SDS-20

Frequency Regression and Smoothing for Noisy Non-stationary Time Series

Seisho Sato and Naoto Kunitomo

May 2021

Statistics & Data Science Series back numbers:
http://www.mims.meiji.ac.jp/publications/datascience.html

Frequency Regression and Smoothing for Noisy Non-stationary Time Series *

Seisho Sato †
and
Naoto Kunitomo ‡

May 8, 2021

Abstract

We develop a new regression method called *Frequency Regression and Smoothing*, which is based on the SIML method. The SIML smoothing method has been developed by Kunitomo and Sato (2021), and Sato and Kunitomo (2020) to estimate the hidden states of random variables and to handle noisy non-stationary (small sample) time series data. Many economic time series include not only trend-cycle, seasonal, and measurement error components, but also there are factors such as structural breaks, abrupt changes, trading-day effects, and institutional changes. The method of frequency regression and smoothing can be applied to handle these factors in non-stationary time series. Our method is simple and applicable to several problems when we analyze non-stationary economic time series and handle seasonal adjustment. An illustrative empirical analysis of macro-consumption in Japan is given.

Key Words

Noisy non-stationary time series, Trend-Cycle, Measurement error and seasonality, SIML-filtering, Regression smoothing, Structural and institutional change, Seasonal adjustment

*Version 2021-5-8. It is a preliminary memorandum.

†Graduate School of Economics, University of Tokyo, Bunkyo-ku,Hongo7-3-1, Tokyo113-0033, seisho@e.u-tokyo.ac.jp

‡Tokyo Keizai University, Minami-cho 1-7-34, Kokubunnji-shi, Tokyo 185-8502, JAPAN, kunitomo@tku.ac.jp

1. Introduction

There has been a vast amount of published research on the use of statistical time series analysis of macro-economic time series. One important feature of macroeconomic time series, which is different from the standard time series analysis, is the fact that the observed time series is an apparent mixture of non-stationary components and stationary components. The second feature is the fact that the measurement errors in economic time series play important roles because macro-economic data are usually constructed from various sources including sample surveys in major official statistics while the statistical time series analysis often ignores measurement errors. Third, to publish the seasonally adjusted data, the official agencies usually apply the X-12-ARIMA program, which uses the univariate reg-ARIMA model to remove the seasonality as the standard filtering procedure. There is yet fourth important issue that the sample size of macro-economic data is rather small and we have 120, say, time series observations for each series when we have quarterly data over 30 years. The quarterly GDP series, which has been the most important data in macro-economy are published since 1994 by the cabinet office of Japan, for instance. Since the sample size is small, it is important to use an appropriate statistical procedure to extract information on trend-cycle and noise (or measurement error) components in a systematic way from data.

In this study we will develop a new regression method called *Frequency Regression and Smoothing*, which is based on the SIML filtering (or smoothing) method. The SIML filtering method has been developed by Kunitomo and Sato (2021) to estimate the hidden states of random variables and to handle multiple time series data. In particular, it can be applicable to small sample economic time series. Kunitomo and Sato (2017), Kunitomo and Sato (2021), and Sato and Kunitomo (2020) have developed the separating information maximum likelihood (SIML) method for estimating the non-stationary errors-in-variables models. They have discussed the asymptotic properties and finite sample properties of the estimation of unknown parameters, and develop the filtering method. We utilize their results to develop the linear regression methods in the frequency domain for non-stationary economic time series. In addition to the fact that there are trend components, cycle components, seasonal components as well as the measurement errors, there are important factors such as structural breaks, trading-day effects, and institutional changes in macroeconomic variables. Since there are many factors in non-stationary time series, there has not been any statistical method to handle them in a systematic and coherent way. The method of frequency regression we are developing can be applied to handle these factors in non-stationary time series. Our method is simple and applicable to several problems when we analyze non-stationary economic time series.

As a classical study, Granger and Hatanaka (1964) have introduced the spectral analysis of economic time series. Engle (1974) introduced the band spectrum regression for stationary time series. Our method of frequency regression could be regarded as their extensions to non-stationary time series.

In Section 2, we explain the non-stationary errors-in-variables model and the SIML-filtering (or smoothing) method. Then in Section 3 we introduce the frequency regression method and as an application, we mention to the method developed by M*ü*ller and Watson (2018) briefly. In Section 4 we discuss the regression smoothing method base on the SIML smoothing method. In Section 5, we discuss the likelihood function. Then in Section 6, we discuss an illustrative empirical analysis of macro-consumption of durable goods in Japan as an illustrative purpose. In Section 7 some concluding remarks are given. Some details of mathematical derivations of the results on frequency regression and figures are given in Appendix.

2. Non-stationary Errors-in-variables models and SIML Filtering

2.1 Non-stationary Errors-in-variables models

Let y_{ji} be the $i-$th observation of the $j-$th time series at i for $i = 1, \cdots, n; j = 1, \cdots, p$. We set $\mathbf{y}_i = (y_{1i}, \cdots, y_{pi})^{'}$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ $(= (y_{ij}))$ be an $n \times p$ matrix of observations and denote $\mathbf{y}_0$ as the initial $p \times 1$ vector. We estimate the model when the underlying non-stationary trend-cycle component $\mathbf{x}_i$ $(= (x_{ji}))$ $(i = 1, \cdots, n)$, but we have the vector of seasonal component $\mathbf{s}_i^{'} = (s_{1i}, \cdots, s_{pi})$ and the vector of noise component $\mathbf{v}_i^{'} = (v_{1i}, \cdots, v_{pi})$, which are independent of $\mathbf{x}_i$. We use the non-stationary errors-in-variables representation

$$(2.1) \qquad \mathbf{y}_i = \mathbf{x}_i + \mathbf{s}_i + \mathbf{v}_i \quad (i = 1, \cdots, n),$$

where $\mathbf{x}_i$, $\mathbf{s}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are sequences of non-stationary I(1), stationary I(0) seasonal process, and stationary I(0) noise process. The trend-cycle component $\mathbf{x}_i$ satisfies

$$(2.2) \qquad \Delta \mathbf{x}_i = (1 - \mathcal{L})\mathbf{x}_i = \mathbf{v}_i^{(x)} ,$$

with the lag-operator $\mathcal{L}\mathbf{x}_i = \mathbf{x}_{i-1}$, $\Delta = 1 - \mathcal{L}$,

$$(2.3) \qquad \mathbf{v}_i^{(x)} = \sum_{j=0}^{\infty} \mathbf{C}_j^{(x)} \mathbf{e}_{i-j}^{(x)} ,$$

and $\mathbf{e}_i^{(x)}$ is a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(x)}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{e}_i^{(x)}\mathbf{e}_i^{(x)'}) = \mathbf{\Sigma}_e^{(x)}$ (a positive-semi-definite matrix). The $p \times p$ coefficient matrices $\mathbf{C}_j^{(x)}$ $(= c_{kl}^{(x)}(j))$ are absolutely summable and $\|\mathbf{C}_j^{(x)}\| = O(\rho^j)$, where $0 \leq \rho < 1$ and $\|\mathbf{C}_j^{(x)}\| =$

$\max_{k,l=1,\cdots,p} |c_{kl}^{(x)}(j)|$.
The random noise component $\mathbf{v}_i$ satisfies

$$\mathbf{v}_i = \sum_{j=0}^{\infty} \mathbf{C}_j^{(v)} \mathbf{e}_{i-j}^{(v)} , \tag{2.4}$$

where the $p \times p$ coefficient matrices $\mathbf{C}_j^{(v)}$ are absolutely summable and $\|\mathbf{C}_j^{(v)}\| = O(\rho^j)$, where $0 \le \rho < 1$ and $\mathbf{e}_i^{(v)}$ is a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(v)}) = \mathbf{0}$, $\mathbf{E}(\mathbf{e}_i^{(v)} \mathbf{e}_i^{(v)'}) = \mathbf{\Sigma}_e^{(v)}$ (a positive definite matrix).
The seasonal component $\mathbf{s}_i$ $(i = 1, \cdots, n)$ is a sequence of stationary process satisfying

$$\mathbf{s}_i = \sum_{j=0}^{\infty} \mathbf{C}_{sj}^{(s)} \mathbf{e}_{i-sj}^{(s)} , \tag{2.5}$$

where the lag-operator is defined by $\mathcal{L}^s \mathbf{s}_i = \mathbf{s}_{i-s}$ $(s \ge 2)$, and $\mathbf{e}_i^{(s)}$ is a a sequence of i.i.d. random vectors with $\mathbf{E}(\mathbf{e}_i^{(s)}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{e}_i^{(s)} \mathbf{e}_i^{(s)'}) = \mathbf{\Sigma}_e^{(s)}$ (a non-negative definite matrix). The $p \times p$ coefficient matrices $\mathbf{C}_j^{(s)}$ are absolutely summable and $\|\mathbf{C}_j^{(s)}\| = O(\rho^j)$, where $0 \le \rho < 1$.

We have the observations of an $n \times p$ matrix $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ and set the $np \times 1$ random vector $(\mathbf{y}_1^{'}, \cdots, \mathbf{y}_n^{'})^{'}$. In particular, when there is no seasonal component, and each pair of vectors $\Delta \mathbf{x}_i$ and $\mathbf{v}_i$ are independently, identically, and normally distributed (i.i.d.) as $N_p(\mathbf{0}, \mathbf{\Sigma}_x)$ and $N_p(\mathbf{0}, \mathbf{\Sigma}_v)$, respectively, we have $\mathbf{\Sigma}_x = \mathbf{\Sigma}_e^{(x)}$ and $\mathbf{\Sigma}_v = \mathbf{\Sigma}_e^{(v)}$. Then, given the initial condition $\mathbf{y}_0$, we have $\mathrm{vec}(\mathbf{Y}_n) \sim N_{n \times p}\left(\mathbf{1}_n \cdot \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \mathbf{\Sigma}_v + \mathbf{C}_n \mathbf{C}_n^{'} \otimes \mathbf{\Sigma}_x\right)$, where $\mathbf{1}_n^{'} = (1, \cdots, 1)$ and

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix}_{n \times n} . \tag{2.6}$$

In the general case with (2.1)-(2.5), we introduce the K_n^*-transformation that from $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = \mathbf{K}_n^* \left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) , \mathbf{K}_n^* = \mathbf{P}_n \mathbf{C}_n^{-1} , \tag{2.7}$$

where

$$\mathbf{C}_n^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{n \times n} , \tag{2.8}$$

and

$$\mathbf{P}_n = (p_{jk}^{(n)}) \ , \ p_{jk}^{(n)} = \sqrt{\frac{2}{n+\frac{1}{2}}} \cos\left[\frac{2\pi}{2n+1}(k-\frac{1}{2})(j-\frac{1}{2})\right] \ . \tag{2.9}$$

By using the spectral decomposition $\mathbf{C}_n^{-1}\mathbf{C}_n^{'-1} = \mathbf{P}_n\mathbf{D}_n\mathbf{P}_n^{'}$, we find that $\mathbf{D}_n$ is a diagonal matrix with the k-th element $d_k = 2[1-\cos(\pi(\frac{2k-1}{2n+1}))]$ $(k = 1,\cdots,n)$ and we can write

$$a_{kn}^{*} \ (= d_k) \ = 4\sin^2\left[\frac{\pi}{2}\left(\frac{2k-1}{2n+1}\right)\right] \ (k = 1,\cdots,n) \ . \tag{2.10}$$

2.2 The SIML Filtering Method

We consider the general filtering procedure based on the $\mathbf{K}_n^{*}-$transformation (2.6). Because the elements of the resulting $n \times p$ random matrix $\mathbf{Z}_n$ by this transformation take real values in the frequency domain, it is easy to interpret their roles in data anlysis. We consider the inversion of a transformation of orthogonal frequency processes. Let an $n \times p$ matrix

$$\hat{\mathbf{X}}_n = \mathbf{C}_n\mathbf{P}_n\mathbf{Q}_n\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \ , \tag{2.11}$$

where $\mathbf{Z}_n = \mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0)$ and $\mathbf{Q}_n$ is an $n \times n$ filtering matrix.
The stochastic process $\mathbf{Z}_n$ are the orthogonal decomposition of the original time series $\mathbf{Y}_n$ (see Section 5 of Kunitomo and Sato (2021)). We give explicit forms of useful examples including the trend-cycle filtering procedure and the seasonal filtering procedure for macro-time series. Kunitomo and Sato (2021) have developed the filtering (or smoothing) method in the form of (2.11) and we give two examples.

Example 1 : Trend Smoothing : Let an $m \times n$ choice matrix $(0 < m < n)$ $\mathbf{J}_m = (\mathbf{I}_m, \mathbf{O})$, and let also $n \times p$ matrix

$$\hat{\mathbf{X}}_n = \mathbf{C}_n\mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \tag{2.12}$$

and an $n \times n$ matrix

$$\mathbf{Q}_n = \mathbf{J}_m^{'}\mathbf{J}_m \ . \tag{2.13}$$

We will construct an estimator of $n \times p$ hidden state matrix $\mathbf{X}_n$ in the frequency domain by using the inverse transformation of $\mathbf{Z}_n$. By deleting the estimated noise parts in the high-frequency, we can recover the trend-cycle component. (See Nishimura, Sato and Takahashi (2019) as a financial application.) Let the $[m+(n-m)] \times [m+(n-m)]$ partitioned matrix

$$\mathbf{P}_n = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix}$$

and

$$(2.14)\qquad \mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n = \begin{pmatrix} \mathbf{P}_{11}^{'} \\ \mathbf{P}_{12}^{'} \end{pmatrix} (\mathbf{P}_{11}, \mathbf{P}_{12}) = \mathbf{I}_n - \begin{pmatrix} \mathbf{P}_{21}^{'} \\ \mathbf{P}_{22}^{'} \end{pmatrix} (\mathbf{P}_{21}, \mathbf{P}_{22})\ .$$

Then the $(j, j^{'})$-th element of $\mathbf{A}_n = \mathbf{P}_n\mathbf{J}_m^{'}\mathbf{J}_m\mathbf{P}_n$ $(= (a_{j,j^{'}}))$ is given by

$$(2.15)\quad a_{j,j} = \frac{2m}{2n+1} + \frac{1}{2n+1}\left[\frac{\sin\frac{2m\pi}{2n+1}(2j-1)}{\sin\frac{\pi}{2n+1}(2j-1)}\right],$$

$$a_{i,j^{'}} = \frac{1}{2n+1}\left[\frac{\sin\frac{2m\pi}{2n+1}(j+j^{'}-1)}{\sin\frac{\pi}{2n+1}(j+j^{'}-1)} + \frac{\sin\frac{2m\pi}{2n+1}(j-j^{'})}{\sin\frac{\pi}{2n+1}(j-j^{'})}\right] \quad (j \neq J^{'})\ .$$

Example 2 : Band Smoothing : We consider the band filtering based on the $\mathbf{K}_n^{*}-$ transformation in (2.7) and use the inversion of only low frequency parts from the random matrix $\mathbf{Z}_n$. The leading example is the seasonal frequency in the discrete time series and we take s (> 1) being a positive integer in this case. Let an $m_2 \times [m_1 + m_2 + (n - m_1 - m_2)]$ choice matrix $\mathbf{J}_{m_1,m_2,n} = (\mathbf{O}, \mathbf{I}_{m_2}, \mathbf{O})$, and let also $n \times p$ matrix

$$(2.16)\qquad \hat{\mathbf{X}}_n = \mathbf{C}_n\mathbf{P}_n\mathbf{J}_{m_1,m_2,n}^{'}\mathbf{J}_{m_1,m_2,n}\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0)$$

and an $n \times n$ matrix

$$(2.17)\qquad \mathbf{Q}_n = \mathbf{J}_{m_1,m_2,n}^{'}\mathbf{J}_{m_1,m_2,n}\ .$$

As an example, when we have the seasonal frequency λ_s $(0 \leq \lambda_s \leq \frac{1}{2})$, we take $m_1 = [2n/s] - h$ and $m_2 = 2h+1$. The $(j, j^{'})$-th element of $\mathbf{A}_n = \mathbf{P}_n\mathbf{J}_{m_1,m_2,n}^{'}\mathbf{J}_{m_1,m_2,n}\mathbf{P}_n$ $(= (a_{j,j^{'}}))$ is given by

$$(2.18)\, a_{j,j} = \frac{2m_2}{2n+1} + \frac{1}{2n+1}\left[\frac{\sin\frac{2(m_1+m_2)\pi}{2n+1}(2j-1) - \sin\frac{2(m_1)\pi}{2n+1}(2j-1)}{\sin\frac{\pi}{2n+1}(2j-1)}\right],$$

$$a_{i,j^{'}} = \frac{1}{2n+1}\left[\frac{\sin\frac{2(m_11+m_2)\pi}{2n+1}(j+j^{'}-1) - \sin\frac{2(m_1)\pi}{2n+1}(j+j^{'}-1)}{\sin\frac{\pi}{2n+1}(j+j^{'}-1)}\right.$$
$$\left. + \frac{\sin\frac{2(m_1+m_2)\pi}{2n+1}(j-j^{'}) - \sin\frac{2(m_1)\pi}{2n+1}(j-j^{'})}{\sin\frac{\pi}{2n+1}(j-j^{'})}\right] \quad (j \neq j^{'})\ .$$

We note that when $m_1 = 0$ and $m_2 = m$, (2.17) becomes (2.13) in Example 1.

When we have seasonality, however, there is a complication in data analysis and we need to use several transformations. (We shall discuss some examples in Section 4 in details.) For quaterly data, 1 year (4 quarters) cycle cannot be distingushed from the 2 quarters cycle. For monthly data, 1 year cycle cannot be distinguished from 6, 4, 3, 2.4 and 2 months cycles.

3. Frequency Regression

In this section, we consider the linear regression model based on observations of $q \times p$ matrix $\mathbf{Z}_m^*$ by

$$\mathbf{Z}_m^* = \mathbf{F}_q \mathbf{P}_n \mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0) = [\mathbf{z}_{1m}^*, \mathbf{Z}_{2m}^*] \ , \tag{3.1}$$

where $\mathbf{F}_q$ is a $q \times n$ matrix and $q\ (> p)$ depends on n as $q = q_n$.
There are several interesting examples. Since we consider the case when the rank of $\mathbf{F}_q$ is p $(p < q)$, we first investigate this case.
When we have non-stationary time series, we often have trend, cycle, seasonal and noise components. To handle these components, we can use a more complicated transformation $\mathbf{F}_q$. In addition to these components, there are trading-day components, leap year effects, structural changes such as the 2008 financial crisis, and institutional changes such as the consumption tax in Japan. When we make seasonal adjusted data, it is important to handle these effects in meaningful ways. Since there are many components, it has been known that the method of handling these effects in official statistics may tend to be *ad hoc* from a standard statistical view.

this case has been considered by Kunitomo and Sato (2021) considered the case when $\mathbf{F}_q = \mathbf{J}_m$. We first investigate this case and we assume that the rank of $\mathbf{F}_q$ is p $(p < q)$. We define $p \times p$ matrices

$$\mathbf{G}_m^* = \frac{1}{m}\mathbf{Z}_m^{*'}\mathbf{Z}_m^* \ , \mathbf{G}_n = \frac{1}{n}\mathbf{Z}_n^{'}\mathbf{Z}_n \tag{3.2}$$

and we denote their probability limits as $m = m_n \to \infty$ $(n \to \infty, m_n/n \to 0)$ when they exist as

$$\mathrm{plim}_{n\to\infty}\mathbf{G}_m^* = \mathbf{\Sigma}_x \ , \mathrm{plim}_{n\to\infty}\mathbf{G}_n = \mathbf{\Sigma}_{\Delta y} \ , \tag{3.3}$$

where

$$\mathbf{\Sigma}_x = (\sum_{j=0}^{\infty}\mathbf{C}_j^{(x)})\mathbf{\Sigma}_e^{(x)}(\sum_{j=0}^{\infty}\mathbf{C}_j^{(x)'}) \ (= \mathbf{f}_{\Delta x}(0)) \ , \tag{3.4}$$

($\mathbf{\Sigma}_x$ is the spectral density matrix of $\Delta\mathbf{x}_i$ at zero frequency) and $\mathbf{\Sigma}_{\Delta y}$ is different from $\mathbf{\Sigma}_x$.

We partition $\mathbf{G}_m^*$ and $\mathbf{\Sigma}_x$ into $(1+k) \times (1+k)$ $(k = p-1)$ submatrices as

$$\mathbf{G}_m^* = \begin{bmatrix} g_{11}^* & \mathbf{g}_{12}^* \\ \mathbf{g}_{21}^* & \mathbf{G}_{22}^* \end{bmatrix} , \mathbf{\Sigma}_x = \begin{bmatrix} \sigma_{11} & \boldsymbol{\sigma}_{12} \\ \boldsymbol{\sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix} . \tag{3.5}$$

Then we investigate the least squares estimator

$$\hat{\boldsymbol{\beta}}_m = \mathbf{G}_{22}^{*-1}\mathbf{g}_{21}^* \ , \tag{3.6}$$

which is an estimator of the vector $\boldsymbol{\beta}_m = \boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\sigma}_{21}$ under the assumption of the inverse matrices of $\mathbf{G}_{22}^*$ and $\boldsymbol{\Sigma}_{22}$ exist. (We need to assume that $\boldsymbol{\Sigma}_{22}$ has a full rank.)
We write

$$\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta} = [\mathbf{Z}_{2m}^{*'}\mathbf{Z}_{2m}^{*}]^{-1}\mathbf{Z}_{2m}^{*'}\mathbf{Z}_m^{*}\begin{pmatrix} 1 \\ -\boldsymbol{\beta} \end{pmatrix} , \tag{3.7}$$

where we have partitioned $\mathbf{Z}_m^*$ into $q \times (1+k)$ submatrices $\mathbf{Z}_m^* = (\mathbf{z}_{1m}^*, \mathbf{Z}_{2m}^*)$.
Then we have the next result on the asymptotic properties of the least squares estimator and the proof is given in the Appendix.

Theorem 3.1 : Let $m = m_n = [n^\alpha]$ and $m \to \infty$ $(n \to \infty)$. In (2.1)-(2.5), assume that the fourth-order moments of $\mathbf{e}_i^{(x)}$, $\mathbf{e}_i^{(s)}$ and $\mathbf{e}_i^{(v)}$ are bounded.
(i) For $0 < \alpha < 1.0$, $\boldsymbol{G}_m^*$ is a consistent estimator of $\boldsymbol{\Sigma}_x$.
(ii) Assume that the rank of $\boldsymbol{\Sigma}_{22}$ is k $(= p-1)$. Let $m = m_n = [n^\alpha]$ and $0 < \alpha < 0.8$. Then when $m \to \infty$ $(n \to \infty)$, $\sqrt{m_n}[\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}]$ is asymptotically and normally distributed as $N(\mathbf{0}, \sigma_{11.2}\boldsymbol{\Sigma}_{22}^{-1})$ and $\sigma_{11.2} = \sigma_{11} - \boldsymbol{\sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\sigma}_{21}$.

We can re-write $\mathbf{u}_m = \mathbf{z}_{1m}^* - \mathbf{Z}_{2m}^*\boldsymbol{\beta}$, that is,

$$\mathbf{z}_{1m}^* = \mathbf{Z}_{2m}^*\boldsymbol{\beta} + \mathbf{u}_m \tag{3.8}$$

and $\mathbf{E}[\mathbf{u}_m] = \mathbf{0}$. This is a linear regression equation, but the error term of $\mathbf{u}_m$ has a specific form of heteroscedasticity.
It is straightforward to Theorem 3.1 to the case when we use $\mathbf{G}_n$ for $\boldsymbol{\Sigma}_{\Delta y}$, which is different from $\boldsymbol{\Sigma}_x$.

One application of Theorem 3.1 would be M*ü*ller and Watson (2018), which have proposed *so-called* the long-run co-variability of macro-economic time series. They have investigated many non-stationary time series using their method and obtained some interesting findings. Kunitomo and Sato (2021) have suggested an interpretation of their method as the relationships among long-run trends in our framework when $p = 2$. Let 2×2 matrices $\boldsymbol{\Sigma}_e^{(x)} = (\sigma_{ij}^{(x)})$, we then define the regression coefficient $\boldsymbol{\beta} = [\sigma_{22}^{(x)}]^{-1}\sigma_{21}^{(x)}$ under the assumption that $\sigma_{22}^{(x)} > 0$.
Let also $\boldsymbol{G}_m^* = (\hat{g}_{ij}^{(x)})$, and an $n \times 2$ matrix

$$(\mathbf{a}_{1n}, \mathbf{a}_{2n}) = \mathbf{C}_n^{-1}(\mathbf{Y}_n - \mathbf{Y}_0) . \tag{3.9}$$

For estimating $\boldsymbol{\beta}$, we define the estimated regression coefficient by

$$\hat{\boldsymbol{\beta}} = [\hat{g}_{22}^{(x)}]^{-1}\hat{g}_{21}^{(x)} = [\mathbf{a}_{2n}^{'}\mathbf{P}_n\mathbf{J}_m\mathbf{J}_m^{'}\mathbf{P}_n\mathbf{a}_{2n}]^{-1}[\mathbf{a}_{2n}^{'}\mathbf{P}_n\mathbf{J}_m\mathbf{J}_m^{'}\mathbf{P}_n\mathbf{a}_{1n}] . \tag{3.10}$$

This quantity can be interpreted as the least squares slope of the transformed vector from $\mathbf{y}_{1n}$ on the transformed vector from $\mathbf{y}_{2n}$ for a $n \times 2$ matrix $\mathbf{Y}_n = (\mathbf{y}_{1n}, \mathbf{y}_{2n})$, which is essentially the same as the one proposed by Müller and Watson (2018) [1]. Then, from Theorem 3.1, we immediately obtain the following result and the proof has been given in Kunitomo and Sato (2021).

Corollary 3.1 : When $p = 2$, we assume that $\mathbf{\Sigma}_e^{(x)}$ is positive semi-definite, $\mathbf{\Sigma}_e^{(s)}$ is positive semi-definite, $\mathbf{\Sigma}_e^{(v)}$ is positive definite, and that that the fourth-order moments of $\mathbf{e}_i^{(x)}$, $\mathbf{e}_i^{(s)}$ and $\mathbf{e}_i^{(v)}$ $(i = 1, \cdots, n)$ are bounded.
(i) We consider a sequence of integers $m = m_n$. Then $\hat{\beta}$ cannot be consistent when $n \to \infty$.
(ii) Set $m_n = [n^\alpha]$ and $0 < \alpha < 1$, then as $n \longrightarrow \infty$ $\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta} \xrightarrow{p} \mathbf{0}$.
(iii) Set $m_n = [n^\alpha]$ and $0 < \alpha < 0.8$, then as $n \longrightarrow \infty$ $\sqrt{m_n}[\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta}]$ is asymptotically normal.

It may be rather straight-forward to incorporate the regression effects of dummy variables in trend relations such as structural breaks.

4. Regression Smoothing

When we have noisy-nonstationary multivariate time series, we often need to remove the seasonality and/or low frequency part. In some applications of official statistics, however, we need to construct the seasonally adjusted data after removing additional effects such as trading-day components including leap year effect, structural changes such as the 2008 financial crisis and institutional changes such as the introduction of consumption tax in Japan. These effects are could be defined in deterministic ways.

Let the observed vector times series $\mathbf{y}_i$ be decomposed as

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{SCO}_i + \mathbf{S}_i + \mathbf{v}_i \quad (i = 1, \cdots, n), \tag{4.1}$$

and $\mathbf{SCO}_i = \mathbf{SC}_i + \mathbf{O}_i$, where $\mathbf{x}_i$ is the trend-cycle component, $\mathbf{SC}_i$ is the structural break component, $\mathbf{S}_i$ is the seasonal component, $\mathbf{v}_i$ is the noise component, and $\mathbf{O}_i$ is the outlier component.
In this section we consider the case when $\mathbf{SC}_i$ and $\mathbf{O}_i$ can be expressed as $\mathbf{SC}_i + \mathbf{O}_i = \mathbf{SCO}_i(w)$, where $\mathbf{w}$ is the set of instrumental variables. If these terms can be expressed as linear relationships, we can write

$$\mathbf{y}_i = \mathbf{B}'\mathbf{w}_i + \mathbf{u}_i \quad (i = 1, \cdots, n), \tag{4.2}$$

where $\mathbf{B}'$ is a $p \times r$ matrix, $\mathbf{w}_i$ is a $r \times 1$ vector of instrumental variables, and $\mathbf{u}_i = \mathbf{x}_i + \mathbf{S}_i + \mathbf{v}_i$ is a sequence of I(1) process. Hence the model is the multivariate regression model when the noise terms are $I(1)$ process with stationary noise term and seasonal

[1] In their notation, m corresponds to q, which is fixed. They did use the (differenced) stationary data, and thus, we can interpret that they actually calculated the linear regression from the filtered data $\hat{\mathbf{X}}_n^* = \mathbf{P}_n'\mathbf{J}_m'\mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \mathbf{Y}_0)$ as a modification of (2.11) in our notation.

terms. To extract or delete some components from the observed time series based on (4.1) we extend the SIML smoothing method developed by Kunitomo and Sato (2021), and Sato and Kunitomo (2020) and incorporate extraneous information such as dummy variables.
In order to find the smoothing procedure of trend and seasonal components, we re-write

$$\mathbf{Y}_n = \mathbf{W}_n\mathbf{B} + \mathbf{U}_n \ , \tag{4.3}$$

where $\mathbf{W}_n = (\mathbf{w}_t^{'})$ be an $n \times r$ matrix of explanatory variables of $\mathbf{w}_t$ ($r \times 1$ vector) and $\mathbf{U}_n$ is an $n \times p$ matrix.

Because (4.2) and (4.3) are the linear regression equations, it is possible to apply Theorem 3.1 by defining a $(p + r) \times 1$ vector

$$\mathbf{y}_i^* = \begin{bmatrix} \mathbf{y}_i \\ \mathbf{w}_i \end{bmatrix} .$$

Then we can estimate the regression coefficients and calculate the residuals from the regression equations. When vectors $\mathbf{w}_i$ $(i = 1, \cdots, n)$ are deterministic, we assume that

$$\lim_{m\to\infty} \frac{1}{m}\mathbf{W}_m^{*'}\mathbf{W}_m^* = \mathbf{\Sigma}_{w^*} \ , \tag{4.4}$$

where $\mathbf{\Sigma}_{w^*}$ is a positive definite matrix and $\mathbf{W}_m^* = \mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{W}_n$ is an $m \times r$ matrix. When the $r \times 1$ instrumental variables $\mathbf{w}_i$ $(i = 1, \cdots, n)$ is exogenous or deterministic, we have the following result from Theorem 3.1.

Corollary 4.1 : We assume the non-singularity condition (4.4), $\mathbf{w}_i$ $(i = 1, \cdots, n)$ is exogenous or deterministic, and the fourth-order moments of $\mathbf{e}_i^{(x)}$, $\mathbf{e}_i^{(s)}$ and $\mathbf{e}_i^{(v)}$ are bounded.
In (4.3) we repress the transformed $\mathbf{Y}_n^* = \mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{Y}_n$ on $\mathbf{W}_n^* = \mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{W}_n$ and $\hat{\mathbf{B}}_m$ is the least suares estimator of $\mathbf{B}$. We set $m_n = [n^\alpha]$ $(0 < \alpha < 0.8)$, and then as $n \longrightarrow \infty$, we have the asymptotic normality as

$$\sqrt{m_n}[\hat{\mathbf{B}}_m - \mathbf{B}] \xrightarrow{w} N(\mathbf{0}, \mathbf{\Sigma}_{w^*}^{-1} \otimes \mathbf{\Sigma}_x) \ . \tag{4.5}$$

Define the general transformed instrumental variables

$$\hat{\mathbf{W}}_n = \mathbf{J}_W\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{W}_n \ , \tag{4.6}$$

where $\mathbf{J}_W$ is a $q \times n$ choice matrix and we denote the idempotent matrix ($q \times q$ matrix)

$$\mathbf{Q}_W = \hat{\mathbf{W}}_n(\hat{\mathbf{W}}_n^{'}\hat{\mathbf{W}}_n)^{-1}\hat{\mathbf{W}}_n^{'} \ . \tag{4.7}$$

We shall utilize the regression information on smoothing by utilizing the projection matrix $\mathbf{Q}_W$ to construct

$$\hat{\mathbf{X}}_n = \mathbf{C}_n \mathbf{P}_n \mathbf{Q}_W \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \ . \tag{4.8}$$

There can be several possibilities how we incorporate the extraneous information in the smoothing procedure. It is reasonable to consider the case when $\mathbf{Q}_W$ is an idempotent matrix such as $\mathbf{Q}_W^2 = \mathbf{Q}_W$. In our study we mainly use two alternative smoothing procedures : *Type-I* and *Type II.* Type-I smoothing may be appropriate for the change-point smoothing in trend component and Type-II smoothing may be appropriate outlier detection in the noise component.

(i) **Type-I Smoothing** : Type-I is based on Example 1 in Section 2. The (trend-cycle) regression part of $\mathbf{Y}_n$ is (4.1) when we take $\mathbf{J}_W = (\mathbf{I}_m, \mathbf{O})$ ($\hat{\mathbf{W}}_n$ is an $m \times r$ matrix, $\mathbf{J}_m^{'} = (\mathbf{I}_m, \mathbf{O})^{'}$ is an $n \times m$ matrix) and an $n \times n$ matrix

$$\mathbf{Q}_n^{(0)} = \mathbf{J}_m^{'} \hat{\mathbf{W}}_n (\hat{\mathbf{W}}_n^{'} \hat{\mathbf{W}}_n)^{-1} \hat{\mathbf{W}}_n^{'} \mathbf{J}_m \ . \tag{4.9}$$

If we want to remove the regression effects and use only trend-cycle part, we need to take $\mathbf{J}_W = \mathbf{J}_m$, $\mathbf{J}_m = (\mathbf{I}_m, \mathbf{O})$ ($m \times n$ choice matrix, $m \leq n$) and

$$\mathbf{Q}_n^{(1)} = \mathbf{J}_m^{'} \mathbf{J}_m - \mathbf{Q}_n^{(0)} = \mathbf{J}_m^{'} [\mathbf{I}_m - \hat{\mathbf{W}}_n (\hat{\mathbf{W}}_n^{'} \hat{\mathbf{W}}_n)^{-1} \hat{\mathbf{W}}_n^{'}] \mathbf{J}_m \ . \tag{4.10}$$

Then we have the decomposition

$$\begin{aligned} \hat{\mathbf{X}}_n &= \mathbf{C}_n \mathbf{P}_n \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \\ &= \mathbf{C}_n \mathbf{P}_n \mathbf{J}_m^{'} [\mathbf{Q}_n^{(0)} + \mathbf{Q}_n^{(1)}] \mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \ . \end{aligned} \tag{4.11}$$

In this case we have the property $\mathbf{Q}_n^2 = \mathbf{Q}_n = \mathbf{Q}_n^{(0)} + \mathbf{Q}_n^{(1)} = \mathbf{J}_m^{'} \mathbf{J}_m$ and we have the decomposition of trend-cycle part and regression part. There is a simple interpretation of this smoothing because we use only regression part at m low frequencies. We first remove the regression part from $\mathbf{Y}_n$ by taking

$$\mathbf{X}_n^{(1)} = \mathbf{C}_n \mathbf{P}_n [\mathbf{I}_n - \mathbf{Q}_n^{(0)}] \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \ . \tag{4.12}$$

By applying the 2nd smoothing to $\mathbf{Y}_n^{(1)}$ as

$$\mathbf{X}_n^{(2)} = \mathbf{C}_n \mathbf{P}_n \mathbf{J}_m^{'} \mathbf{J}_m \mathbf{P}_n \mathbf{C}_n^{-1} \mathbf{X}_n^{(1)} \tag{4.13}$$

by taking another transformation.
Then the resulting transformation is (4.8) with $\mathbf{Q}_W = \mathbf{Q}_n^{(1)}$ after an iteration. We note that since Sato and Kunitomo (2020) have developed an iterated smoothing procedure, there should be some way of further iterations.

(ii) **Type-II Smoothing** : Type-II is based on Example 2 in Section 2. When we do need to estimate not only the trend component, but also the noise component, it is important to estimate the structural changes and outlier components consistently. For instance, we need to estimate the seasonal component to obtain the seasonally adjusted series and it is related to Example 2 in Section 2. For this purpose, we construct an $q \times n$ choice matrix $\mathbf{F}_q$ such that the seasonal components can be removed in their frequencies.

When $s = 4$, we often want to remove the data with frequencies around $\lambda_s = 1/4, 1/2$ (1/2 corresponds to the cycle of 2 quarters and 1/4 corresponds to the cycle of 4 quarters). But we cannot distinguish the 4 quarters cycle from 2 quarters cycle by using quarterly observations. We set $m_1 = [2n/s] - h$, an $(n-2h-1) \times n$ choice matrix and an $(n-3h-2) \times (n-2h-1)$ choice matrix such that

$$\mathbf{J}_1^Q = \begin{bmatrix} \mathbf{I}_{m_1} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{n-m_1-2h-1} \end{bmatrix} , \ \mathbf{J}_2^Q = [\mathbf{I}_{n-3h-2} \ , \ \mathbf{O}] \ , \tag{4.14}$$

and a $q \times n$ matrix

$$\mathbf{F}_q^Q = \mathbf{J}_2^Q \mathbf{J_1}^Q \tag{4.15}$$

with a small positive integer $h > 0$.

When $s = 12$, we need a more complicated transformation to remove seasonality because we cannot distinguish the 12 month cycle from 6, 4, 3, 2.4, 2 months cycles by using monthly observations with frequencies around $\lambda_s = 1/12, 2/12, 3/12, 4/12, 5/12, 6/12$. We set $m_i = [2n/s] - (i-1)(2h+1)$ and $(n - i(2h+1)) \times (n - (i-1)(2h+1))$ choice matrices $(i = 1, \cdots, 5)$ and an $(n - 5(2h+1) - (h+1)) \times (n - 5(2h+1))$ choice matrix such that

$$\mathbf{J}_i^M = \begin{bmatrix} \mathbf{I}_{m_i} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_{n-m_i-i(2h+1)} \end{bmatrix} , \ \mathbf{J}_6^M = \left[\mathbf{I}_{n-5(2h+1)} \ , \ \mathbf{O}\right] \ , \tag{4.16}$$

with a small positive integer $h > 0$. To remove the data with seasonal frequencies around λ_{js} $(j = 2, 3, 4, 5)$ by using $m_s = [2n/s] - jh$ and $(n - 2jh) \times n$ matrices $\mathbf{J}_j^M$ $(j = 1, \cdots, 6)$, we set a $q \times n$ matrix $\mathbf{F}_q$

$$\mathbf{F}_q^M = \prod_{j=1}^{6} \mathbf{J}_{7-j}^M \ . \tag{4.17}$$

Although we do not know the unknown coefficient matrix $\mathbf{B}$, we can incorporate the estimated coefficient by regressing

$$\mathbf{Y}_m^* = \mathbf{F}_q \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \tag{4.18}$$

to

$$\mathbf{W}_m^* = \mathbf{F}_q \mathbf{P}_n \mathbf{C}_n^{-1} (\mathbf{W}_n - \bar{\mathbf{W}}_0) \ , \tag{4.19}$$

where $\mathbf{F}_q$ is either $\mathbf{F}_q^Q$ or $\mathbf{F}_q^M$.

Type-II smoothing is defined by

$$\mathbf{Q}_n^{(2)} = \mathbf{W}_n^* (\mathbf{W}_n^{*'} \mathbf{W}_n^*)^{-1} \mathbf{W}_n^{*'} \ . \tag{4.20}$$

and

$$\mathbf{Q}_n^{(3)} = \mathbf{F}_q^{'}\mathbf{F}_q - \mathbf{Q}_n^{(2)} \tag{4.21}$$

Then we have the decomposition

$$\begin{aligned}\hat{\mathbf{X}}_n^* &= \mathbf{C}_n\mathbf{P}_n\mathbf{F}_q^{'}\mathbf{F}_q\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0) \\ &= \mathbf{C}_n\mathbf{P}_n[\mathbf{Q}_n^{(2)} + \mathbf{Q}_n^{(3)}]\mathbf{P}_n\mathbf{C}_n^{-1}(\mathbf{Y}_n - \bar{\mathbf{Y}}_0)\ .\end{aligned} \tag{4.22}$$

In this case we have the decomposition $\mathbf{Q}_n^{(2)} + \mathbf{Q}_n^{(3)} = \mathbf{F}_q^{'}\mathbf{F}_q$ and we have the corresponding decomposition of trend-cycle part and regression part.

Examples of Dummy Variables :
There are some examples of outlier dummies and trend dummies. For non-stationary time series we should be careful on normalization because there can be significant effects on smoothing. Although there are many other possible dummy variables. we give some examples, which have been used in official data handling such as official seasonal adjustment. Let w_s $(s = 1, \cdots, n)$ be the dummy variable.

Example 1 :
The level shift variable can be defined as $w_s = 0$ if $s < t$ and $w_t = 1$ if $s \geq t$ for $s = 1, \cdots, n$. This can be handle as Type-I.

Example 2 :
The outlier variable can be defined as $w_s = 1$ if $s = t$ and $w_t = 0$ if $s \neq t$ for $s = 1, \cdots, n$.

Example 3 :
The ramp variable can be defined by $w_s = 1$ if $s < t_0$, $w_s = 1 - (t - t_0)/(t_1 - t_0)$ if $t_0 \leq t \leq t_1$ and $w_t = 0$ if $s \geq t_1$.

Example 4 :
The double ramp variable can be defined by $w_s = 1$ if $s < t_0$, $w_s = 1-(t-t_0)/(t_1-t_0)$ if $t_0 \leq t \leq t_1$, $w_s = (t - t_1)/(t_2 - t_1)$ if $t_1 \leq t \leq t_2$, and $w_t = c$ if $s \geq t_2$.

5. Frequency Domain and Likelihood

We have considered the additive decomposition model $\mathbf{y}_i = \mathbf{x}_i + \mathbf{s}_i + \mathbf{v}_i$ $(i = 1, \cdots, n)$ and we take positive integers s $(s > 1)$, N, and $n = sN$ for the resulting simplicity and arguments.
Let $\mathbf{f}_{\Delta x}(\lambda)$, $\mathbf{f}_s(\lambda)$, and $\mathbf{f}_v(\lambda)$ be the spectral density $(p \times p)$ matrices of $\Delta\mathbf{x}_i$, $\mathbf{s}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ and

$$\mathbf{f}_s(\lambda) = (\sum_{j=0}^{\infty}\mathbf{C}_{sj}^{(s)}e^{2\pi i\lambda sj})\mathbf{\Sigma}_e^{(s)}(\sum_{j=0}^{\infty}\mathbf{C}_{sj}^{(s)'}e^{-2\pi i\lambda sj}) \quad (-\frac{1}{2} \leq \lambda \leq \frac{1}{2})\ , \tag{5.1}$$

where we set $\mathbf{C}_0^{(s)} = \mathbf{I}_p$ as normalizations and $i^2 = -1$. Then the $p \times p$ spectral density matrix of the transformed vector process, which are observable, the spectral density of the difference series $\Delta\mathbf{y}_i$ $(= \mathbf{y}_i - \mathbf{y}_{i-1})$ can be represented as

$$\mathbf{f}_{\Delta y}(\lambda) = \mathbf{f}_{\Delta x}(\lambda) + (1 - e^{2\pi i\lambda})[\mathbf{f}_s(\lambda) + f_v(\lambda)](1 - e^{-2\pi i\lambda}) \ . \tag{5.2}$$

We denote the long-run variance-covariance matrices of trend components and stationary components for $g, h = 1, \cdots, p$ as

$$\mathbf{\Sigma}_e^{(x)} = \mathbf{f}_{\Delta x}(0) \ (= (\sigma_{gh}^{(x)})) \ , \ \mathbf{\Sigma}_e^{(v)} = f_v(0) \ = (\sigma_{gh}^{(v)}) \ . \tag{5.3}$$

Let $f_v^{(SR)}(\lambda_k)$, $f_s^{(SR)}(\lambda_k)$ and $f_{\Delta x}^{(SR)}(\lambda_k)$ be the symmetrized $p \times p$ spectral matrices of $\mathbf{v}_i$, $\mathbf{s}_i$ and $\Delta\mathbf{x}_i$ at λ_k $(= (k - \frac{1}{2})/(2n+1))$ for $k = 1, \cdots, n$, that is, $f_v^{(SR)}(\lambda_k) = (1/2)[f_v^{(SR)}(\lambda_k) + \bar{f}_v^{(SR)}(\lambda_k)]$, $f_s^{(SR)}(\lambda_k) = (1/2)[f_s^{(SR)}(\lambda_k) + \bar{f}_s^{(SR)}(\lambda_k)]$ and $f_{\Delta x}^{(SR)}(\lambda_k) = (1/2)[f_{\Delta x}^{(SR)}(\lambda_k) + \bar{f}_{\Delta x}^{(SR)}(\lambda_k)]$.
Proposition 1 of Kunitomo and Sato (2021) gives the condition that the orthogonal processes are approximately distributed as the Gaussian distribution. Then, (-2) times the log-likelihood function in the general model can be approximated as

$$\begin{aligned}(-2)l_n(\boldsymbol{\theta}) \ &= \ \sum_{k=1}^{n} \log |a_{kn}^*(f_v^{(SR)}(\lambda_k) + f_s^{(SR)}(\lambda_k)) + f_{\Delta x}^{(SR)}(\lambda_k))| \\ &\quad + \sum_{k=1}^{n} \mathbf{z}_k^{'}[a_{kn}^*(f_v^{(SR)}(\lambda_k) + f_s^{(SR)}(\lambda_k)) + f_{\Delta x}^{(SR)}(\lambda_k)]^{-1}\mathbf{z}_k \ .\end{aligned} \tag{5.4}$$

We further consider the case when $\Delta\mathbf{x}_i$, $\mathbf{s}_i$ and $\mathbf{v}_i$ are sequence of independent random vectors. Then we have $\mathbf{\Sigma}_e^{(x)} = f_{\Delta x}^{(SR)}(\lambda_k)$ and $\mathbf{\Sigma}_e^{(v)} = f_v^{(SR)}(\lambda_k)$ for $k = 1, \cdots, n$, and $\mathbf{\Sigma}_e^{(s)} = f_s^{(SR)}(\lambda_k)$ for some index set $k \in I_n(s)$.
When we have some dummy variables $\mathbf{W}_n$, we assume that they are independent of other components of noise, cycle, seasonal, and trend components. Then, given the initial condition and the information set of explanatory variables $\mathbf{W}_n$, the conditional log-likelihood ca be written as

$$\begin{aligned}(-2)l_n(\boldsymbol{\theta}|\mathbf{W}_n) \ &= \ \sum_{k=1}^{n} \log |a_{kn}^*(\mathbf{\Sigma}_e^{(v)}(w) + \mathbf{\Sigma}_e^{(s)}(w)) + \mathbf{\Sigma}_e^{(x)}(w)| \\ &\quad + \ \sum_{k=1}^{n} \mathbf{z}_k^{*'}[a_{kn}^*(\mathbf{\Sigma}_e^{(v)}(w) + \mathbf{\Sigma}_e^{(s)}(w)) + \mathbf{\Sigma}_e^{(x)}(w)]^{-1}\mathbf{z}_k^* \ ,\end{aligned} \tag{5.5}$$

where we denote the vector $\boldsymbol{\theta}$, which contains the parameters in $\mathbf{\Sigma}_e^{(x)}$, $\mathbf{\Sigma}_e^{(s)}$ and $\mathbf{\Sigma}_e^{(v)}$.
When we use the explanatory variables $\mathbf{W}_n$, $\mathbf{z}_k^*$ $(k = 1, \cdots, n)$ may depend on $\mathbf{W}_n$ and denote $\mathbf{z}_k^*(w)$ $(k = 1, \cdots, n)$.
To estimate $\mathbf{\Sigma}_x$, it is reasonable to use

$$\mathbf{G}_m^*(w) = \frac{1}{m_n}\sum_{k=1}^{m_n} \mathbf{z}_k^*(w)\mathbf{z}_k^{*'}(w) \tag{5.6}$$

with $m/n \to 0$ and $m \to \infty$.

There are two remarks. First, the ML estimation in the non-stationary errors-in-variables models may have some difficulty when $p > 1$ without some strong restrictions of the parameter space as illustrated by Kunitomo, Awaya and Kurisu (2019). Second, the likelihood functions in this section can be interpreted as the *Wittle-type* likelihood function, which does not depend on the Gaussian distributions as utilized by Hosoya (1997), for instance.

6. An Example of macro-consumption of durable goods

To illustrate the regression smoothing, we use the official macro-consumption data of durable goods in Japan from 1994Q1 to 2019Q4 [2]. To deal with the original quarterly time series, we need to estimate the trend component, seasonal component and noise component. We have applied the SIML smoothing procedure with $m = 29$ and $h = 2$, which gives the minimum numbers of AIC. (Kunitomo and Sato (2021), and Sato and Kunitomo (2020) have explained some aspect of the choice problem of m.) All figures are in the Appendix-B.

Figure 1 is a summary of the SIML smoothing for the log-transformed data. (It was because the original series has a significant heteroscedastic seasonality.) The original time series has a typical characteristics of macro-economic time series in Japan, that is, it is a realization of non-stationary time series and it exhibits rather clear trend, cycle, seasonal and irregular components. We applied the SIML filtering with $m = 29$ and the red-curve gives the estimated trend-cycle component. Although the estimated seasonal component gives regular seasonal pattern, the estimated noise component suggests there are some abrupt changes, which may be different from the usual noise component. Then we applied two ao-dummy variable at 2011Q1 and 2014Q1. In these period, there was a large effect due to the 2011 earthquake in Japan and there was an introduction of consumption tax, which were significant effects on the macro-economy and consumption in Japan. Third, we applied the ramp-dummy variable from 2008Q3 to 2009Q1. In this period, there was a rapid down-ward effect due to the 2008 financial crisis and we may regard that it may be appropriate to use the ramp-dummy at 2008Q3-2009Q1. Figure 2 and Figure 3 are the summary of the SIML smoothing and frequency regression results in these cases.

Finally, Figure 4 gives a summary of the SIML smoothing and frequency regression with three dummy variables at the same time. Based on the criteria of AIC, we have chosen the last case for the best modelling for macro-consumption of durable goods and these effects are captured by our method explained in the previous sec-

[2] We have taken the data from https://www.esri.cao.go.jp/jp/sna/menu.html (Economic and Social Research Institute (ESRI), Cabinet Office, Japan). They are the original series in real terms and ESRI uses the X-12-ARIMA smoothing program for constructing seasonal adjusted official data.

tions. By using the transformed data of (4.16) and (4.17), the criterion AIC has been calculated based on the regression equation by

$$AIC = n \log \hat{\sigma}_w^2 + 2r \tag{6.1}$$

where $\hat{\sigma}_w^2$ is calculated from the residuals of the dummy regression ((4.2) with $p = 1$) and r is the number of dummy variables.
In our example we have $p = 1$ and two AIC's have been calculated: the first one was calculated by using m low frequencies while the AIC in the parenthesis was calculated by all frequency data except data around the seasonal frequency. By using the model selection criteria of minimizing these AIC's, we find that the SIML smoothing with three dummy variables is the best one. This empirical analysis may suggest that we need to consider an important role of incorporating the effects of change point problem and abrupt changes in the seasonal adjustment procedure.

7. Concluding Remarks

In many official time series, it is common to observe non-stationary trend, cycles, seasonal, and measurement errors at the same time. In addition to these components, we further observe abrupt changes, trading-day effects, and other irregular components. Then it seems to be difficult to remove seasonal and construct macro-index, which involve multiple non-stationary time series. This paper develops a new way to handle the non-stationary time series by using the frequency regression based on the SIML modelling in a systematic way. Our method may shed a new light on some practical ways to handle published economic time series, which have been practically used in official seasonal adjustments without formal justification. There would be many empirical examples and we have reported an application of constructing a monthly macro-consumption index in Kunitomo, Sato and Sakurai (2021) in some detail as a real illustration.

There are some problems remained to be investigated. The present study is based on the standard time series decomposition in (2.1) and (5.1), and assume that $\mathbf{v}_i^{(x)}$, $\mathbf{v}_i^{(s)}$ and $\mathbf{v}_i^{(x)}$ are i.i.d. sequences of random variables with mean zero and variance-covariance matrix. It means that their spectral densities are constant in the frequency domain. There might be another way to formulate the problem and decompose the time series. For instance, it may be reasonable to consider the case when the spectral density of $\mathbf{v}_i^{(s)}$ is zero except the region around zero frequency.

Another issue would be the computation of the procedure we have explained in this paper. Since we have developed R-programs (Sato (2020), Kunitomo, Sato and Sakurai (2021)), which will be available in future.

References

[1] Anderson, T. W. (1971), *The Statistical Analysis of Time Series*, Wiley.

[2] Brockwell, P. and R. Davis (1990), *Time Series : Theory and Methods*, 2nd Edition, Wiley.

[3] Engle, R. (1974), "Band Spectrum Regression," *Econometrica*, 15-1, 1-11.

[4] Granger, C. and Hatanaka, M. (1964), *Spectral Analysis of Economic Time Series*, Princeton University Press.

[5] Hosoya, Y. (1997), "A limit theory for long range dependence and statistical inference on related models," *The Annals of Statistics*, vol.25, no.1, pp.105 37, 1997

[6] Kitagawa, G. (2010), *Introduction to Time Series Analysis*, CRC Press.

[7] Kunitomo, N., N. Awaya and D. Kurisu (2020), "Some Properties of Estimation Methods for Structural Relationships in Non-stationary Errors-in-Variables Models," Open Access, *Japanese Journal of Statistics and Data Science*, Springer.

[8] Kunitomo, N. and S. Sato (2017), "Trend, Seasonality and Economic Time Series : the Non-stationary Errors-in-variables Models," SDS-4, MIMS, Meiji University, http://www.mims.meiji.ac.jp/publications/2017-ds.

[9] Kunitomo, N. , Sato and D. Kurisu (2018), *Separating Information Maximum Likelihood Estimation for High Frequency Financial Data*, Springer.

[10] Kunitomo, N. and S. Sato (2021), "A Robust-filtering Method for Noisy Non-Stationary Multivariate Time Series with an Application to Macro-consumption," Open Access, *Japanese Journal of Statistics and Data Science*, Springer.

[11] Kunitomo, N. , S. Sato and T. Sakurai (2021), "Estimating economic time series state and constructing a macro index," (in Japenese) SDS-19, MIMS, Meiji University.

[12] Sato, S. (2020), "x12siml6," In preparation (Unpublished Program).

[13] Sato, S. and Kunitomo, N. (2020), "On Backward Smoothing for Noisy Non-stationary Time Series," SDS-15, MIMS, Meiji University.

[14] Müller, U. and M. Watson (2018), "Long-run Covariability," *Econometrica*, 86-3, 775-804.

APPENDIX A : Mathematical Derivations

We present the details of derivation of Theorem 3.1 as an application of Theorem A.1 below, which is an extension of Proposition 2 in Kunitomo and Sato (2021) and Chapter 5 of Kunitomo, Sato and Kurisu (2018). Since some details are essentially the same in the existing literature, we omit some of them and only refer to them. We denote them as KS (2021) and KSK (2018), and use notations and some arguments in their proof. We first give an intuition for our result based on KS (2021) and then give our proof.

A-I A Heuristic Derivation : We use the arguments in Section 5 of KS (2021). Let $\theta_{jk} = \frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})$, $p_{jk}^{(n)} = \frac{1}{\sqrt{2n+1}}(e^{i\theta_{jk}} + e^{-i\theta_{jk}})$ and for $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ we write $\mathbf{z}_k$ $(k=1,\cdots,n)$ as

(A.1) $$\mathbf{z}_n(\lambda_k^{(n)}) = \sum_{j=1}^{n} p_{jk}^{(n)} \mathbf{r}_j \ , \ \mathbf{r}_j = \mathbf{y}_j - \mathbf{y}_{j-1} \ ,$$

which is a (real-valued) Fourier-type transformation and $\mathbf{y}_0$ is fixed.
Then, we find that $\mathbf{z}_n(\lambda_k^{(n)})$ $(k=1,\cdots,n)$ are the (real-valued) Fourier-transformation of data at the frequency $\lambda_k^{(n)}$ $(=(k-1/2)/(2n+1))$,, which is a (real-part of) estimate of the orthogonal incremental process $\mathbf{z}(\lambda)$ $(0 \leq \lambda \leq 1/2)$, which is continuous in the frequency domain. They are asymptotically uncorrelated random variables. (See Chapters 8-9 of Anderson (1971) or Section 5 of KS (2021).)
Then, by using a similar argument as in Proposition 1 of KS (2021), we can find that for $k \neq k^{'}$

(A.2) $$\mathbf{E}[\mathbf{z}_{in}(\lambda_k^{(n)})\mathbf{z}_{jn}(\lambda_k^{(n)})\mathbf{z}_{hn}(\lambda_{k^{'}}^{(n)})\mathbf{z}_{ln}(\lambda_{k^{'}}^{(n)})] = \sigma_{ij}(\lambda_k^{(n)})\sigma_{hl}(\lambda_{k^{'}}^{(n)}) + o(1)$$

and for $k = k^{'}$

(A.3) $$\begin{aligned} & \mathbf{E}[\mathbf{z}_{in}(\lambda_k^{(n)})\mathbf{z}_{jn}(\lambda_k^{(n)})\mathbf{z}_{hn}(\lambda_k^{(n)})\mathbf{z}_{ln}(\lambda_k^{(n)})] \\ = \ & \sigma_{ij}(\lambda_k^{(n)})\sigma_{hl}(\lambda_k^{(n)}) + \sigma_{ih}(\lambda_k^{(n)})\sigma_{jl}(\lambda_k^{(n)}) + \sigma_{il}(\lambda_k^{(n)})\sigma_{jh}(\lambda_k^{(n)}) + o(1) \ , \end{aligned}$$

where

(A.4) $$\sigma_{ij}(\lambda_k^{(n)}) = \sum_{h=-(n-1)}^{n-1} \left[\cos 2\pi\lambda_k^{(n)} h\right] \mathbf{\Gamma}_{ij}(h) \ ,$$

$\mathbf{E}[\mathbf{r}_j\mathbf{r}_{j-h}^{'}] = \mathbf{\Gamma}(h)$, $\mathbf{r}_j = \mathbf{y}_j - \mathbf{y}_{j-1}$ $(j=1,\cdots,n)$ and $\mathbf{y}_0$ is a fixed vector.
As $n \rightarrow \infty$ and $m/n \rightarrow 0$, we have $\lambda_k^{(n)} \rightarrow 0$ for $1 \leq k \leq m$. We write for $k=1,\cdots,m$ and as $m/n \rightarrow 0$,

(A.5) $$\lim_{n\rightarrow\infty} \sigma_{ij}(\lambda_k^{(n)}) = \sigma_{ij}^{(x)} \ (i,j=1,\cdots,p)$$

and $\mathbf{\Sigma}_x = (\sigma_{ij}^{(x)})$. Then

(A.6) $$\mathbf{Var}[\frac{1}{m}\sum_{k=1}^{m} \mathbf{z}_{in}(\lambda_k^{(n)})\mathbf{z}_{jn}(\lambda_k^{(n)})] \longrightarrow \sigma_{ii}^{(x)}\sigma_{jj}^{(x)} + \sigma_{ij}^{(x)2} \ .$$

We construct a sequence of random variables, which are approximately uncorrelated (see Proposition 1 of KS (2021)) and for $i, j = 1, \cdots, p$

$$s_{ij}(t) = \mathbf{z}_{in}(\lambda_t^{(n)})\mathbf{z}_{jn}(\lambda_t^{(n)}) - \mathbf{E}[\mathbf{z}_{in}(\lambda_t^{(n)})\mathbf{z}_{jn}(\lambda_t^{(n)})]$$

and

$$M_{ij}(n,k) = \sum_{t=1}^{k} s_{ij}(t) \ .$$

Then, heuristically, we can apply the CLT for stationary process to obtain the asymptotic normality. But, to show this argument in a rigorous way, we need further developments.

A-II Proof of the Main Results : We first prepare a general result of the consistency and asymptotic normality of the SIML estimation in non-stationary time series, which is new and it is an extension of Proposition 2 of KS (2021).

Theorem A.1: Assume that the fourth order moments of each element of $\mathbf{v}_i^{(x)}$ and $\mathbf{v}_i$ in (2.1)-(2.5) are bounded. Let

(A.7) $$\hat{\mathbf{\Sigma}}_x (= (\hat{\sigma}_{gh}^{(x)})) = \frac{1}{m}\mathbf{Z}_m^{*\prime}\mathbf{Z}_m^{*} \ ,$$

which is $\mathbf{G}_m^*$ in (3.2). Then
(i) For $m_n = [n^\alpha]$ and $0 < \alpha < 1$, as $n \longrightarrow \infty$

(A.8) $$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O} \ .$$

(ii) We set $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$. For $m_n = [n^\alpha]$ and $0 < \alpha < 0.8$, as $n \longrightarrow \infty$

(A.9) $$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{\mathcal{L}} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right) \ .$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g, h, k, l = 1, \cdots, p)$.

The proof of Theorem A.1 : The proof consists of two steps.
(**Step 1**) : Let $\mathbf{z}_k^{(x)} = (z_{kj}^{(x)})$ and $Z_k^{(s+v)} = (z_{kj}^{(s+v)})$ $(k = 1, \cdots, n)$ be the k-th row vector elements of $n \times p$ matrices

(A.10) $$\mathbf{Z}_n^{(x)} = \mathbf{K}_n^*(\mathbf{X}_n - \bar{\mathbf{Y}}_0) \ , \ \mathbf{Z}_n^{(s+v)} = \mathbf{K}_n^*(\mathbf{S}_n + \mathbf{V}_n) \ , \ \mathbf{K}_n^* = \mathbf{P}_n\mathbf{C}_n^{-1} \ ,$$

respectively, where we denote $\mathbf{X}_n = (\mathbf{x}_k^{'}) = (x_{kg})$, $\mathbf{S}_n = (\mathbf{s}_k^{'}) = (s_{kg})$, $\mathbf{V}_n = (\mathbf{v}_k^{'}) = (v_{kg})$, $\mathbf{Z}_n = (\mathbf{z}_k^{'})$ $(= (z_{kg}))$ are $n \times p$ matrices with $z_{kg} = z_{kg}^{(x)} + z_{kg}^{(s+v)}$. We write $z_{kg}, z_{kg}^{(x)}, z_{kg}^{(s+v)}$ as the $g-$th component of $\mathbf{z}_k, \mathbf{z}_k^{(x)}, \mathbf{z}_k^{(s+v)}$ $(k = 1, \cdots, n; g = 1, \cdots, p)$. We use $z_{kg}^{(f)}$ $(f = x, s+v)$ and decompose $\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x (= (\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)})_{gh})$ for $g, h = 1, \cdots, p)$. We re-write

$$
\begin{aligned}
\text{(A.11)}\quad & \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k\mathbf{z}_k^{'} - \mathbf{\Sigma}_x\right] \\
&= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'} - \mathbf{\Sigma}_x\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathbf{E}[\mathbf{z}_k^{(s+v)}\mathbf{z}_k^{(s+v)'}] \\
&+ \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{z}_k^{(s+v)}\mathbf{z}_k^{(s+v)'} - \mathbf{E}[\mathbf{z}_k^{(s+v)}\mathbf{z}_k^{(s+v)'}]\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[\mathbf{z}_k^{(x)}\mathbf{z}_k^{(s+v)'} + \mathbf{z}_k^{(s+v)}\mathbf{z}_k^{(x)'}\right].
\end{aligned}
$$

Then we can show that three terms except the first one of (A.11) are $o_p(1)$ (as in Theorem 4.1 of KS (2021), see Chapter KSK (2018) also) and the dominant term

$$
\text{(A.12)}\qquad \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'} - \mathbf{\Sigma}_x\right]
$$

is asymptotically normal as $m_n \to \infty$ $(n \to \infty)$, where $\mathbf{\Gamma}(h) = \mathbf{E}[\Delta\mathbf{x}_t\Delta\mathbf{x}_{t-h}^{'}]$ and

$$
\text{(A.13)}\qquad \mathbf{\Sigma}_x = \mathbf{f}_{\Delta x}(0) = \sum_{h=-\infty}^{+\infty}\mathbf{\Gamma}(h)\ .
$$

Then the second term of (A.5) is $o_p(1)$ if $m = [n^{\alpha}]$ $(0 < \alpha < 0.8)$.

We shall show that the second, third and fourth terms in the right-hand-side of (A.11) are asymptotically negligible as $n \to \infty$. By modifying the proof of Proposition 2 of KS (2021), it is straightforward to show these conditions because of the independence assumption among $\mathbf{\Delta x}_i$, $\mathbf{s}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$. We have utilized the relation that

$$
\text{(A.14)}\qquad \sqrt{m_n}\frac{1}{m_n}\sum_{k=1}^{m_n}a_{kn}^{*} = \frac{1}{\sqrt{m_n}}2\sum_{k=1}^{m_n}\left[1 - \cos(\pi\frac{2k-1}{2n+1})\right] = O(\frac{m_n^{5/2}}{n^2})\ ,
$$

and

$$
\text{(A.15)}\qquad \frac{1}{m}\sum_{k=1}^{m}2\cos(\pi\frac{2k-1}{2n+1}) = \frac{1}{m}\sum_{k=1}^{m}[e^{i\frac{2\pi}{2n+1}(k-\frac{1}{2})} + e^{-i\frac{2\pi}{2n+1}(k-\frac{1}{2})}] = \frac{1}{m}\frac{\sin(\frac{2\pi}{2n+1}m)}{\sin(\frac{\pi}{2n+1})}\ .
$$

Then we find that (A.14) is $o(1)$ when $m_n = [n^{\alpha}]$ $(0 < \alpha < 0.8)$ while (A.15) is bounded when $m_n/n \to 0$ and $n \to \infty$.
Because of (A.14), $(1/m)\sum_{k=1}^{m}a_{kn}^{*} = O([m/n]^2)$. By using (A.15), we have the

consistency result of $\hat{\Sigma}_x$ in (i) under the condition $m/n \to 0$ as $n \to \infty$ as in Kunitomo and Sato (2017) and KSK (2018).

(**Step 2**) : When we have the condition $0 < \alpha < 0.8$, we only need to evaluate the limiting distribution of the first term of (A.11) because of (A.14). Instead of (A.12), however, we shall consider the asymptotic distribution of

$$s_{ij}^{(m)*} = \frac{1}{\sqrt{m}}[g_{ij}^{(m*)} - \mathbf{E}(g_{ij}^{(m*)})] \tag{A.16}$$

and

$$g_{ij}^{(m*)} = (\frac{1}{m}\sum_{k=1}^{m}\mathbf{z}_k^{(x)}\mathbf{z}_k^{(x)'})_{ij} \; (i,j = 1,\cdots,p) \; . \tag{A.17}$$

Then we decompose

$$\begin{aligned} s_{ij}^{(m)*} &= \frac{1}{\sqrt{m}}\sum_{k=1}^{m}[\sum_{s=t=1}^{n} p_{ks}^2(r_{is}^* r_{js}^* - \mathbf{E}(r_{is}^* r_{js}^*))] \\ &\quad + \frac{1}{\sqrt{m}}\sum_{k=1}^{m}[\sum_{s\neq t=1}^{n} p_{ks}^2(r_{is}^* r_{jt}^* - \mathbf{E}(r_{is}^* r_{jt}^*))] \end{aligned} \tag{A.18}$$

and

$$\mathbf{r}_i^* = \Delta\mathbf{x}_i = \sum_{s=0}^{\infty}\mathbf{\Gamma}_s\mathbf{w}_{i-s} \; , \tag{A.19}$$

where $\mathbf{\Gamma}_s \; (= (\gamma_{is}))$ are $p \times p$ matrices with $\mathbf{\Gamma}(h) = O(\rho^{|h|}) \; (0 \leq \rho < 1)$, and we take $\mathbf{w}_i \; (= \mathbf{e}_i^{(x)})$ as a sequence of mutually independent random variables with $\mathbf{E}[\mathbf{w}_i] = 0$, $\mathbf{E}[\mathbf{w}_i\mathbf{w}_i^{'}] = \mathbf{\Sigma}_v^{(x)} \; (> 0)$.

When we have the condition $m_n/n \to 0$ as $n \to \infty$, the proof of Proposition 1 of KS (2021) implies that

$$\frac{1}{\sqrt{m_n}}[\sigma_{ij}^{(x)} - \mathbf{E}(g_{ij}^{(m*)})] = o(1) \; . \tag{A.20}$$

We note that the evaluation of the limiting distribution of (A.16) or (A.18) is considerably simpler than the one for (A.12).

We set $c_{st} = [(2n+1)/2m]a_{st}$ for a_{st} $(s,t = 1,\cdots,n)$ in (2.15). Then the first term of (A.18) is asymptotically negligible because $\sum_{k=1}^{n} p_{ks}^2 = [2m/[2n+1]c_{ss}$ and $\sum_{s=1}^{n} c_{ss}^2 = O(n)$ as given in Chapter 5 of KSK (2018). We show the asymptotic normality of the leading term

$$s_{ij}^{(m)**} = \frac{2\sqrt{m}}{2n+1}\sum_{s\neq t=1}^{n} c_{st}[r_{is}^* r_{jt}^* - \mathbf{E}(r_{is}^* r_{jt}^*)] \; , \tag{A.21}$$

where

$$c_{st} = \frac{2}{m}\sum_{k=1}^{m} s_{sk}s_{tk} \;\; , \; s_{jk} = \cos\frac{2\pi}{2n+1}(j - \frac{1}{2})(k - \frac{1}{2}) \; . \tag{A.22}$$

Under the stationarity condition of (A.19), the difference between (A.21) and the second term of (A.18) is asymptotically negligible.
The proof of the asymptotic normality requires a lengthy derivation, which is a modification of the method for the spectral density estimation used in the proof of Theorem 9.4.1 of Anderson (1971). Because some of our arguments are quite similar, we only give the essential arguments and differences. We give the proof for the case when $p = 1$ and use the notation $\mathbf{\Gamma}_s = \gamma_s$ $(s = 0, 1, \cdots)$ and $s^{(m)**} = s_{ij}^{(m)**}$ because the proof of the general case when $p \geq 1$ can be done by using the standard device of $r_j^{**} = \mathbf{a}^{'}\mathbf{r}_j^{*}$ $(j = 1, \cdots, n)$ with an arbitrary ($p \times 1$ non-zero constant) vector $\mathbf{a}$.

Let $K_n = [n/m]$ be a sequence of positive integers and $K_n \to \infty$ $(n \to \infty)$. Then given s, $c_{st} \to 0$ for $t - s > K_n$ as $m, n \to \infty$ and $m/n \to 0$. Then by taking $t = s + k$ $(k = 1, \cdots, n - s)$ we re-write

$$\text{(A.23)} \quad s^{(m)**} = \frac{4\sqrt{m}}{2n+1}\sum_{t>s=1}^{n} c_{st}[r_s r_t - \mathbf{E}(r_s r_t)] \ ,$$
$$= \frac{4\sqrt{m}}{2n+1}\sum_{l,l'=1}^{\infty}\gamma_l\gamma_{l'}\sum_{s=1}^{n}\sum_{t=s+1,s-l\neq t-l'}^{n} c_{st}w_{s-l}w_{t-l'} \ .$$
$$= \frac{4\sqrt{m}}{2n+1}\sum_{l,l'=1}^{\infty}\gamma_l\gamma_{l'}\sum_{s=1}^{n}\sum_{k=1}^{n+1-s}\sum_{s=1,s-l\neq s+k-l'}^{n} c_{s,s+k}w_{s-l}w_{s+k-l'} \ .$$

We truncate the sum $\sum_{l,l'=1}^{\infty}[\,\cdot\,]$, which decomposes as $(\sum_{l=1}^{r_n} + \sum_{l=r_n+1}^{\infty})\,(\sum_{l'=1}^{r_n} + \sum_{l'=r_n+1}^{\infty})[\,\cdot\,]$, by a sequence of sums $\sum_{l,l'=1}^{r_n}[\,\cdot\,]$ such that $r_n \to \infty$ and $\sum_{l=r_n+1}^{\infty}|\gamma_l| \to 0$. We can approximate the infinite sum by a finite sum because the remaing terms are of smaller order. The main term is

$$\text{(A.24)} \quad s_1^{(m)**} = \frac{4\sqrt{m}}{2n+1}\sum_{l,l'=1}^{r_n}\gamma_l\gamma_{l'}\sum_{s=1}^{n}\sum_{k=1,s-l\neq s+k-l'}^{n+1-s} c_{s,s+k}w_{s-l}w_{s+k-l'}$$
$$= \frac{4\sqrt{m}}{2n+1}\sum_{l,l'=1}^{r_n}\gamma_l\gamma_{l'}\sum_{h=l-l'+1,h\neq 0}^{n-q-l'}\sum_{q=1-l}^{n-l} c_{q+l,q+h+l'}w_q w_{q+h} \ .$$

Since some parts of the above summation (i.e., the terms in $\sum_{l-l'\leq h<1}[\,\cdot\,]$, $\sum_{n-q-l'\leq h<n-q}[\,\cdot\,]$, $\sum_{1-l\leq q<0}[\,\cdot\,]$ and $\sum_{n-l\leq q\leq n-1}[\,\cdot\,]$) can be of negligible order, we can approximate the summation as

$$\text{(A.25)} \quad s_{11}^{(m)**} = \frac{4\sqrt{m}}{2n+1}[\sum_{l=1}^{r_n}\gamma_l\sum_{l'=1}^{r_n}\gamma_{l'}]\sum_{h=l-l'+1}^{n-q-l'}\sum_{q=1-l}^{n-l} c_{q+l,q+h+l'}w_q w_{q+h}$$
$$\sim \frac{4\sqrt{m}}{2n+1}[\sum_{l=1}^{r_n}\gamma_l\sum_{l'=1}^{r_n}\gamma_{l'}]\sum_{h=1}^{n-q}\sum_{q=1}^{n} c_{q+l,q+h+l'}w_q w_{q+h} \ ,$$

where we denote $c_{s,t} = 0$ $(s > n$ or $t > n)$ for the resulting notational convenience.

We take $m = [n^{\alpha}]$ $(0 < \alpha < 0.8)$, $K_n = [n/m]$, $N_n = [n^{\delta/2}]$ $(\delta > 0)$, $M_n = [n^{1-\delta/2}]$ such that $1 - \delta/2 > 0$ and $\alpha + \delta/2 > 1$. Then we find that $K_n/N_n \to 0$, $N_n/n \to 0$ $\sqrt{m}/n \sim [1/\sqrt{n}][1/\sqrt{K_n}]$ and $M_n \sim n/N_n$ as $n \to \infty$. In the following we utilize the relation that $c_{q+l,q+h+l'} - c_{q,q+h} = o(1)$ for $l, l' = 1, \cdots, r_n$ if we take r_n such that $r_n \times m_n/n \to 0$ as $n, m_n \to \infty$. It is because

$$
\begin{aligned}
& \sin 2\pi m[\frac{2q+h+l+l'}{2n+1}] - \sin 2\pi m[\frac{2q+h}{2n+1}] \\
= \; & \sin 2\pi m[\frac{2q+h}{2n+1}][\cos 2\pi m(\frac{l+l'}{2n+1}) - 1] + \cos 2\pi m[\frac{2q+h}{2n+1}] \sin 2\pi m[\frac{l+l'}{2n+1}] \to 0
\end{aligned}
$$

as $n \to \infty$, for instance.

Furthermore, by using that some parts of (A.25) are of smaller orders as $n \to \infty$ (the terms in $\sum_{h=K_n+1}[\,\cdot\,]$), we can apply the CLT to

$$
s_{11}^{(m)***} = 2[\sum_{l=1}^{r_n} \gamma_l]^2 \frac{1}{\sqrt{n}} \frac{1}{\sqrt{K_n}} \sum_{q=1}^{n} \sum_{h=1}^{K_n} c_{q,q+h} w_q w_{q+h} \; , \tag{A.26}
$$

where we denote $c_{q,q+h} = 0$ $(q + h > n)$ for the notational convenience. We notice that $c_{q,q+h}$ $(q = 1, \cdots, n)$ is a sequence of bounded real numbers.

Let

$$
W_{qn} = \frac{1}{\sqrt{K_n}} \sum_{h=1}^{K_n} c_{q,q+h} w_q w_{q+h} \tag{A.27}
$$

and

$$
U_{jn} = \frac{1}{\sqrt{N_n}} [W_{(j-1)N_n+1,n} + \cdots + W_{jN_n - K_n,n}] \;\; (j = 1, \cdots, M_n) \; . \tag{A.28}
$$

Then we find that $\mathbf{E}[W_{q,n}] = 0$, $\mathbf{E}[W_{q,n} W_{q+h,n}] = 0$ (h is any non-zero integer), $\mathbf{E}[W_{q,n}^2]$ are bounded. Also we have that $U_{1,n}, \cdots, U_{M_n,n}$ are mutually independent and $\mathbf{E}[U_{i,n}^4]$ $(i = 1, M_n)$ are uniformly bounded by using the assumption of boundedness of 4-th order moments of W_q $(q = 1, \cdots, n)$. Since other terms except the leading term are stochastically of smaller order, which we can ignore for evaluating the limiting distribution, we apply the Liyaponov-type CLT(central limit theorem).

By using the relation that

$$
\frac{1}{\sqrt{n}} \sum_{q=1}^{n} W_{qn} - \frac{1}{\sqrt{M_n}} \sum_{j=1}^{M_n} U_{jn} \xrightarrow{p} 0 \tag{A.29}
$$

as $n \to \infty$, and the remaining terms are of smaller order (i.e. K_n terms in each U_{jn} $(j = 1, \cdots, M_n)$) when $m, n \to \infty$ and $m = n^{\alpha}, 0 < \alpha < 0.8$ because of

$K_n/N_n \to 0$. Then we have the asymptotic normality of (A.19) when $p = 1$. By using the relation $\sum_{s,t=1}^{n} c_{st}^2 = (n+1/2)^2/m$ ((5.16) of KSK (2018)) and

$$4[\sum_{j=-\infty}^{\infty} \gamma_j]^2 \sum_{g=1}^{n} \sum_{h=1}^{K_n} c_{g,g+h}[\sigma_v^{(x)}]^4 \sim 2[\sum_{j=-\infty}^{\infty} \gamma_j]^2 \sum_{s,t=1}^{n} c_{st}^2 [\sigma_v^{(x)}]^4 \ , \tag{A.30}$$

we have the desired result of the asymptotic variance when $p = 1$.
When $p \geq 1$, we can evaluate the asymptotic covariance by calculating the covariance of $\sum_{a,b,q,h} c_{q,q+h} \boldsymbol{\gamma}_{as} \mathbf{w}_q \boldsymbol{\gamma}_{bt} \mathbf{w}_{q+h}$ and $\sum_{c,d,q',h'} c_{q',q'+h'} \boldsymbol{\gamma}_{cs} \mathbf{w}_{q'} \boldsymbol{\gamma}_{bt} \mathbf{w}_{q'+h'}$, where $\boldsymbol{\gamma}_{ac}$ is the a-th row vector of $\boldsymbol{\Gamma}_s$. Then, after a straightforward evaluation, we finally find the asymptotic covariance in Theorem A.1 as $\sigma_{ac}^{(x)} \sigma_{bd}^{(x)} + \sigma_{ad}^{(x)} \sigma_{bc}^{(x)}$ $(a, b, c, d = 1, \cdots, p)$. (We note that the argument here is essentially the same to the one in Chapter 5 of KSK (2018) in the high-frequency financial formulation.)
(**Q.E.D**)

The proof of Theorem 3.1 : We use the representation

$$\hat{\boldsymbol{\beta}}_m - \boldsymbol{\beta} = \mathbf{G}_{22}^{-1}[(\mathbf{0}, \mathbf{I}_k)\mathbf{G} \begin{pmatrix} 1 \\ -\boldsymbol{\beta} \end{pmatrix} \ , \tag{A.31}$$

Because $(1/m)\mathbf{G}_m \xrightarrow{p} \boldsymbol{\Sigma}_{22}$ $(m/n \to 0, n \to \infty)$ and under the assumption that $\boldsymbol{\Sigma}_{22}$ is a positive definite matrix, we investigate the asymptotic distribution of

$$\sqrt{m_n}[\hat{\boldsymbol{\beta}}_m^* - \boldsymbol{\beta}] = \boldsymbol{\Sigma}_{22}^{-1} \frac{1}{\sqrt{m}} (\mathbf{0}, \mathbf{I}_k)\mathbf{G} \begin{pmatrix} 1 \\ -\boldsymbol{\beta} \end{pmatrix} \ , \tag{A.32}$$

Then the asymptotic variance-covariance matrix can be written as

$$\mathbf{AV}[\hat{\boldsymbol{\beta}}_m] = \boldsymbol{\Sigma}_{22}^{-1} \mathbf{Cov}[(\mathbf{0}, \mathbf{I}_k)\mathbf{S}\mathbf{b} \ , \ \mathbf{b}'\mathbf{S} \begin{pmatrix} \mathbf{0}' \\ \mathbf{I}_k \end{pmatrix}] \boldsymbol{\Sigma}_{22}^{-1} \ , \tag{A.33}$$

where $\mathbf{S} = \sqrt{m_n}[\mathbf{G} - \boldsymbol{\Sigma}]$ $(= (s_{jk}))$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -\boldsymbol{\beta} \end{pmatrix}$ $(= (b_j))$.
By using Theorem A.1, we can evaluate the (l, l')-th element $(l, l' = 2, \cdots, k+1)$ as

$$\begin{aligned} \mathbf{Cov}[\sum_{j=1}^{k+1} b_j s_{jl} \sum_{j'=1}^{k+1} b_{j'} s_{j'l'}] &= \sum_{j,j'=1}^{k+1} b_j b_{j'} (\sigma_{j,j'} \sigma_{l,l'} + \sigma_{j,l'} \sigma_{j',l}) \\ &= \sigma_{l,l'} \sum_{j=1}^{k+1} b_j [\sum_{l'=1}^{k+1} b_{j'} \sigma_{j,j'}] + [\sum_{j=1}^{k+1} b_j \sigma_{j,l'}][\sum_{j'=1}^{k+1} b_{j'} \sigma_{l,j'}] \\ &= \sigma_{l,l'} \sigma_{11.2} \end{aligned}$$

because $[\boldsymbol{\sigma}_{21}, \boldsymbol{\Sigma}_{22}]\mathbf{b} = \mathbf{0}$ and

$$[\boldsymbol{\sigma}_{11}, \boldsymbol{\sigma}_{12}]\mathbf{b} = \sigma_{11} - \boldsymbol{\sigma}_{12} \boldsymbol{\Sigma}_{22}^{-1} \boldsymbol{\sigma}_{21} \ . \tag{A.34}$$

Then we have the result of the asymptotic variance-covariance matrix.
(**Q.E.D**)

The proof of Corollary 4.1 : We use the representation

$$\hat{\mathbf{B}}_m - \mathbf{B} = (\mathbf{W}_n^{*'}\mathbf{W}_n^*)^{-1}\mathbf{W}_n^{*'}\mathbf{U}_n^* \ , \tag{A.35}$$

where $\mathbf{U}_n^* = \mathbf{J}_m\mathbf{P}_n\mathbf{C}_n^{-1}\mathbf{U}_n$. By using a similar argument as the proof of Theorem 3.1 under the assumption of (4.4), we find that

$$\mathbf{AV}[\hat{\mathbf{B}}_m] = \mathbf{\Sigma}_{w^*}^{-1}\mathbf{Cov}[\frac{1}{\sqrt{m}}\mathbf{W}_n^{*'}\mathbf{U}_n^*, \frac{1}{\sqrt{m}}\mathbf{W}_n^{*'}\mathbf{U}_n^*]\mathbf{\Sigma}_{w^*}^{-1} \ . \tag{A.36}$$

Then by using Theorem A.1 and Theorem 3.1 we have the result.
(**Q.E.D**)

APPENDIX B : Figures

In this Appendix, we gather some figures in Section 6. All computations have been done by x12siml6 written in R (Sato (2020)), which will be available in the near future.

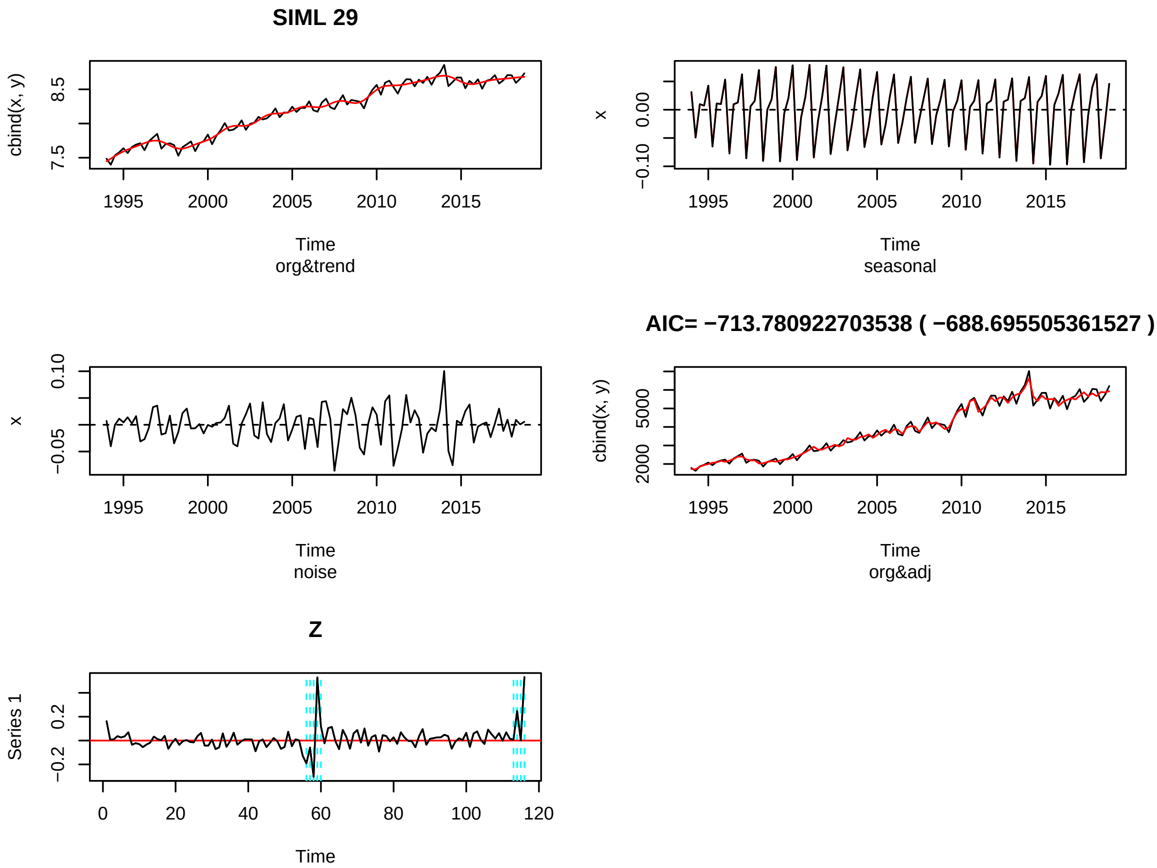

Figure 1: Macro-consumption (Data are the Quarterly real consumption of durable goods (after log-transformation) between 1994Q1-2018Q4, which were published by the Economic Social Research Institute (ESRI), Cabinet Office, Japan.)

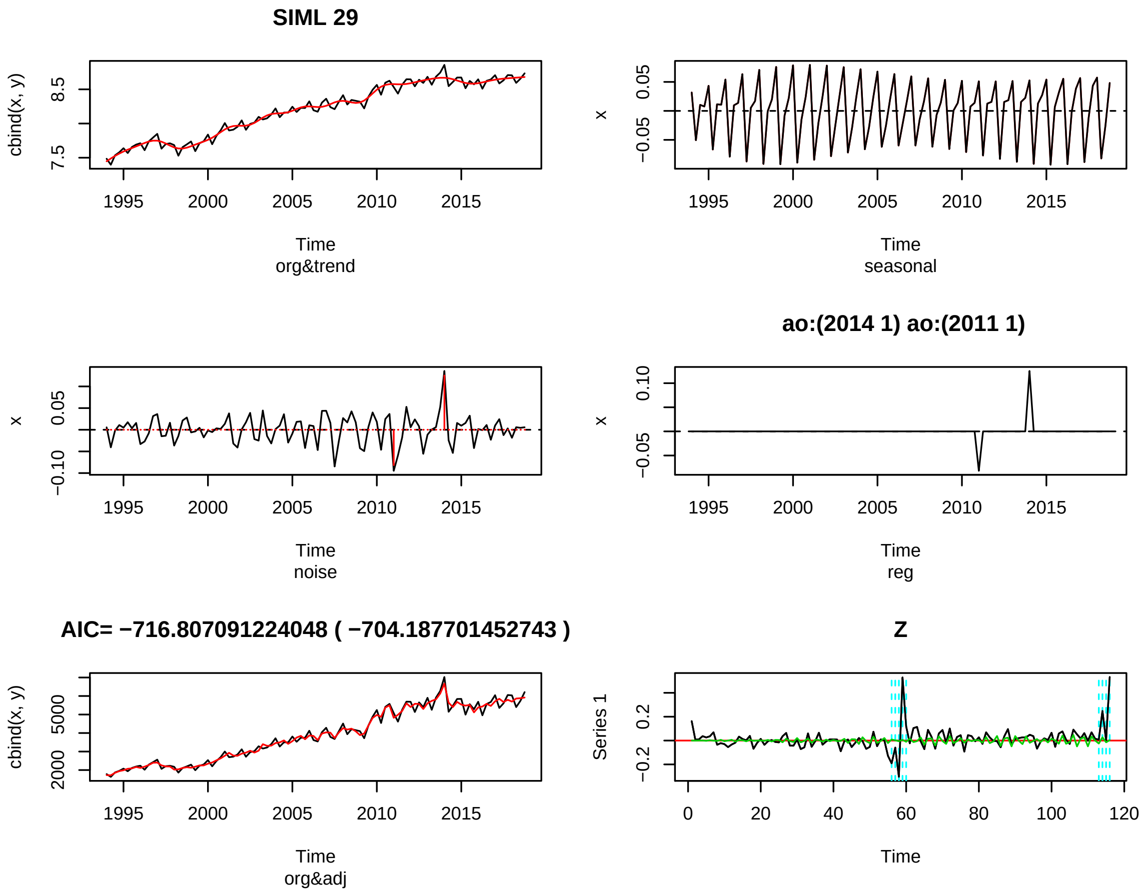

Figure 2: Macro-consumption (Data are the Quarterly real consumption of durable goods (after log-transformation) between 1994Q1-2018Q4, which were published by the Economic Social Research Institute (ESRI), Cabinet Office, Japan.)

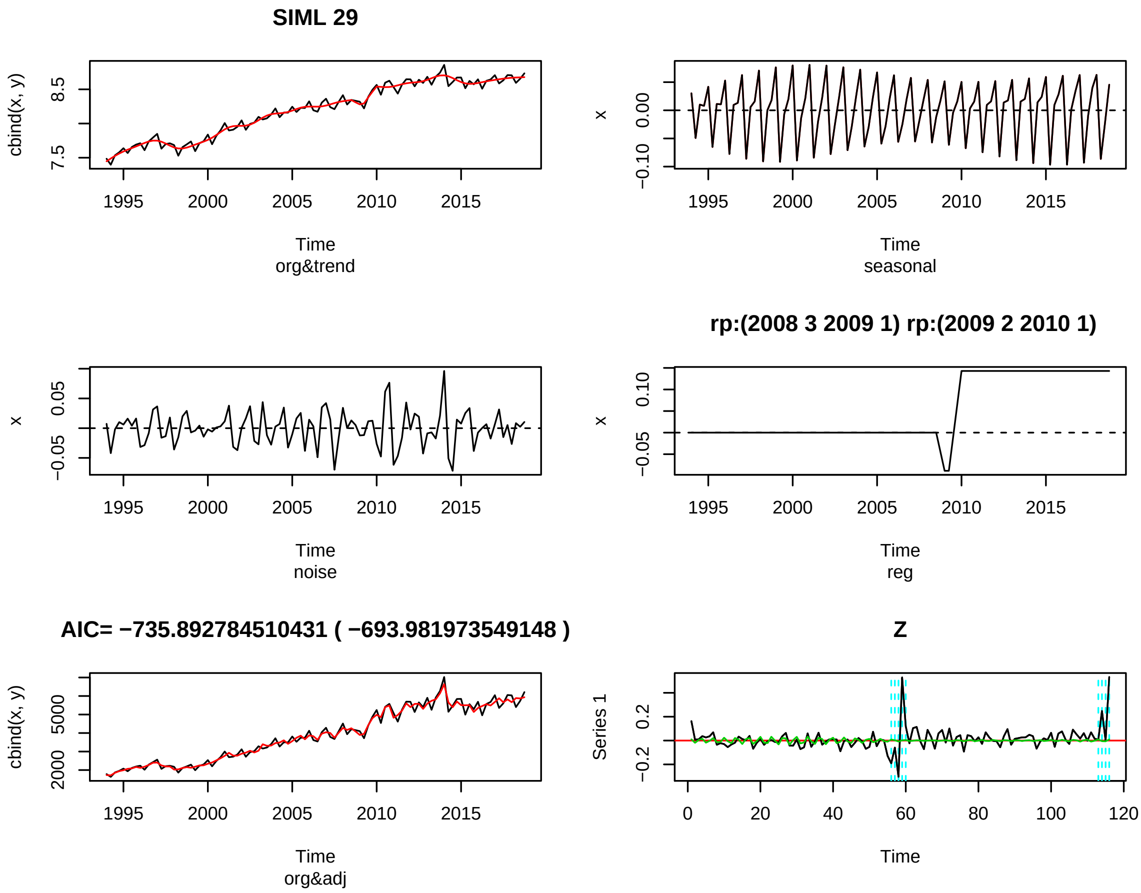

Figure 3: Macro-consumption (Data are the Quarterly real consumption of durable goods (after log-transformation) between 1994Q1-2018Q4, which were published by the Economic Social Research Institute (ESRI), Cabinet Office, Japan.)

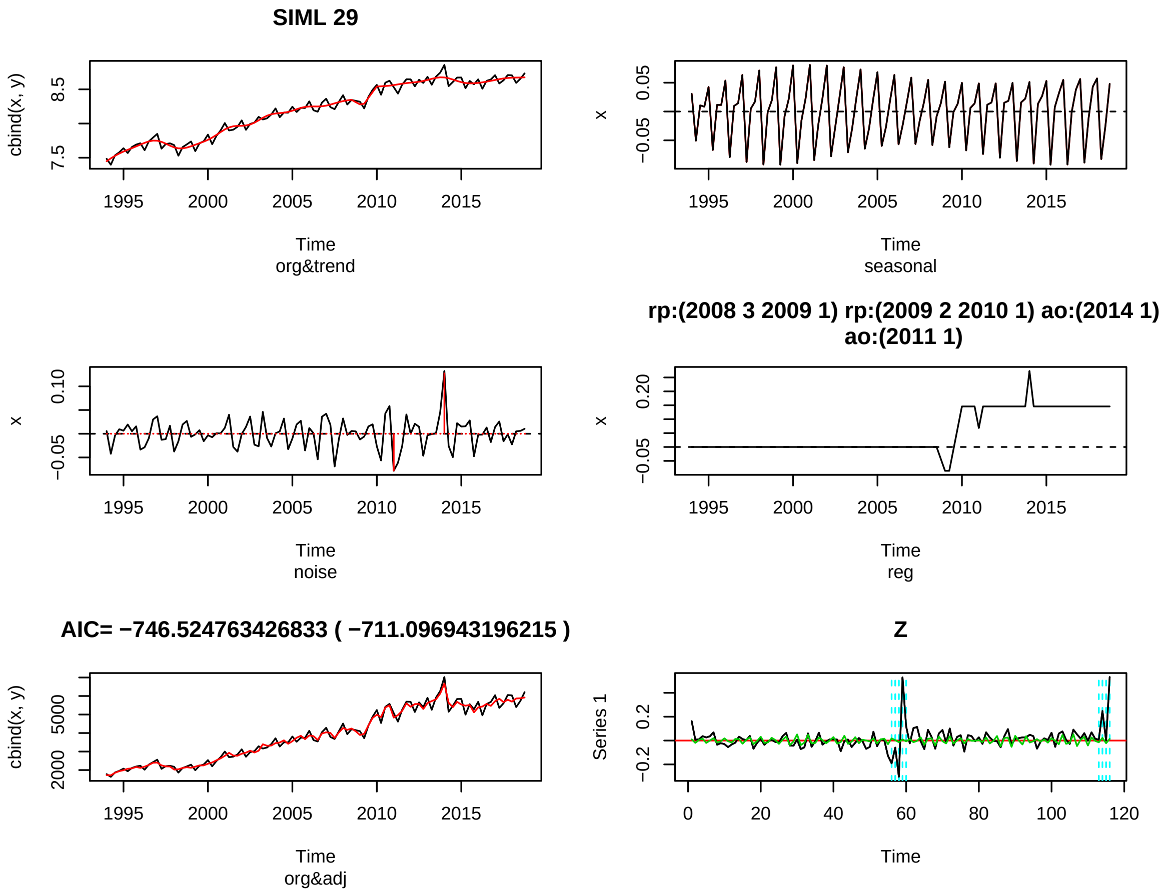

Figure 4: Macro-consumption (Data are the Quarterly real consumption of durable goods (after log-transformation) between 1994Q1-2018Q4, which were published by the Economic Social Research Institute (ESRI), Cabinet Office, Japan.)