^{(i)} = \boldsymbol{H}\boldsymbol{x}_{n|n-1}^{(i)}, \quad w_{n|n-1}^{(i)} = \boldsymbol{H}\boldsymbol{C}_{n|n-1}^{(i)}\boldsymbol{H}^{\mathrm{T}} + \sigma_i^2.$$

Thus, for a given value of λ_i, the estimates of the hyperparameters $\widetilde{\eta}_i^2$ and $\boldsymbol{\beta}^{(i)}$ can be obtained numerically using the maximum likelihood method. That is, the hyperparameters can be estimated numerically by maximizing $LL_i(\widetilde{\eta}_i^2, \boldsymbol{\beta}^{(i)}|\sigma_i^2, \lambda_i)$ in Eq. (16) together with Eq. (17). Furthermore, based on Kitagawa (2010), the estimate $\widehat{\sigma}_i^2$ for σ_i^2 is obtained analytically by:

$$\widehat{\sigma}_i^2 = \frac{1}{N} \sum_{n=1}^{N} \frac{(y_i(n) - \widehat{y}_{n|n-1}^{(i)})^2}{w_{n|n-1}^{(i)}}. \tag{18}$$

It is notable that we can increase the efficiency of estimating $\boldsymbol{\beta}^{(i)}$ when we apply the fast recursive estimation method proposed by Kyo and Noda (2011). Additionally, let $\widehat{\widetilde{\tau}}_i^2$ and $\widehat{\lambda}_i$ denote the estimates of $\widetilde{\tau}_i^2$ and λ_i, respectively. Then, from the above settings, the estimates for η_i^2 and τ_i^2 are given, respectively, by:

$$\widehat{\eta}_i^2 = \widehat{\sigma}_i^2 \widehat{\widetilde{\eta}}_i^2, \quad \widehat{\tau}_i^2 = \widehat{\lambda}_i \widehat{\eta}_i^2. \tag{19}$$

It is obvious that the estimates for the hyperparameters $\widetilde{\eta}_i^2$ and $\boldsymbol{\beta}^{(i)}$ depend on the value of λ_i. Furthermore, from the algorithm of the Kalman filter, we can see that the estimates for the elements of the state vector are a set of functions of λ_i. That is, under a given value of λ_i, we can estimate the hyperparameters $\widetilde{\eta}_i^2$ and $\boldsymbol{\beta}^{(i)}$. Thus, as a function of λ_i, the estimate for the cyclical component $a_i(n)$ can be obtained using the Kalman filter and fixed-interval smoothing. Assume that the estimates of the cyclical components for the other indicators (except that of the i-th) are given in advance. Then, we apply the principal component method for normalized estimates of the cyclical components and we can obtain the first principal component as the IBC $b(n)$, as defined in Eq. (10).

As mentioned in subsection 3.1, as a signal of business fluctuations, we expect that our IBC will have larger variations. Therefore, we estimate the hyperparameter λ_i by numerically maximizing the variance $\mathrm{Var}\{b(n)\}$ of $b(n)$. In the same way, the value of the AR model order q_i is determined by the maximization of $\mathrm{Var}\{b(n)\}$.

4.3 The algorithm

To summarize the scheme for parameter estimation, the algorithm for the CCS approach is obtained as follows:

1. Assume $s = 0$, set an appropriate value for L, and then give the initial estimates:
$$\{\widehat{a}_i(n)^{(0)}; n = 1, 2, \ldots, N; i = 1, 2, \ldots, m\}$$
for the cyclical components using Eq. (12), with N being the length of the time series data.

2. For $i = 1, 2, \ldots, m$, perform the following steps.

3. Replace the value of s with $s + 1$.

   (a) Estimate the hyperparameters $\widetilde{\eta}_i^2$ and $\boldsymbol{\beta}^{(i)}$ numerically by maximizing the log-likelihood in Eq. (16) together with Eq. (17) under the given value of λ_i.

   (b) Compute the estimate of σ_i^2 using Eq. (18).

   (c) Obtain the estimates $\{\widehat{a}_i(n)^{(s)}; n = 1, 2, \ldots, N\}$ for the cyclical component $a_i(n)$ by the Kalman filter and a fixed-interval smoothing algorithm.

(d) Compute the normalized values $\{\widehat{a}_i^*(n)^{(s)}; n = 1, 2, \ldots, N\}$ for $\{\widehat{a}_i(n)^{(s)}; n = 1, 2, \ldots, N\}$ using Eq. (9).

(e) Perform the principal component analysis for:

$$\boldsymbol{a}^*(n)^{(i,s)} = \begin{cases} (\widehat{a}_1^*(n)^{(s)}, \widehat{a}_2^*(n)^{(s-1)}, \ldots \widehat{a}_m^*(n)^{(s-1)})^{\mathrm{T}} & \text{if } i = 1, \\ (\widehat{a}_1^*(n)^{(s)}, \widehat{a}_2^*(n)^{(s)}, \widehat{a}_3^*(n)^{(s-1)}, \ldots \widehat{a}_m^*(n)^{(s-1)})^{\mathrm{T}} & \text{if } i = 2, \\ (\widehat{a}_1^*(n)^{(s)}, \ldots, \widehat{a}_i^*(n)^{(s)}, & \\ \qquad \widehat{a}_{i+1}^*(n)^{(s-1)}, \ldots \widehat{a}_m^*(n)^{(s-1)})^{\mathrm{T}} & \text{if } 2 < i < m-1, \\ (\widehat{a}_1^*(n)^{(s)}, \ldots, \widehat{a}_{m-1}^*(n)^{(s)}, \widehat{a}_m^*(n)^{(s-1)})^{\mathrm{T}} & \text{if } i = m-1, \\ (\widehat{a}_1^*(n)^{(s)}, \widehat{a}_2^*(n)^{(s)}, \ldots, \widehat{a}_m^*(n)^{(s)})^{\mathrm{T}} & \text{if } i = m, \end{cases}$$

Treat $\{\boldsymbol{a}^*(n)^{(i,s)}; n = 1, 2, \ldots, N\}$ as a set of m-variate time series data, and then obtain:

$$b(n)^{(i,s)} = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{a}^*(n)^{(i,s)}$$

as the first principal component of $\boldsymbol{a}^*(n)^{(i,s)}$ with $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$ being the vector of the principal component loadings, and $\mathrm{Var}\{b(n)^{(i,s)}\}$ being its variance.

(f) Estimate λ_i and q_i by numerically maximizing $\mathrm{Var}\{b(n)^{(i,s)}\}$.

4. When the value of $\mathrm{Var}\{b(n)^{(m,s)}\}$ almost converges to a fixed level, proceed to the next step; otherwise, return to the beginning of step 2.

5. End the calculation and take the estimates corresponding to the values of λ_i and q_i as the final results.

6. Calculate the estimates for η_i^2 and τ_i^2 using Eq. (19).

5 Evaluation of the IBC for the analysis of Japanese business cycles

5.1 Comparing the performance with the coincident CI

The main aim of the coincident CI is to measure the speed and magnitude of economic fluctuations in a particular period. It is designed to be a useful tool to analyze current business conditions. Thus, it is important that we compare the performance of the IBC with that of the coincident CI.

Figure 1 shows the coincident CI in Japan from January 1985 to December 2015. We can observe that the CI varies around a horizontal level of 100; that is, the CI is almost stationary in mean for the observed period.

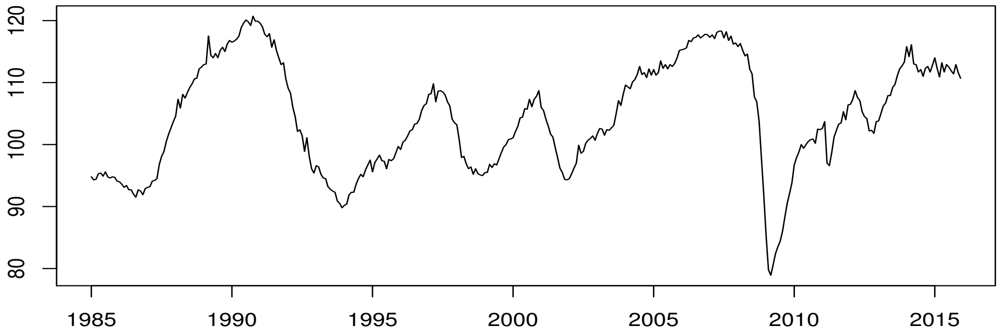

Figure 1: Time series data for the CI in Japan (January 1985 to December 2015); Source: ESRI, Cabinet Office, Government of Japan

The coincident CI comprises the quantitative changes in indicators that are sensitive to business cycle movements. The following 10 indicators were used to construct the CI:

C1: Index of industrial production (mining and manufacturing)

C2: Index of producers' shipments (producer goods for mining and manufacturing)

C3: Index of producers' shipments of durable consumer goods

C4: Index of nonscheduled working hours (industries covered)

C5: Index of producers' shipments (investment goods excluding transport equipment)

C6: Retail sales value (change from previous year)

C7: Wholesale sales value (change from previous year)

C8: Operating profits (all industries)

C9: Index of shipments in small and medium-sized enterprises

C10: Effective job offer rate (excluding new graduates)

The functions used to transform the indicators in the model in Eq. (6) are given in Table 1. To express each function $g(\cdot)$ using the additive multi-components form in Eq. (6) and to equalize the error variance for each component over time, a

Table 1: Functions for transforming the indicators

Name of indicator	C1	C2	C3	C4	C5
Indicator	y_1	y_2	y_3	y_4	y_5
Function $g(\cdot)$	$\log(y_1)$	$\log(y_2)$	$\log(y_3)$	$\log(y_4)$	$\log(y_5)$
Name of indicator	C6	C7	C8	C9	C10
Indicator	y_6	y_7	y_8	y_9	y_{10}
Function $g(\cdot)$	y_6	y_7	$\log(y_8)$	$\log(y_9)$	$\log(y_{10})$

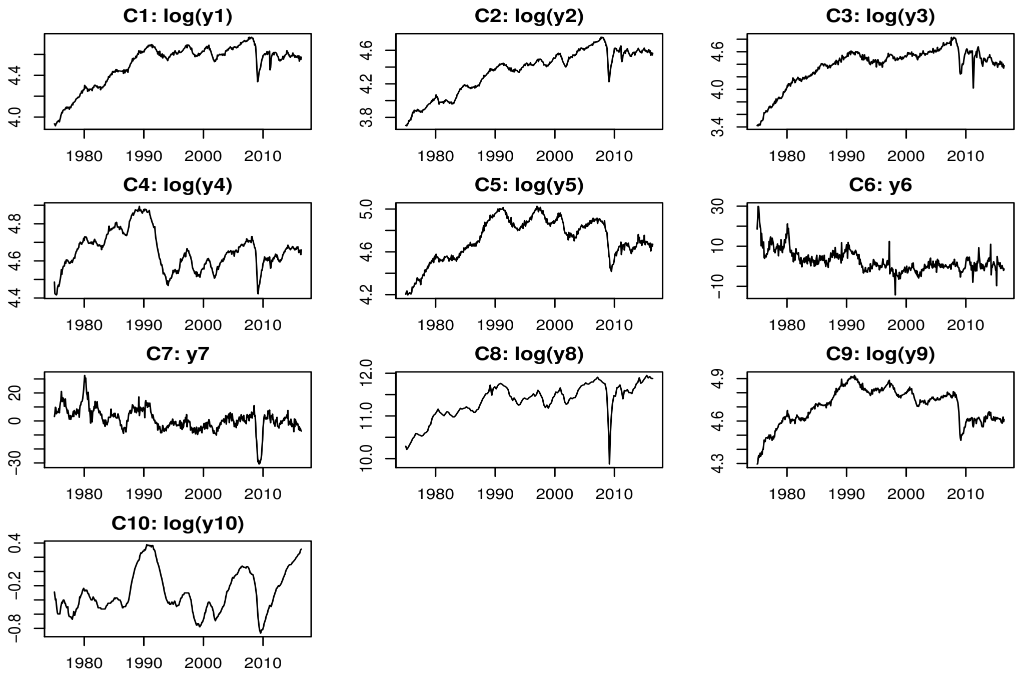

Figure 2: Time series data for $g(y_i(n))$ (January 1975 to June 2016); Source for $y_i(n)$: ESRI, Japanese Government's Cabinet Office

logarithmic transformation is employed for most indicators, excluding C6 and C7. (Note that for C6 and C7, the identical transformation is used because there are some negative values in the data sets, which means that a logarithmic transformation cannot be applied.)

Figure 2 shows the transformed time series data for each indicator for the period January 1975 to June 2016. Note that to correspond with the end time point of the CI series, in the following analysis we only used part of the data set, namely the data for the period January 1975 to December 2015.

As mentioned in the algorithm in subsection 4.3, the computation for parameter estimation is based on a set of initial estimates of the cyclical components. The value of L used in Eq. (12) to obtain the initial estimates of the cyclical components is determined as $L = 12$.

The computation for parameter estimation is achieved by repetition. Figure 3 shows the variation of the CR for the first principal component with respect to the number of repetitions. It can be seen that the CR converges to a limited larger value (approximately 0.725) and it steadies from the sixth repetition.

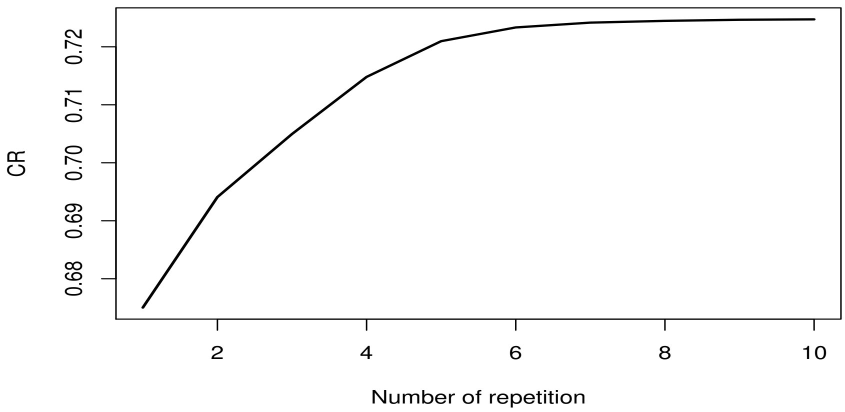

Figure 3: The contribution rate for each iteration

For reference purposes, the estimated hyperparameters at each iteration are listed in Table 2.

Table 2: Main results for hyperparameter estimation

i	1	2	3	4	5
q_i	10	10	10	1	2
$\widehat{\sigma}_i^2$	3.75×10^{-8}	8.74×10^{-8}	1.04×10^{-4}	4.80×10^{-6}	1.03×10^{-7}
$\widehat{\tau}_i^2$	5.23×10^{-8}	4.24×10^{-8}	6.42×10^{-8}	1.90×10^{-7}	3.79×10^{-7}
$\widehat{\eta}_i^2$	3.03×10^{-4}	3.12×10^{-4}	1.30×10^{-3}	1.39×10^{-4}	4.81×10^{-4}
i	6	7	8	9	10
q_i	2	12	7	1	1
$\widehat{\sigma}_i^2$	5.36×10^{-2}	7.19×10^{-4}	1.69×10^{-5}	6.11×10^{-8}	4.90×10^{-5}
$\widehat{\tau}_i^2$	5.58×10^{-5}	1.58×10^{-4}	2.35×10^{-7}	1.69×10^{-7}	1.02×10^{-6}
$\widehat{\eta}_i^2$	5.48	7.19	1.07×10^{-3}	2.53×10^{-4}	4.63×10^{-4}

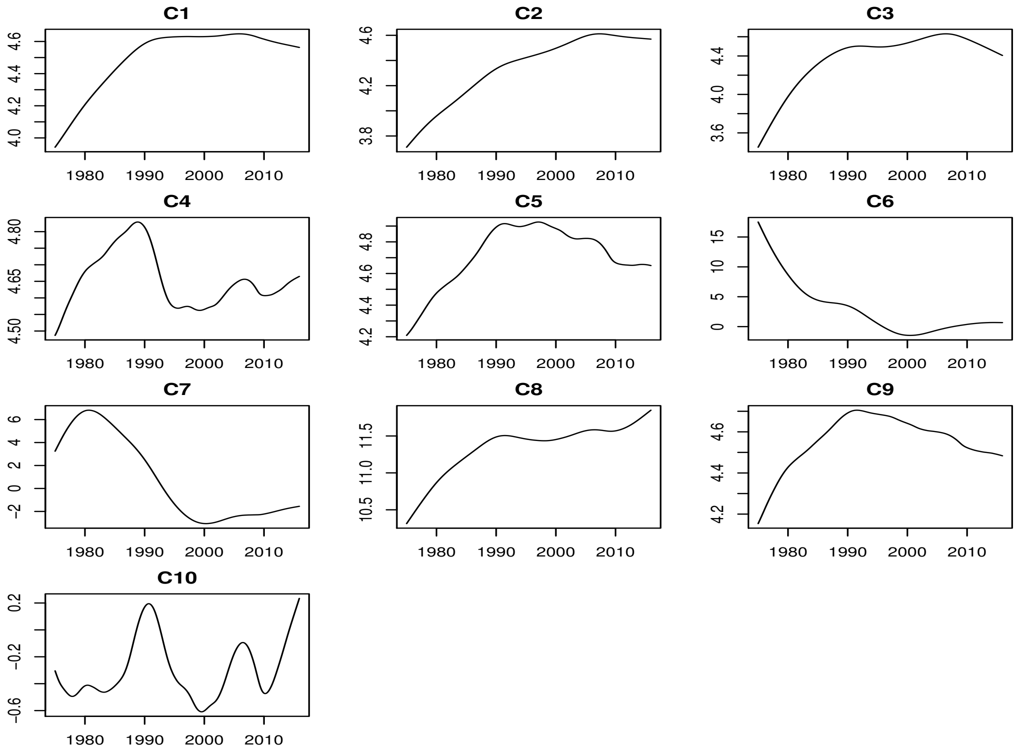

Figure 4: Estimated trend components (January 1975 to December 2015)

Figure 4 shows the estimated trend components for each transformed indicator. We can see that the behavior of each trend is distinct and, therefore, it may be difficult to extract a common factor from the estimated trend components.

Figure 5 shows estimated cyclical components for each transformed indicator. The correlation matrix for the estimated cyclical components is given in Table 3. (Please note that because the matrix is symmetric, only the elements below the diagonal are shown.) This table illustrates that there is a very high positive correlation between pairs among the estimated cyclical components. Thus, it is possible to extract a common factor using the first principal component composed of the estimated cyclical components with a very high CR.

Thus, to construct the IBC, we use principal component analysis based on the correlation matrix shown in Table 3. The largest eigenvalue for the correlation matrix is 7.247, and the corresponding eigenvector is:

$$\boldsymbol{w} = (0.363, 0.341, 0.296, 0.335, 0.336, 0.164, 0.294, 0.338, 0.342, 0.309)^{\mathtt{T}}.$$

From these results, we can see that the CR for the first principal component is 72.47%

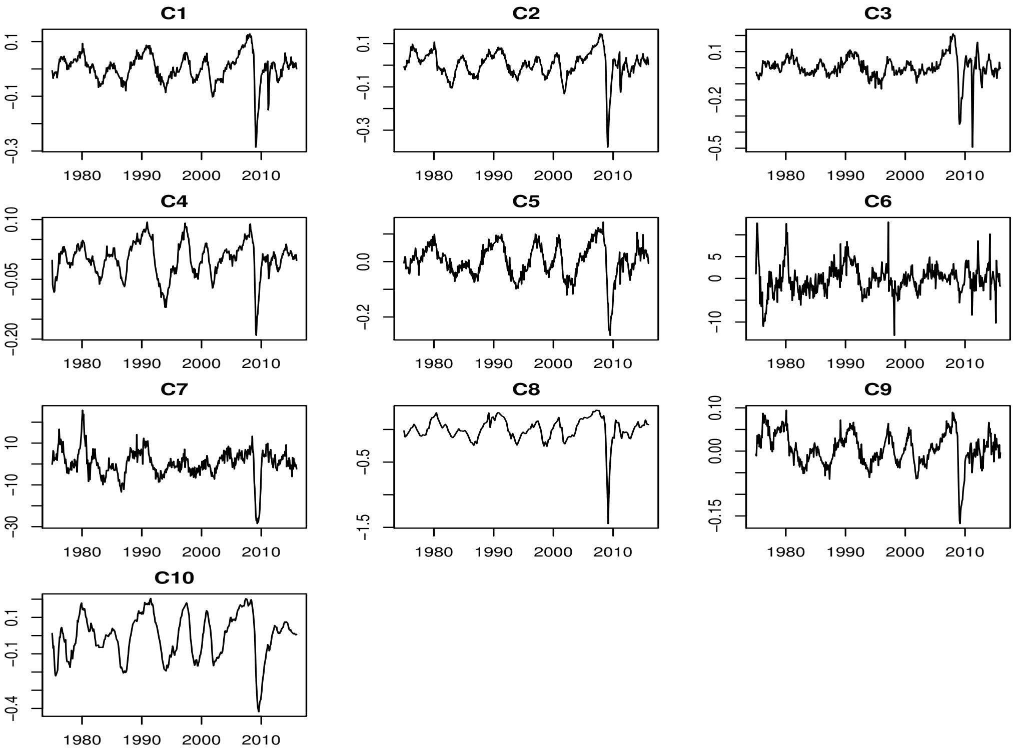

Figure 5: Estimated cyclical components (January 1975 to December 2015)

Table 3: Correlation matrix for the estimated AR components

	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
C1	1									
C2	0.942	1								
C3	0.827	0.746	1							
C4	0.865	0.794	0.667	1						
C5	0.880	0.750	0.653	0.813	1					
C6	0.358	0.324	0.289	0.375	0.344	1				
C7	0.707	0.709	0.523	0.646	0.657	0.429	1			
C8	0.876	0.852	0.702	0.798	0.767	0.376	0.732	1		
C9	0.919	0.900	0.688	0.788	0.851	0.297	0.735	0.774	1	
C10	0.769	0.622	0.587	0.800	0.851	0.331	0.572	0.737	0.694	1

because the sum of the variances for each of the normalized cyclical components is 10. Moreover, $\boldsymbol{w}$ is equivalent to the vector of the principal component loadings for the IBC. An element in the vector of the principal component loadings expresses how the IBC depends on the corresponding indicator. For example, $w_1 = 0.363$ is the maximum element and $w_9 = 0.342$ is the second maximum element in the vector of principal component loadings. Hence, we can say that the index of industrial production (C1) has the largest effect on the IBC, the index of shipments in small and medium-sized enterprises (C9) has the second largest effect, and so on.

Based on the normalized estimates for the cyclical components and the vector of principal component loadings $\boldsymbol{w}$, the estimated IBC, say $b(n)$, is computed using Eq. (12). Figure 6 shows the time series of the estimated IBC for the period January 1975 to December 2015. Here, the estimated IBC is temporarily called the IBC-10 because it is obtained based on the 10 indicators. It can be observed that the IBC-10 varies around zero, and then drops suddenly in the first half of 2009 after the collapse of the Lehman Brothers.

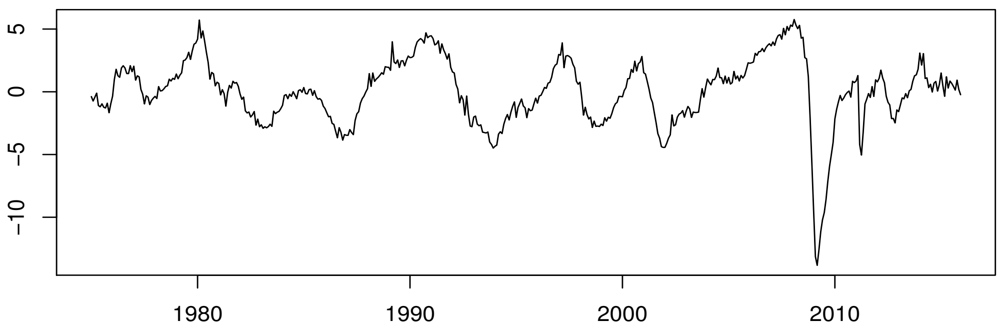

Figure 6: Time series for IBC-10 (January 1975 to December 2015)

To illustrate how the IBC-10 correlates with the coincident CI, we create a scatter diagram (see Figure 7) for the period January 1985 to December 2015. Note that the correlation coefficient between these two indexes is 0.8966. This implies that there is a very high positive correlation between the IBC-10 and the CI.

An important issue is: how well does the IBC-10 express the state of business fluctuations? It is difficult to examine the performance of a business index because we have no objective standard that expresses the level of business fluctuations. However, as mentioned in subsection 2.3, it is widely accepted that GDP is an inclusive measure of economic performance and Mariano and Murasawa (2010) used real GDP to measure business cycles. Thus, we use real GDP as an expediential reference for business cycles.

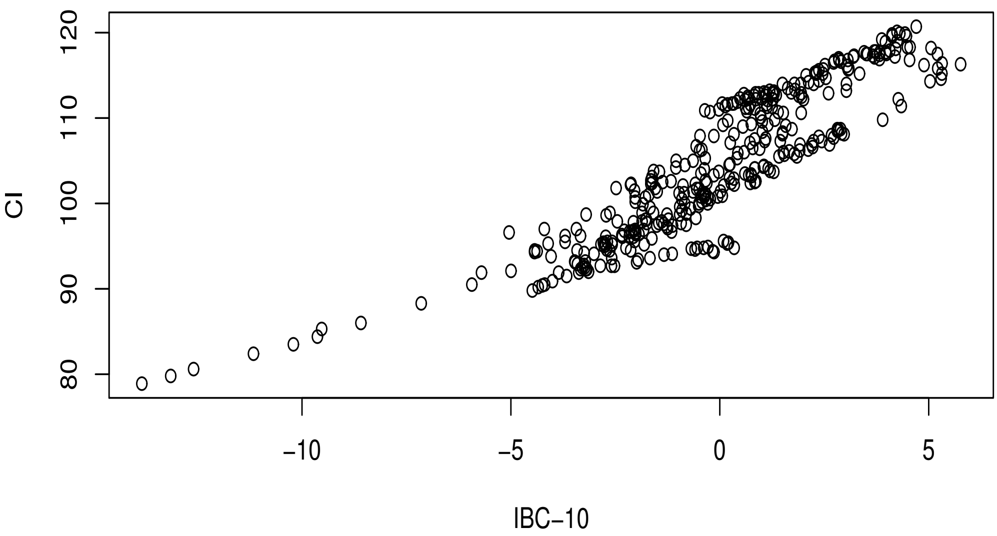

Figure 7: Scatter diagram of IBC-10 and CI (January 1985 to December 2015)

Similar to the model in Eq. (1), we employ the following model:

$$\log(y_n) = t_n + s_n + a_n + \varepsilon_n, \quad \varepsilon_n \sim \mathrm{N}(0, \sigma^2),$$

where y_n denotes the quarterly time series data on real GDP in Japan, and t_n, s_n, and a_n express the trend, seasonal, and cyclical components, respectively. Furthermore, ε_n is the irregular component. We use the following trend component, seasonal component, and AR models, respectively:

$$t_n = 2t_{n-1} - t_{n-2} + v_{n1}, \quad v_{n1} \sim \mathrm{N}(0, \tau_1^2),$$

$$s_n = -(s_{n-1} + s_{n-2} + s_{n-3}) + v_{n2}, \quad v_{n2} \sim \mathrm{N}(0, \tau_2^2),$$

$$a_n = \sum_{j=1}^{q} \beta_j a_{n-j} + v_{n3}, \quad v_{n3} \sim \mathrm{N}(0, \tau_3^2),$$

to express the priors for each component. Then, the logarithm of real GDP (log-GDP) can be decomposed into the trend, seasonal, and cyclical components together with the irregular component. A detailed explanation of the modeling and parameter estimation can be found in Kitagawa (2010).

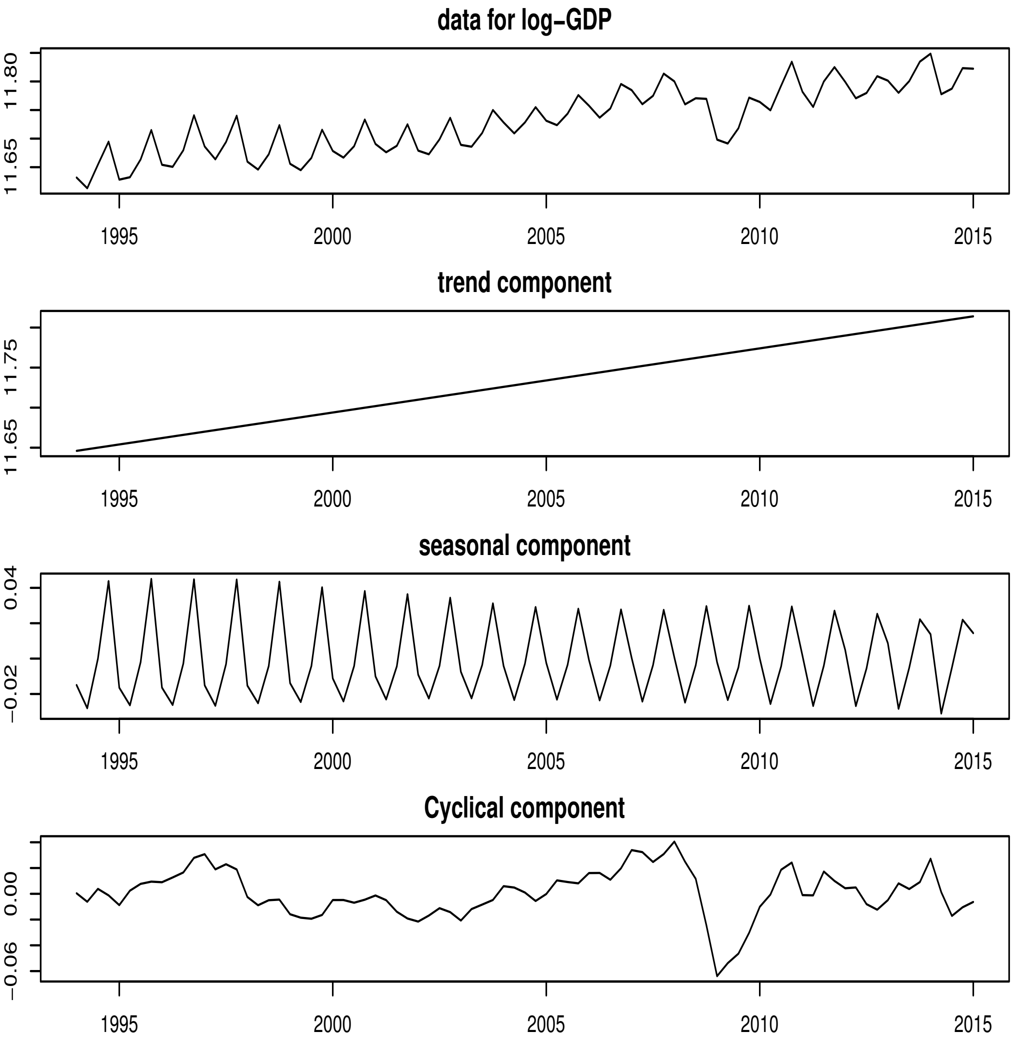

Figure 8: Results for decomposition of log-GDP in Japan (Q1, 1994 to Q1, 2015)

Figure 8 shows the results for the decomposition of log-GDP in Japan for the first quarter (Q1) of 1994 to Q1 of 2015.

Here, we take the cyclical component of log-GDP as a measure of a business cycle, abbreviated to cyclical log-GDP. Thus, the performance of an IBC can be measured using its correlation with the cyclical log-GDP. We compare the performance of the IBC-10 and CI by comparing their correlation coefficients with the cyclical log-GDP from Q1 1994 to Q1 2015. Note that for the correspondence to the quarterly GDP data, we take the average for the 3 months as the value for each quarter for the IBC-10 and CI. The line graphs for the time series data on the cyclical log-GDP, the IBC-10, and the CI are shown in Figure 9. The graphs show that the behaviors for these three time series have a high level of similarity.

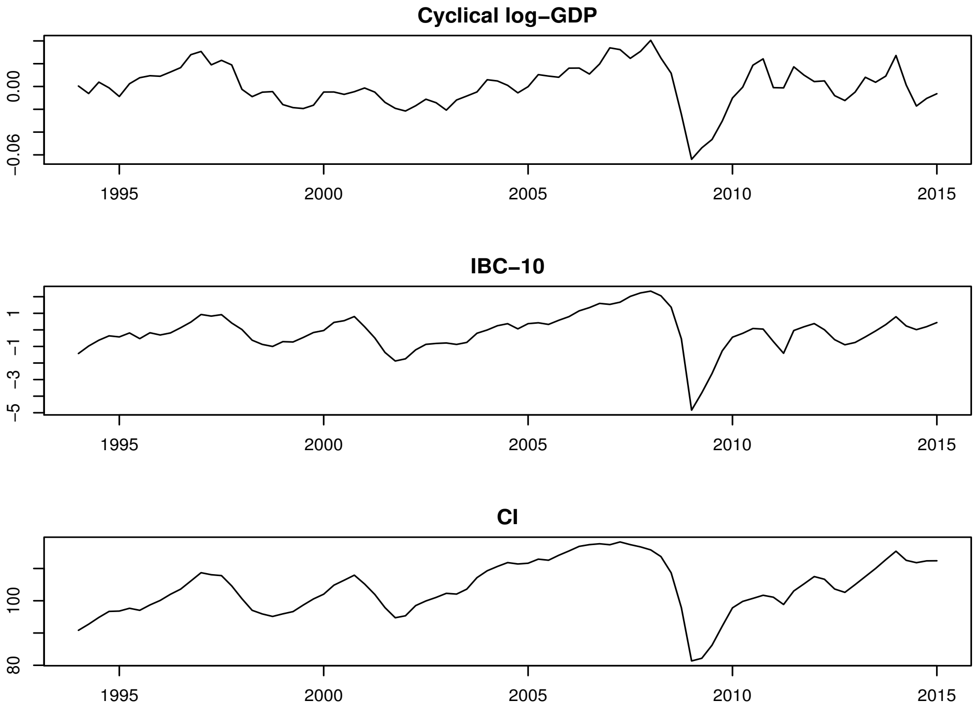

Figure 9: Cyclical log-GDP, IBC-10, and CI (Q1, 1994 to Q1, 2015

Figure 10 shows scatter diagrams for the cyclical log-GDP with the CI (left diagram) and the IBC-10 (right diagram). The correlation coefficients of the cyclical log-GDP with the CI and IBC-10 are 0.7181 and 0.8524, respectively. Thus, the IBC-10 has a higher correlation with the cyclical log-GDP than the CI, which implies that the IBC-10 performs better than the coincident CI.

5.2 Comparing the performance with the NBI

Now, we compare the performance of the proposed IBC, based on the CCS approach, with a Stock–Watson index. The NBI is constructed based on the Stock–Watson dynamic factor model by the Nihon Keizai Shimbun Inc. in Japan. Thus, we can compare the performance of our CCS approach with the Stock–Watson approach, using the dynamic factor modeling method, by comparing the performance of the IBC with that of the NBI.

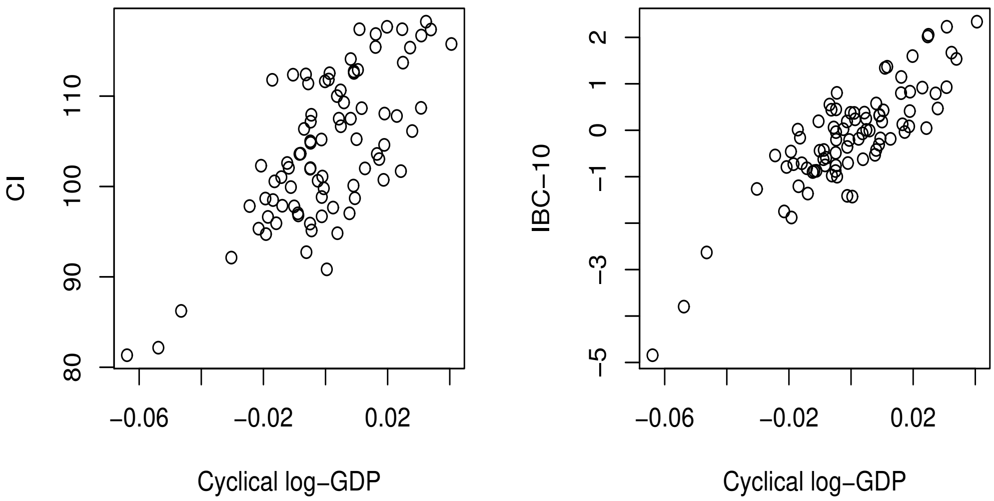

Figure 10: Scatter diagrams of the cyclical log-GDP with CI and IBC-10 (Q1, 1994 to Q1, 2015)

Figure 11 shows a set of monthly time series data for the NBI for January 1973 to June 2016. This data set was obtained from the Nihon Keizai Shimbun, Inc. website (`http://www.nikkei.com/biz/report/nkidx/`). From the figure, we can see that the NBI time series is not globally stationary in mean; that is, a trend component exists in the NBI series. Thus, compared with the IBC, the time series for NBI is difficult to explain because its behavior is somewhat complicated.

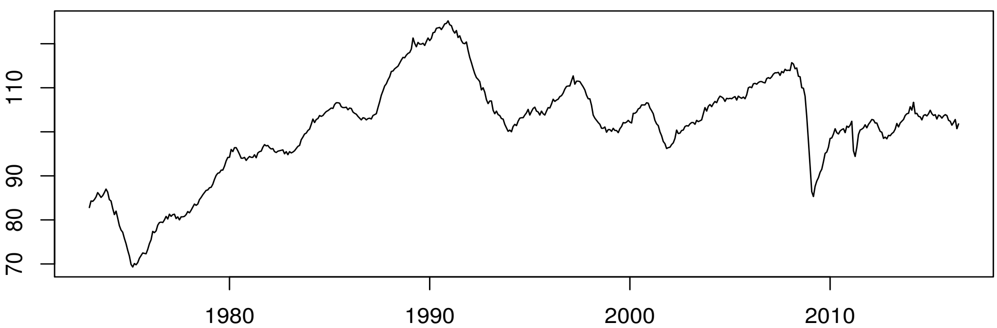

Figure 11: Time series data for NBI (January 1973 to June 2016); Source: Nihon Keizai Shimbun, Inc.

The NBI is constructed using time series data for the following four indicators:

NBI-y1: Index of industrial production (mining and manufacturing), C1

NBI-y2: Index of nonscheduled working hours (industries covered), C4

NBI-y3: Effective job offer rate (excluding new graduates), C10

NBI-y4: Total retail and wholesale sales value (ratio to the same month of the previous year, in %)

Note that the first three indicators, NBI-y1, NBI-y2, and NBI-y3, are the same as C1, C4, and C10, respectively. Because these indicators were used to construct the current CI and IBC-10, which we have illustrated in Figure 2, we need only show the data for NBI-y4. The monthly time series data for NBI-y4 from January 1980 to June 2016 are shown in Figure 12. Note that this data set is obtained from the website of the Ministry of Economy, Trade, and Industry (`http://www.meti.go.jp/statistics/index.html`). The logarithmic transformation is applied to all four indicators.

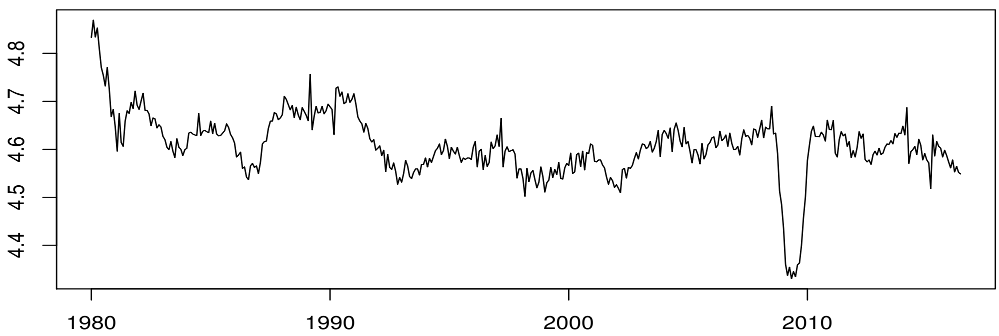

Figure 12: Time series data for NBI-y4 (January 1980 to June 2016); Source: Ministry of Economy, Trade and Industry

To compare the performance of the IBC with the NBI under the same conditions, we construct the IBC following our CCS approach. The IBC is based on the data for the above four indicators for January 1980 to June 2016, corresponding with the period of the NBI-y4 data. The initial settings are very similar to those for constructing the IBC-10. The main estimates for the hyperparameters are given in Table 4 for the models of $g(y_i(n))(i = 1, 2, 3, 4)$.

Figures 13 and 14 show the estimated trend and cyclical components for all indicators.

Table 4: Main results for hyperparameter estimation

i	1	2	3	4
q_i	10	10	12	1
$\widehat{\sigma}_i^2$	3.31×10^{-8}	2.74×10^{-5}	5.41×10^{-8}	8.35×10^{-5}
$\widehat{\tau}_i^2$	4.40×10^{-7}	2.42×10^{-7}	2.68×10^{-7}	2.83×10^{-6}
$\widehat{\eta}_i^2$	3.31×10^{-4}	7.03×10^{-5}	5.41×10^{-4}	2.53×10^{-4}

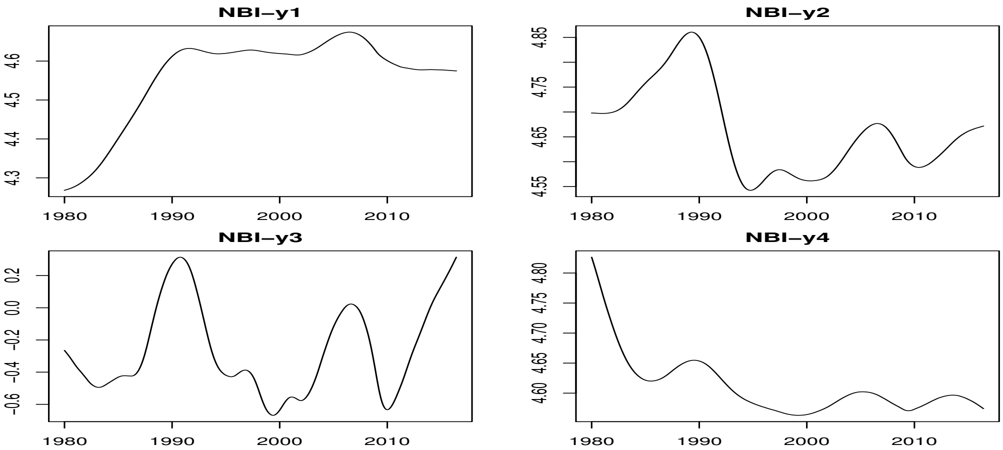

Figure 13: Estimated trend components (January 1980 to June 2016)

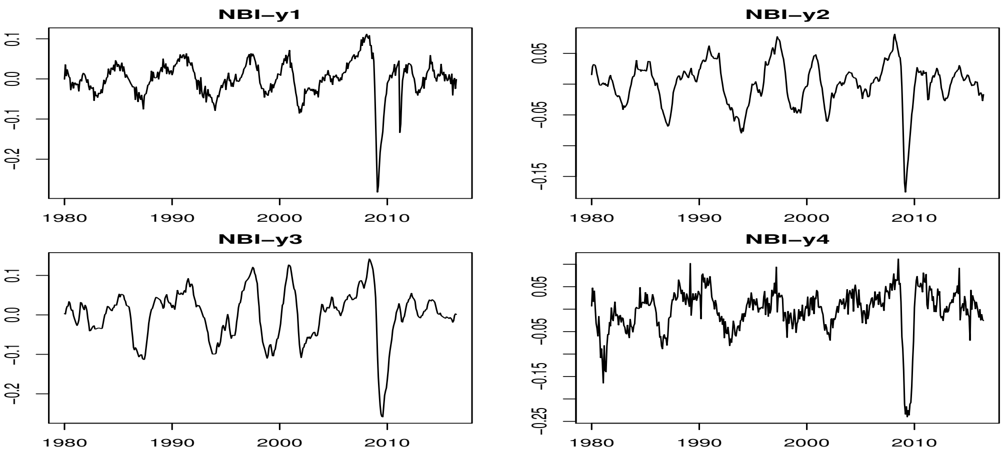

Figure 14: Estimated cyclical components (January 1980 to June 2016)

Figure 15 shows the estimated time series of the IBC, termed the IBC-4. Figure 16 shows a scatter diagram for the IBC-4 and NBI from January 1980 to

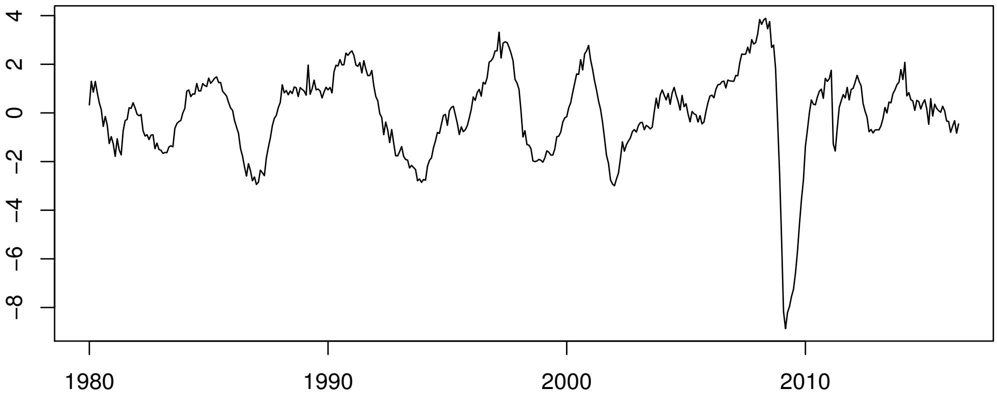

Figure 15: Time series for IBC-4 (January 1980 to June 2016)

June 2016. Note that the correlation coefficient between these two indexes is 0.6509. These results indicate that the correlation coefficient is not very high, which may be a result of the trend component of the NBI.

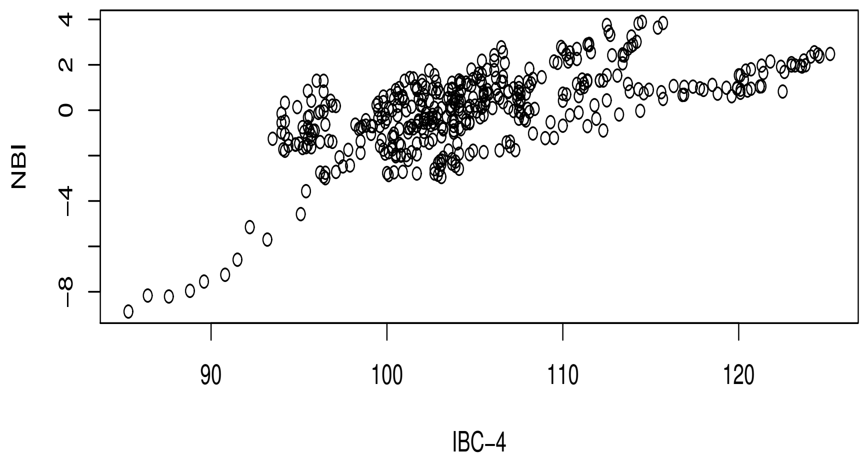

Figure 16: Scatter diagram for IBC-4 and NBI (January 1980 to June 2016)

We compare the performance of the IBC-4 and NBI by using the cyclical log-GDP as a reference. However, to obtain the stationary part of the NBI and, thus, to ensure that the NBI has a higher correlation with the cyclical log-GDP, we remove the trend component from the NBI series.

The top graph in Figure 17 shows the quarterly time series data for the NBI, which are obtained as the averages over the corresponding 3 months from Q1 1994 to Q1 2015. From this graph, we can see a declining trend in the NBI time series.

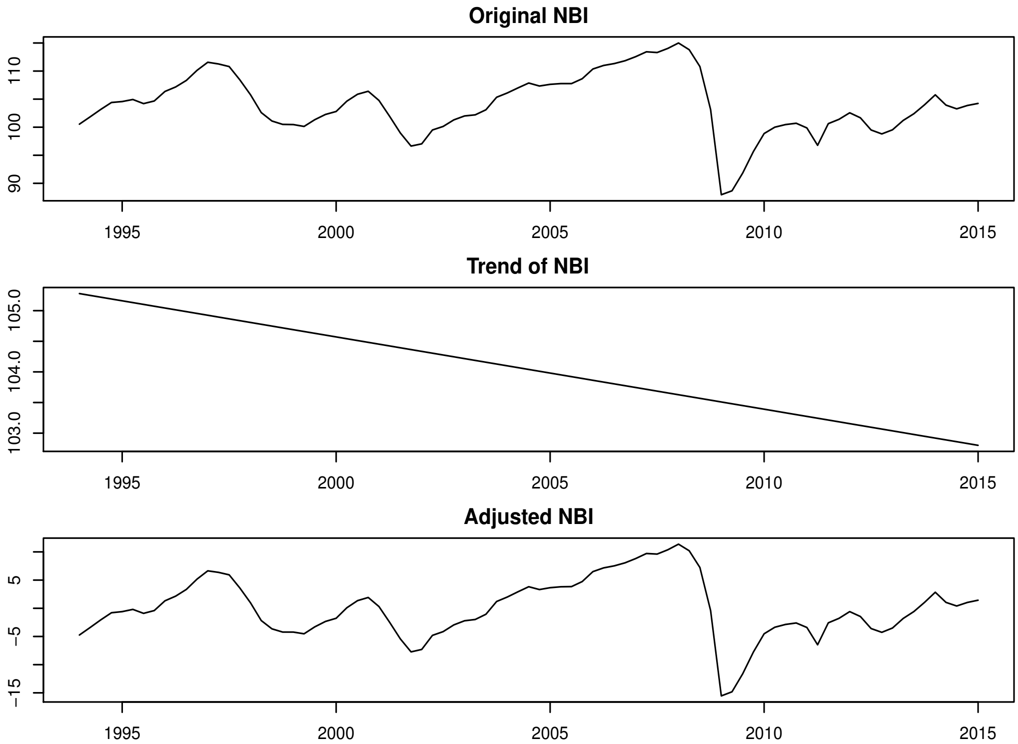

Figure 17: Result for decomposing the NBI (Q1, 1994 to Q1, 2015)

Let y_n and x_n be the values of the NBI and the trend at time point n (quarter), respectively. Then, we can compute the adjusted NBI by:

$$z_n = y_n - x_n, \quad (n = 1, 2, \ldots, N_1)$$

with $N_1 = 85$ being the length of the time series data from Q1 1994 to Q1 2015. Thus, to ensure that the short-term variations in the adjusted NBI remain as powerful as possible, we express the trend of the NBI as a straight line:

$$x_n = a + bn, \quad (n = 1, 2, \ldots, N_1).$$

Then, we estimate the parameters a and b by maximizing the correlation coefficient between the adjusted NBI and the cyclical log-GDP under the condition that:

$$\sum_{n=1}^{N1} z_n = 0.$$

Note that the estimates for the parameters are $\widehat{a} = 105.31$ and $\widehat{b} = -0.0295$.

The graphs for the estimates of the trend in the series of the NBI and the adjusted NBI are plotted in Figure 17.

Thus, we can employ correlation analysis for the cyclical log-GDP with the NBI, adjusted NBI, and IBC-4. Figure 18 shows the scatter diagrams of the cyclical log-GDP with the NBI (left diagram), adjusted NBI (center), and IBC-4 (right). Note that the correlation coefficients of the cyclical log-GDP with these three indexes are 0.8277, 0.8355, and 0.8466, respectively. That is, the correlation coefficient of the cyclical log-GDP with the adjusted NBI is higher than that with the NBI, and the correlation coefficient with the IBC-4 takes the highest value. These results reveal that our CCS approach performs well.

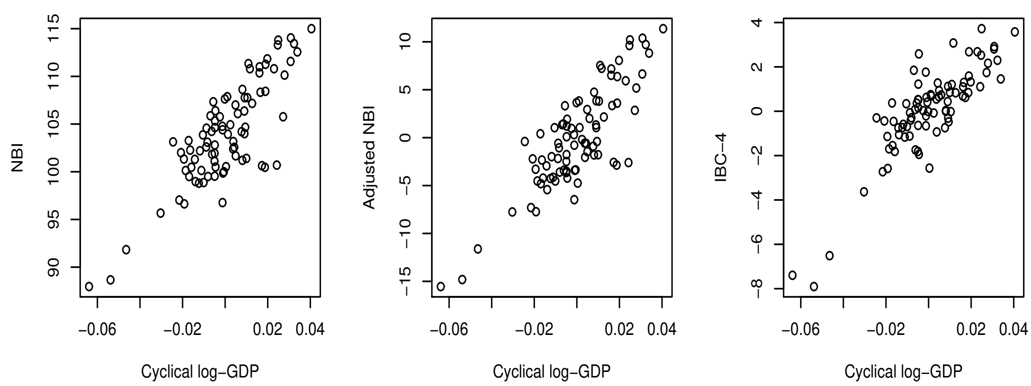

Figure 18: Scatter diagrams of the cyclical log-GDP with all indexes (Q1, 1994 to Q1, 2015)

6 Discussions

To summarize Sections 4 and 5, our results show that our CCS approach has many advantages when compared with existing approaches. These are discussed below.

1. Our approach is systematic. As previously mentioned, when using the Stock–Watson dynamic factor modeling approach, it is necessary to transform the data to be stationary by differencing the time series data. However, it is often difficult to determine whether it is appropriate to do this and, even when differencing is necessary, there is no clear criterion to determine the order of difference. Thus, the Stock–Watson dynamic factor modeling approach is

somewhat ad hoc regarding the use of data. Moreover, the MTV modeling method is ad hoc in the interpretation of results. As shown in Figure 4, under the CCS approach, the model in Eq. (2) can generally be applied for indicators with various types of trends, so long as these are smooth. Even where the time series data are stationary in mean, it can be ensured that the estimates for the trend component are obtained as a constant.

2. The results of our approach are clear and easy to apply. As shown in Eq. (4), in most cases following the Stock–Watson approach, the behavior of an estimated IBC is complicated because it comprises stationary and nonstationary parts. In contrast, the structure of the IBC is very simple, ensuring greater understanding because of global stationarity.

3. Our approach has a strong performance in analyzing and predicting business conditions. The results shown in the preceding section imply that the IBC can better explain the variation in GDP than the CI and NBI. Moreover, the models using the CI approach do not have clear structures, making it difficult to predict future business conditions. In contrast, each cyclical component of our approach is estimated using the Kalman filter, which is well known for its good predictive abilities.

In addition, if we use the model in Eq. (5) instead of that in Eq. (6) and add a prior model for the seasonal component, we can process the time series data in advance without the need for a seasonal adjustment. Thus, our CCS approach can be widely applied as a more general method.

7 Conclusions

In this paper, to construct a coincident index of growth cycles from a given set of indicators, we proposed an alternative approach, which we refer to as the CCS approach, to develop an IBC of coincident economic indications.

The framework of the CCS approach can be summarized as follows: (1) We used the same time series data as the CI and DI compiled by the ESRI; (2) Seasonally adjusted data were decomposed into a trend, cyclical, and irregular components; and (3) We constructed the IBC based on the first principal component of the normalized estimates for the cyclical components.

We examined whether the constructed coincident IBC performs better than existing indexes. The correlation coefficients of the cyclical component of real GDP were 0.7181 and 0.8524 for the CI and IBC, respectively, which indicates that the IBC performed better than the CI. Focusing on the adjusted NBI and IBC, the correlation coefficients of the cyclical component of GDP were 0.8355 and 0.8466, respectively, indicating that IBC performed better than the NBI. Therefore, the results show that the CCS approach has a number of advantages over existing methods.

Acknowledgments

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