**MIMS**
**Meiji Institute for Advanced Study of Mathematical Sciences**

# International Conference
# on
# COMMUTATIVE ALGEBRA

**Yokohama Port Opening Memorial Hall**

**March 17 - 21, 2008**

# Program and Abstracts

# International Conference on Commutative Algebra

## Yokohama, March 17 - 21, 2008

The first International Conference on Commutative Algebra in Yokohama was organized by Shiro Goto and Kei-ichi Watanabe and held in August of 2001. This is the second time, which is performed as one of the activities of Meiji Institute for Advanced Study of Mathematical Sciences (MIMS). The fund for the conference comes from Research Project Grant (A) of Institute of Science and Technology of Meiji University. The students of Meiji University among the participants are partially supported by Support Program for Improving Graduate School Education (GP).

**Host Institution:** Meiji Institute for Advanced Study of Mathematical Sciences (MIMS)

**Organizers:** Shiro Goto, Kei-ichi Watanabe, Koji Nishida, Kazuhiko Kurano

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# Time Table

The rooms **Room 6, 8** of Yokohama Port Opening Memorial Hall are available at **9:00 AM**. Room 6 is for the conference and Room 8 is for registration.

| **17 (Mon)** | **18 (Tue)** | **19 (Wed)** | **20 (Thu)** | **21 (Fri)** |
|---|---|---|---|---|
| 9:10–9:25 *Registration* | 9:10–9:50 **Sharp** | 9:10–9:50 **Yoshino** | 9:10–9:50 **Vasconcelos** | 9:10–9:50 **Bruns** |
| 9:25–9:30 *Opening* | 10:05–10:45 **Yoshida** | 10:05–10:45 **Leuschke** | 10:05–10:45 **Verma** | 10:05–10:45 **Yanagawa** |
| 9:30–10:10 **Huneke** | 10:45–11:15 *Coffee Break* | 11:05–11:45 **Iyama** | 10:45–11:15 *Coffee Break* | 10:45–11:15 *Coffee Break* |
| 10:25–11:05 **Takagi** | 11:15–11:55 **Hoa** | 12:00–12:40 **Takahashi** | 11:15–11:55 **Corso** | 11:15–11:55 **Nishida** |
| 11:05–11:35 *Coffee Break* | 12:10–12:35 **Ozeki** | | 12:10–12:35 **Ionescu** | |
| 11:35–12:15 **Brenner** | | | | |
| 14:00–14:40 **Rossi** | 14:20–15:00 **Roberts** | 13:30– *Excursion* | 14:20–15:00 **Polini** | 13:30–14:10 **Ulrich** |
| 14:55–15:35 **Elias** | 15:15–15:55 **Srinivas** | | 15:15–15:55 **Kurano** | 14:25–15:05 **Hyry** |
| 15:35–16:05 *Coffee Break* | 15:55–16:25 *Coffee Break* | | 15:55–16:25 *Coffee Break* | 15:05–15:35 *Coffee Break* |
| 16:05–16:30 **Terai** | 16:25–17:05 **De Negri** | | 16:25–16:50 **Miyazaki** | 15:35–16:00 **Iai** |
| 16:45–17:25 **Ghezzi** | 17:20–18:00 **Planas-Vilanova** | | 17:00–17:40 **Hashimoto** | 16:15–16:55 **Trung** |
| 17:45–18:25 **Cuong** | 19:00– *Banquet* | | 17:50–18:30 **Kuroda** | 17:15–17:55 **Zarzuela** |

# International Conference
# on
# Commutative Algebra

## March 17 - 21 , 2008

**Room 6, Yokohama Port Opening Memorial Hall**

# Schedule of Scientific Program

**March 17 (Mon)**
**Chairman : Shiro Goto**

| | |
|---|---|
| 9:10 – 9:25 | *Registration* at **Room 8** |
| 9:25 – 9:30 | *Opening* |
| 9:30 – 10:10 | **Craig Huneke** (University of Kansas)<br>Symbolic powers, tight closure, and uniformity |
| 10:25 – 11:05 | **Shunsuke Takagi** (Kyushu University)<br>Finiteness properties of rings with finite F-representation type |
| 11:05 – 11:35 | *Coffee Break* |
| 11:35 – 12:15 | **Holger Brenner** (University of Osnabrueck)<br>Tight closure is dead - long live tight closure |
| 12:15 – 14:00 | *Lunch* |

**Chairman : Koji Nishida**

| | |
|---|---|
| 14:00 – 14:40 | **Maria E. Rossi** (University of Genova)<br>Extremal Betti numbers of ideals in a local regular ring |
| 14:55 – 15:35 | **Juan Elias** (University of Barcelona)<br>Structure theorems and classification of certain Gorenstein ideals |
| 15:35 – 16:05 | *Coffee Break* |
| 16:05 – 16:30 | **Naoki Terai** (Saga University)<br>Arithmetical rank of Stanley-Reisner ideals with 2-linear resolution |
| 16:45 – 17:25 | **Laura Ghezzi** (New York City College of Technology)<br>Generalizations of the strong Castelnuovo lemma |

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|---|---|
| 17:45 – 18:25 | **Nguyen Tu Cuong** (Institute of Mathematics, Hanoi)<br>Asymptotic Behaviour of parameter ideals in generalized Cohen-Macaulay modules |

**March 18 (Tue)**
**Chairman : Ngo Viet Trung**

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|---|---|
| 9:10 – 9:50 | **Rodney Y. Sharp** (University of Sheffield)<br>Frobenius structures on injective modules and tight closure test ideals |
| 10:05 – 10:45 | **Ken-ichi Yoshida** (Nagoya University)<br>Some properties of generalized test ideals |
| 10:45 – 11:15 | *Coffee Break* |
| 11:15 – 11:55 | **Le Tuan Hoa** (Institute of Mathematics, Hanoi)<br>Gröbner bases of simplicial toric ideals |
| 12:10 – 12:35 | **Kazuho Ozeki** (Meiji University)<br>The structure of Sally module of rank one |
| 12:35 – 14:20 | *Lunch* |

**Chairman : Kazuhiko Kurano**

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|---|---|
| 14:20 – 15:00 | **Paul C. Roberts** (University of Utah)<br>Fontaine rings defined by noetherian rings |
| 15:15 – 15:55 | **Vasudevan Srinivas** (Tata Institute)<br>Finite Schur filtration dimension for modules over an algebra with Schur filtration |
| 15:55 – 16:25 | *Coffee Break* |
| 16:25 – 17:05 | **Emanuela De Negri** (University of Genova)<br>Contracted ideals and their associated graded ring |
| 17:20 – 18:00 | **Francesc Planas-Vilanova** (Universitat Polit ecnica de Catalunya)<br>Irreducible affine space curves and the uniform Artin-Rees property on the prime spectrum |
| 19:00 – | *Banquet* |

**March 19 (Wed)**
**Chairman : Craig Huneke**

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|---|---|
| 9:10 – 9:50 | **Yuji Yoshino** (Okayama University)<br>Deforming a chain complex to non-commutative direction |
| 10:05 – 10:45 | **Graham J. Leuschke** (Syracuse University)<br>Non-commutative desingularizations |
| 11:05 – 11:45 | **Osamu Iyama** (Nagoya University)<br>Cluster tilting for one-dimensional hypersurface singularities |
| 12:00 – 12:40 | **Ryo Takahashi** (Shinshu University)<br>Approximating modules by resolving subcategories |
| 13:30 – | *Excursion* |

**March 20 (Thu)**
**Chairman : Maria E. Rossi**

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|---|---|
| 9:10 – 9:50 | **Wolmer V. Vasconcelos** (Rutgers University)<br>The Chern Coefficients of Local Rings |
| 10:05 – 10:45 | **Jugal K. Verma** (Indian Institute of Technology Bombay)<br>Hilbert polynomials and powers of ideals |
| 10:45 – 11:15 | *Coffee Break* |
| 11:15 – 11:55 | **Alberto Corso** (University of Kentucky)<br>Specializations of Ferrers ideals |
| 12:10 – 12:35 | **Cristodor Ionescu** (Institute of Mathematics of the Romanian Academy)<br>Some algebraic invariants of mixed product ideals |
| 12:35 – 14:20 | *Lunch* |

**Chairman : Paul C. Roberts**

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|---|---|
| 14:20 – 15:00 | **Claudia Polini** (University of Notre Dame)<br>Blowups and fibers of morphisms |
| 15:15 – 15:55 | **Kazuhiko Kurano** (Meiji University)<br>Symbolic Rees rings of space monomial curves in characteristic $p$ and existence of negative curves in characteristic 0 |
| 15:55 – 16:25 | *Coffee Break* |

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|---|---|
| 16:25 – 16:50 | **Mitsuhiro Miyazaki** (Kyoto University of Education)<br>Doset Hibi rings and application |
| 17:00 – 17:40 | **Mitsuyasu Hashimoto** (Nagoya University)<br>Equivariant local cohomology |
| 17:50 – 18:30 | **Shigeru Kuroda** (Tokyo Metropolitan University)<br>Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism |

**March 21 (Fri)**
**Chairman : Bernd Ulrich**

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|---|---|
| 9:10 – 9:50 | **Winfried Bruns** (University of Osnabrueck)<br>Algebras over monoidal complexes |
| 10:05 – 10:45 | **Kohji Yanagawa** (Kansai University)<br>Squarefree modules over a toric face ring |
| 10:45 – 11:15 | *Coffee Break* |
| 11:15 – 11:55 | **Koji Nishida** (Chiba University)<br>An upper bound on the reduction number of an ideal |
| 11:55 – 13:30 | *Lunch* |

**Chairman : Kei-ichi Watanabe**

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|---|---|
| 13:30 – 14:10 | **Bernd Ulrich** (Purdue University)<br>Criteria for Integral Dependence |
| 14:25 – 15:05 | **Eero Hyry** (University of Tampere)<br>On the Jacobian ideal of the Rees algebra |
| 15:05 – 15:35 | *Coffee Break* |
| 15:35 – 16:00 | **Shin-ichiro Iai** (Hokkaido University of Education, Sapporo)<br>On Gorenstein Rees algebras |
| 16:15 – 16:55 | **Ngo Viet Trung** (Institute of Mathematics, Hanoi)<br>Combinatorial characterizations of normal monomial ideals |
| 17:15 – 17:55 | **Santiago Zarzuela** (Universitat de Barcelona)<br>Micro-invariants of a one dimensional Cohen-Macaulay ring and invariants of its tangent cone |

# International Conference on Commutative Algebra

## Yokohama, March 17 - 21 , 2008

# Abstracts

## Tight closure is dead - long live tight closure

Holger Brenner

We describe an example, based on a joint work with Paul Monsky, showing that tight closure does not commute with localization. The example is given by a normal hypersurface domain in dimension three in characteristic two and the ideal is generated by three elements. We consider the ring as a family of graded two-dimensional rings parameterized by the affine line. It turns out that a certain element belongs to the tight closure of the ideal in the generic (transcendental) fiber ring, but never in any special (algebraic) fiber ring. This contradicts the localization property.

The geometry in the background of this example is the existence of a vector bundle on a family of smooth projective curves parameterized by the affine line, such that the bundle on the generic curve is strongly semistable, but not so on any special curve.

If time permits we will also discuss some new developments, discussed with Helena Fischbacher-Weitz, indicating that a certain generic ideal inclusion, which is based on the Froeberg conjecture and which holds for the polynomial ring in three variables, has a tight closure version for graded three-dimensional Cohen-Macaulay domains.

## Algebras over monoidal complexes

Winfried Bruns

Monoidal complexes provide a combinatorial structure that generalizes both simplicial complexes and affine monoids: affine monoids are glued according to a combinatorial pattern that is provided by a complex of rational cones. Standard examples are provided by Stanley's toric face rings. Moreover, such algebras appear naturally in the Gröner basis theory of affine monoid rings.

Algebras over monoidal complexes have been studied intensively by the Osnabrük commutative algebra group (W. Bruns, B. Ichim, T. Römer, in corporation with M. Brun and J. Gubeladze). We will in particular discuss formulas for local cohomology and Betti numbers that generalize Hochster's well-known formulas for Stanley-Reisner rings.

# Specializations of Ferrers ideals

Alberto Corso

We introduce a specialization technique in order to study monomial ideals that are generated in degree two by using our earlier results about Ferrers ideals. It allows us to describe explicitly a cellular minimal free resolution of various ideals including any strongly stable and any squarefree strongly stable ideal whose minimal generators have degree two. In particular, this shows that threshold graphs can be obtained as specializations of Ferrers graphs, which explains their similar properties.

# Asymptotic Behaviour of Parameter Ideals in Generalized Cohen-Macaulay Modules

Nguyen Tu Cuong

This is a joint work with Hoang Le Truong.

We give in this talk affirmative answers to two open questions as follows. Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay Noetherian local ring. Both questions, the first question was raised by M. Rogers [2] and the second one is due to S. Goto and H. Sakurai [1], ask whether for every parameter ideal $\mathfrak{q}$ contained in a high enough power of the maximal ideal $\mathfrak{m}$ the following statements are true: (1) The index of reducibility $N_R(\mathfrak{q}; R)$ is independent of the choice of $\mathfrak{q}$; and (2) $I^2 = \mathfrak{q}I$, where $I = \mathfrak{q} :_R \mathfrak{m}$.

## References

[1] S. Goto and H. Sakurai, *The equality $I^2 = QI$ in Buchsbaum rings*, Rend. Sem. Mat. Univ. Padova, **110** (2003), 25-56.

[2] S. Goto and N. Suzuki, *Index of Reducibility of Parameter Ideals in a Local Ring,* J. Algebra, **87** (1984), 53-88.

[3] J. C. Liu and M. Rogers, *The index of reducibility of parameter ideals and mostly zero finite local cohomologies*, Comm. Algebra **34** (2006), no. 11, 4083-4102.

[4] M. Rogers, *The index of reducibility for parameter ideals in low dimension,* J. Algebra, **278** (2004), 571-584.

# Contracted ideals and their associated graded ring

Emanuela De Negri

The study of ideals in a regular local ring of dimension 2 has a long and important tradition dating back to Zariski. The class of contracted ideals plays an important role in this theory. Every integrally closed ideal is contracted and in the homogeneous case being contracted is equivalent to being componentwise linear. One of the main results in this setting is the unique factorization theorem for integrally closed ideals as a product of simple integrally closed ideals.

Zariski himself raised the question of how this theory could be generalized to regular rings of higher dimension.

We first present some results on the depth of the associated graded ring of homogeneous contracted ideals in a polynomial ring of dimension 2. A special role in the class of contracted ideals is played by lex-segment ideals $L$. The study of the associated graded ring of $L$ has an unexpected relation with the study of the Gröbner fan of the ideal of the rational normal curve.

In the second part of the talk we investigate the interplay among integrally closed, contracted and componentwise linear ideals in a polynomial ring of any dimension. By using the interaction among these different concepts we give some classes of ideals that are closed under the product. Within these classes, Zariski theory can be extended, and in particular unique factorization theorems can be proved.

This is a joint work with A. Conca and M.E. Rossi.

# Structure theorems and classification of certain Gorenstein ideals

Juan Elias

This is a joint work with G. Valla. We give structure theorems for the minimal system of generators of Gorenstein ideals with some special Hilbert functions: stretched and Gorenstein almost stretched. We give a complete analytic classification of stretched ideals in terms of the Cohen-Macaulay type, and in the Gorenstein almost stretched case we classify the ideals with a big socle degree.

# Generalizations of the Strong Castelnuovo Lemma

Laura Ghezzi

Consider a set of distinct points in the $n$-dimensional projective space over an algebraically closed field. The Strong Castelnuovo Lemma (SCL) shows that if the points are in general position, then there is a linear syzygy of order $n-1$ if and only if the points are on a rational normal curve.

Cavaliere, Rossi and Valla conjectured that if the points are not in general position the possible extension of the SCL should be the following: There is a linear syzygy of order $n-1$ if and only if either the points are on a rational normal curve or in the union of two linear subspaces whose dimensions add up to $n$.

In this talk we discuss recent developments in the proof of the conjecture. This is work in progress.

# Equivariant local cohomology

Mitsuyasu Hashimoto

This is a joint work with Masahiro Ohtani.

We define an equivariant counterpart of local cohomology using sheaf theory over diagrams of schemes.

Let $I$ be a small category, and $X$ an $I^{\mathrm{op}}$-diagram of schemes (i.e., a contravariant functor from $I$ to the category of schemes). Let $U$ be an open subdiagram of schemes of $X$, and $V$ an open subdiagram of schemes of $V$. Let $f : U \to X$ and $g : V \to U$ be the inclusion maps. Then the kernel of the canonical map $f_*f^* \to f_*g_*g^*f^*$ is denoted by $\underline{\Gamma}_{U,V}$. It is a left exact functor. The right derived functor $R^i\,\underline{\Gamma}_{U,V}$ is denoted by $\underline{H}^i_{U,V}$, and we call it the local cohomology functor.

An $\mathcal{O}_X$-module $\mathcal{M}$ is said to be locally quasi-coherent if the restriction $\mathcal{M}_i$ is quasi-coherent for each $i \in I$.

**Theorem 1.** *Let $I$ be a small category, and*

$$\begin{tikzcd} V' \arrow[r, "g'"] \arrow[d, "h_V"'] \arrow[dr, phantom, "\text{(a)}"] & U' \arrow[r, "f'"] \arrow[d, "h_U"'] \arrow[dr, phantom, "\text{(b)}"] & X' \arrow[d, "h"] \\ V \arrow[r, "g"] & U \arrow[r, "f"] & X \end{tikzcd} \tag{1.1}$$

*a commutative diagram of $I^{\mathrm{op}}$-diagrams of schemes. Assume that $f$, $g$, $f'$, and $g'$ are inclusions of open subdiagrams, and* (a) *and* (b) *are cartesian squares. Then we have*

**(Independence)** $R\,\underline{\Gamma}_{U,V}\,Rh_* \cong Rh_*R\,\underline{\Gamma}_{U',V'}$.

**(Flat base change)** *Assume moreover that there exist some closed cartesian subdiagrams of schemes $Z \subset Y \subset X$ such that $U = X \setminus Z$ and $V = X \setminus Y$, and $h$ is flat. Then there is an isomorphism $h^*R\,\underline{\Gamma}_{U,V} \cong R\,\underline{\Gamma}_{U',V'}\,h^*$ of functors $D_{\mathrm{Lqc}}(X) \to D_{\mathrm{Lqc}}(X')$, where* Lqc *denotes "locally quasi-coherent".*

Changing notation, let $S$ be a scheme, $G$ a flat $S$-group scheme, and $X$ a $G$-scheme (i.e., an $S$-scheme with a left $G$-action) now. An equivariant counterpart of quasi-coherent $\mathcal{O}_X$-module is a $G$-linearized quasi-coherent $\mathcal{O}_X$-module, defined by Mumford [5]. In [3], we considered the diagram of schemes

$$B_G^M(X) := \left( G \times_S G \times_S X \begin{array}{c} \xrightarrow{1_G \times a} \\ \xrightarrow{\mu \times 1_X} \\ \xrightarrow{p_{23}} \end{array} G \times_S X \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{p_2} \end{array} X \right),$$

where $a : G \times X \to X$ is the action, $\mu : G \times G \to G$ the product, and $p_{23}$ and $p_2$ are appropriate projections. The category of $G$-linearized quasi-coherent $\mathcal{O}_X$-modules is equivalent to the category $\mathrm{Qch}(G, X) := \mathrm{Qch}(B_G^M(X))$, the category of quasi-coherent modules over $B_G^M(X)$.

For a $G$-stable closed subscheme $Y$ of $X$, we define $\underline{\Gamma}_Y$ to be

$$\underline{\Gamma}_{B_G^M(X), B_G^M(X) \setminus B_G^M(Y)} : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X).$$

The right derived functor $R^i \underline{\Gamma}_Y$ is denoted by $\underline{H}_Y^i$, and we call it the *equivariant local cohomology*.

As a first application, we prove the following:

**Theorem 2.** *Let $k$ be field, and $G$ a linearly reductive $k$-group scheme. Let $X$ be a noetherian Cohen–Macaulay $G$-scheme, and $f : X \to Y$ a geometric quotient which is an affine morphism. Then $Y$ is noetherian Cohen–Macaulay.*

If $G$ is a finite group and $X = \operatorname{Spec} B$ is affine (so $Y = \operatorname{Spec} B^G$), then this theorem is a special case (that $B$ contains a field) of a theorem of Hochster and Eagon [4].

For the proof, we use the notion of $G$-localness, which is an equivariant counterpart of localness of a scheme. Let $G$ be quasi-compact quasi-separated over $S$. We say that a $G$-scheme $X$ is *$G$-local* if $X$ has a unique minimal non-empty closed $G$-stable subscheme $Y$. In this case, we say that $(X, Y)$ is $G$-local.

**Example 3.** Let $S = \operatorname{Spec} k$, where $k$ is a field, and $G$ an affine $k$-group scheme of finite type, and $H$ a closed subgroup scheme. Then $(G/H, G/H)$ is $G$-local. In general, $G/H$ is not affine.

**Example 4.** Let $S = \operatorname{Spec} \mathbb{Z}$ and $G = \mathbb{G}_m^n$, the split torus over $S$. Let $X = \operatorname{Spec} A$ be affine. Then $A$ is a $\mathbb{Z}^n$-graded ring in a natural way [2, (II.1.2)]. By definition, $X$ is $G$-local if and only if $A$ is $H$-local in the sense of Goto–Watanabe [1].

The next lemma is used in the proof of Theorem 2.

**Lemma 5.** *Let $R$ and $G$ be as in* Theorem 2. *Let $A$ be a $G$-algebra, and $\mathfrak{p}$ a prime ideal of $A^G$. Set $A_\mathfrak{p} := A \otimes_{A^G} A^G_\mathfrak{p}$. Then* $\operatorname{Spec} A_\mathfrak{p}$ *is $G$-local.*

We can also formulate and prove equivariant analogue of Matlis duality and local duality.

Let $S = \operatorname{Spec} R$ be noetherian, $G$ smooth of finite type with connected fibers, and $(X, Y)$ a $G$-local noetherian $G$-scheme.

We say that $\mathbb{I} \in D_{\mathrm{Coh}}(G, X)$ is a $G$-dualizing complex of $X$ if $\mathbb{I}$, viewed as an object of $D(X)$ by restriction, is a dualizing complex on $X$, where Coh denotes "coherent". If there is a separated of finite type $G$-morphism $f : X \to X'$ such that $X'$ is Gorenstein of finite Krull dimension, then $X$ has a $G$-dualizing complex. Note that $\underline{H}^i_Y(\mathbb{I}) \neq 0$ for exactly one $i$. If this $i$ is 0, then we say that $\mathbb{I}$ is $G$-normalized. Let $\mathbb{I}$ be $G$-normalized. We call the nonzero cohomology $\underline{H}^0_Y(\mathbb{I})$ the *G-sheaf of Matlis*, and denote it by $\mathcal{E}_X$ (note that $\mathcal{E}_X$ may depend on the choice of $\mathbb{I}$).

Note that $Y$ is integral. Let $\eta$ be the generic point of $Y$. Then $\mathcal{E}_{X,\eta}$ is the injective hull of the residue field of $\mathcal{O}_{X,\eta}$.

**Theorem 6** ($G$-Matlis duality)**.** *Let* $\mathrm{Coh}(G, X)$ (*resp.* $\mathcal{F}$) *denote the full subcategory of* $\mathrm{Qch}(G, X)$ *consisting of noetherian objects* (*resp. objects of finite length*)*. Let* $\mathbb{D}$ *denote the functor*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(?, \mathcal{E}_X) : \mathrm{Mod}(G, X) \to \mathrm{Mod}(G, X).$$

*Then*

**1** $\mathbb{D}$ *is an exact functor on* $\mathrm{Coh}(G, X)$*.*

**2** $\mathbb{D}(\mathcal{F}) \subset \mathcal{F}$*.*

**3** *The canonical map* $\mathcal{M} \to \mathbb{D}\mathbb{D}\mathcal{M}$ *is an isomorphism for* $\mathcal{M} \in \mathcal{F}$*.*

**4** $\mathbb{D} : \mathcal{F} \to \mathcal{F}$ *is an equivalence.*

**Theorem 7** ($G$-local duality)**.** *Let* $\mathbb{F}$ *be a bounded below complex in* $\mathrm{Mod}(G, X)$ *with coherent cohomology groups. Then there is an isomorphism in* $\mathrm{Qch}(G, X)$

$$\underline{H}^i_Y(\mathbb{F}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\underline{\mathrm{Ext}}^{-i}_{\mathcal{O}_X}(\mathbb{F}, \mathbb{I}), \mathcal{E}_X).$$

# References


[1] S. Goto and K.-i. Watanabe, On graded rings, II ($\mathbb{Z}^n$-graded rings), *Tokyo J. Math.* **1** (1978), 237–261.

[2] M. Hashimoto, *Auslander-Buchweitz Approximations of Equivariant Modules, London Mathematical Society Lecture Note Series* **282**, Cambridge (2000).

[3] M. Hashimoto, *Equivariant Twisted Inverses*, preprint (2007).

[4] M. Hochster and J. A. Eagon, Cohen–Macaulay rings, invariant theory, and the generic perfections of determinantal loci, *Amer. J. Math.* **93** (1971), 1020–1059.

[5] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory,* third edition, Springer (1994).

## Gröbner bases of simplicial toric ideals

Lê Tuân Hoa

This is a joint work with M. Hellus and J. Stückrad. We will give bounds for the maximal degree of certain Gröbner bases of simplicial toric ideals. These bounds are close to the bound stated in Eisenbud-Goto's Conjecture on the Castelnuovo-Mumford regularity.

## Symbolic powers, tight closure, and uniformity

Craig Huneke

This talk will discuss several open questions concerning symbolic powers of ideals, and give some history and new results concerning them. Particular attention will be devoted to the following question: if $R$ is a complete local domain, does there exist an integer $k$ such that for all primes $P$ and for all $n$, the $kn$-th symbolic power of $P$ is contained in the nth power of $P$?

## On the Jacobian ideal of the Rees algebra

Eero Hyry

This is a joint work with Lauri Ojala. Given a normal ideal in a regular local ring we write down the formulas for the canonical module of the Rees algebra and the extended Rees algebra in terms of the corresponding Jacobian ideals. We then look for a formula for the core of the ideal. We also get information about the a-invariant. Moreover, we calculate the canonical class of the Rees algebra.

## On Gorenstein Rees algebras

Shin-ichiro Iai

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $d = \dim A$. Assume that $A$ is a homomorphic image of a Gorenstein local ring and that the field $A/\mathfrak{m}$ is infinite. In the talk we will discuss the Gorenstein Rees algebra $\mathrm{R}(I) := \bigoplus_{i \geq 0} I^i$ of an ideal $I$ in $A$. To state our result let us set up some notation. Put $\mathfrak{a}(A) := \Pi_{i=0}^{d-1}(0) :_A \mathrm{H}^i_{\mathfrak{m}}(A)$ and $\mathrm{NCM}(A) := \{\mathfrak{p} \in \mathrm{Spec} A \mid A_{\mathfrak{p}} \text{ is not a Cohen-Macaulay ring}\}$. Then $\mathrm{NCM}(A) = \mathrm{V}(\mathfrak{a}(A))$. Assume that $\dim \mathrm{NCM}(A) \leq 1$. So there is a system of parameters $x_1, x_2, \ldots, x_d$ of $A$ such that $x_2, x_3 \ldots, x_d \in \mathfrak{a}(A)$. Set $J = (x_2, x_3, \ldots, x_d)$. Taking enough large power of the element $x_1$, we may assume $J : x_1 = J : x_1^2$. And we put $Q = J : x_1$. With this notation the main result in the talk can be stated as follows.

**Theorem 1.** *Let $d \geq 3$. Assume that $A$ is a quasi-Gorenstein ring and* $\dim \mathrm{NCM}(A) \leq 1$*. Let $k$ be a positive integer. Then the following two conditions are equivalent.*

(1) $\mathrm{R}(Q^k)$ *is a Gorenstein ring.*

(2) *$A$ has finite local cohomology modules and $k = d - 2$.*

The Rees algebra $\mathrm{R}(Q^k)$ is Cohen-Macaulay for all integers $k \geq d - 2$ whenever $\mathrm{depth} A \geq 2$ by Kawasaki. The implication (2) $\Rightarrow$ (1) says that a certain ideal of Kawasaki's arithmetic Macaulayfication gives the Gorenstein Rees algebra.

# Some algebraic invariants of mixed product ideals

Cristodor Ionescu

This is a joint work with G. Rinaldo.

We compute some algebraic invariants(e.g. depth, Castelnuovo-Mumford regularity) for a special class of monomial ideals, namely the ideals of mixed products. As a consequence, we characterize the Cohen-Macaulay ideals of mixed products.

# Cluster tilting for one-dimensional hypersurface singularities

Osamu Iyama

This is a joint work with I. Burban, B. Keller and I. Reiten [BIKR].

Let $R = k[[x, y]]/(f)$ be a one-dimensional reduced hypersurface singularity over an algebraically closed field $k$ of characteristic zero. We denote by $\mathrm{Cohen-Macaulay}(R)$ the category of maximal Cohen-Macaulay $R$-modules. From Auslander-Reiten duality and Eisenbud matrix factorization, $\mathrm{Cohen-Macaulay}(R)$ is *2-Calabi-Yau* in the sense that there exists a functorial isomorphism $\mathrm{Ext}^1_R(X, Y) \simeq D\,\mathrm{Ext}^1_R(Y, X)$ for any $X, Y \in \mathrm{Cohen-Macaulay}(R)$ [1]. We call $M \in \mathrm{Cohen-Macaulay}(R)$

- *rigid* if $\mathrm{Ext}^1_R(M, M) = 0$,
- *cluster tilting* if $\mathrm{add}\, M = \{X \in \mathrm{Cohen-Macaulay}(R) \mid \mathrm{Ext}^1_R(M, X) = 0\}$.
- *basic* if it is isomorphic to a direct sum of non-isomorphic indecomposable $R$-modules,

Our aim in general is to classify rigid objects and cluster tilting objects in the category of Cohen-Macaulay modules [IY]. We write $f = f_1 \cdots f_n$ for irreducible formal power series $f_i \in (x, y)$ $(1 \le i \le n)$. Let $\mathfrak{S}_n$ be the symmetric group of degree $n$. For $w \in \mathfrak{S}_n$ and $I \subseteq \{1, \cdots, n\}$, we put

$$S_i^w := S/(f_{w(1)} \cdots f_{w(i)}),\ M_w := \bigoplus_{i=1}^{n} S_i^w \ \text{ and }\ S_I := S/(\prod_{i \in I} f_i).$$

**Theorem 2.** *There exists a cluster tilting object in* $\mathrm{Cohen-Macaulay}(R)$ *if and only if* $f_i \notin (x,y)^2$ *holds for any* $1 \le i \le n$. *In this case the following assertions hold.*

*(a) There are exactly* $n!$ *basic cluster tilting objects* $M_w$ *(*$w \in \mathfrak{S}_n$*) in* $\mathrm{Cohen-Macaulay}(R)$.

*(b) There are exactly* $2^n - 1$ *indecomposable rigid objects* $S_I$ *(*$\emptyset \ne I \subseteq \{1, \cdots, n\}$*) in* $\mathrm{Cohen-Macaulay}(R)$.

The ingredient of our proof is

- a 2-*almost split sequence* [I] which is an exact sequence $0 \to X \xrightarrow{g} M_1 \to M_0 \xrightarrow{f} X \to 0$ in $\operatorname{add} M_w$ with a sink map $f$ and a source $g$ in $\operatorname{add} M_w$,
- *cluster tilting mutation* [BMRRT] which is a process to construct another cluster tilting object from a given one by replacing an indecomposable direct summand,
- *tilting mutation* [RS] which is a process to construct another tilting module from a given one by replacing an indecomposable direct summand,
- *non-commutative crepant resolution* [V] which gives a bridge between birational geometry and cluster tilting theory.

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# Symbolic Rees rings of space monomial curves in characteristic $p$ and existence of negative curves in characteristic $0$

Kazuhiko Kurano

This is a joint work with Naoyuki Matsuoka (Meiji University).

We shall study finite generation of symbolic Rees rings of space monomial curves $(t^a, t^b, t^c)$ for pairwise coprime positive integers $a$, $b$ and $c$ such that $(a, b, c) \neq (1, 1, 1)$. If such a ring is not finitely generated over a base field, then it is a counterexample to Hilbert's fourteenth problem (Theorem 1). Finite generation of such rings is deeply related to existence of negative curves on certain normal projective surfaces. We prove that, if $(a + b + c)^2 > abc$, then a negative curve exists (Theorem 4). Furthermore, using a computer, we show that a negative curve exists in the case of characteristic 0 if $1 \leq a, b, c \leq 300$ (Theorem 5). As a corollary, we know that the symbolic Rees rings of space monomial curves are Noetherian if the base field is of positive characteristic and all of $a$, $b$, $c$ are less than or equal to 300 (Corollary 6).

Let $P$ be a prime ideal of a ring $R$. The $r$th symbolic power of $P$ is defined as $P^{(r)} = P^r R_P \cap R$. The ring $R_s(P) = \oplus_{r \geq 0} P^{(r)} T^r$ is called the symbolic Rees ring of $P$.

Let $k$ be a field and $m$ be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the $k$-algebra homomorphism

$$\phi : k[x_1, \ldots, x_m] \longrightarrow k[t]$$

given by $\phi(x_i) = t^{a_i}$ for $i = 1, \ldots, m$. Let $\mathfrak{p}_k(a_1, \ldots, a_m)$ be the kernel of $\phi$.

**Theorem 3.** *Let $k$ be a field and $m$ be a positive integer. Let $a_1, \ldots, a_m$ be positive integers. Consider the prime ideal $\mathfrak{p}_k(a_1, \ldots, a_m)$ of the polynomial ring $k[x_1, \ldots, x_m]$.*

*Let $\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t, T$ be indeterminates over $k$. Consider the following injective $k$-homomorphism*

$$\xi : k[x_1, \ldots, x_m, T] \longrightarrow k(\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t)$$

*given by $\xi(T) = \alpha_2/\alpha_1$ and $\xi(x_i) = \alpha_1 \beta_i + t^{a_i}$ for $i = 1, \ldots, m$.*

*Then, the following equality holds:*

$$k(\alpha_1\beta_1 + t^{a_1}, \cdots, \alpha_1\beta_m + t^{a_m}, \alpha_2/\alpha_1) \cap k[\alpha_1, \alpha_2, \beta_1, \ldots, \beta_m, t] = \xi(R_s(\mathfrak{p}_k(a_1, \ldots, a_m)))$$

Goto, Nishida and Watanabe proved that $R_s(\mathfrak{p}_k(7n - 3, (5n - 2)n, 8n - 3))$ is not finitely generated over $k$ if the characteristic of $k$ is zero, $n \geq 4$ and $n \not\equiv 0$ (3). By the above theorem, it is a counterexample to Hilbert's fourteenth problem.

From now on, we restrict ourselves to the case $m = 3$. For the simplicity of notation, we write $x$, $y$, $z$, $a$, $b$, $c$ for $x_1$, $x_2$, $x_3$, $a_1$, $a_2$, $a_3$, respectively. We regard the polynomial ring $S = k[x, y, z]$ as a $\mathbb{Z}$-graded ring by $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$.

By a result of Herzog, we know that $\mathfrak{p}_k(a, b, c)$ is generated by at most three elements. We are interested in finite generation of the symbolic Rees ring $R_s(\mathfrak{p}_k(a, b, c))$. It is well-known that this problem is reduced to the case where $a$, $b$ and $c$ are pairwise coprime, i.e., $(a, b) = (b, c) = (c, a) = 1$ .

We always assume that $a$, $b$ and $c$ are pairwise coprime.

Let $\mathbb{P}_k(a, b, c)$ be the weighted projective space $\mathrm{Proj}(S)$. Let $\pi : X_k(a, b, c) \to \mathbb{P}_k(a, b, c)$ be the blow-up at the smooth point $V_+(\mathfrak{p}_k(a, b, c))$. Let $E$ be the exceptional divisor, i.e., $E = \pi^{-1}(V_+(\mathfrak{p}_k(a, b, c)))$ .

**Definition 4.** A curve $C$ on $X_k(a, b, c)$ is called a *negative curve* if $C \neq E$ and $C^2 < 0$.

Existence of a negative curve is deeply related to finite generation of $R_s(\mathfrak{p}_k(a, b, c))$ as follows:

(1) If $R_s(\mathfrak{p}_k(a, b, c))$ is finitely generated over $k$ with $\sqrt{abc} \notin \mathbb{Z}$, then there exists a negative curve on $X_k(a, b, c)$. (Cutkosky)

(2) If there exists a negative curve on $X_k(a, b, c)$ for a field $k$ of positive characteristic, then $R_s(\mathfrak{p}_k(a, b, c))$ is finitely generated over $k$. (Cutkosky)

(3) If there exists a negative curve on $X_{k_0}(a, b, c)$ for a field $k_0$ of characteristic 0, so does on $X_k(a, b, c)$ for any field $k$. (mod $p$ reduction)

By (2) and (3), we know that, *if there exists a negative curve on $X_{\mathbb{C}}(a, b, c)$, then $R_s(\mathfrak{p}_k(a, b, c))$ is finitely generated over $k$ for any field $k$ of positive characteristic.*

**Definition 5.** Let $a$, $b$, $c$ be pairwise coprime positive integers. Let $k$ be a field.

Consider the following three conditions.

(C1) There exists a negative curve on $X_k(a, b, c)$, i.e., $[\mathfrak{p}_k(a, b, c)^{(r)}]_n \neq 0$ for some positive integers $n$, $r$ satisfying $n/r < \sqrt{abc}$.

(C2) There exist positive integers $n$, $r$ satisfying $n/r < \sqrt{abc}$ and $\dim S_n > r(r + 1)/2$.

(C3) There exist positive integers $q$, $r$ satisfying $abcq/r < \sqrt{abc}$ and $\dim S_{abcq} > r(r + 1)/2$.

Since $\dim[S/\mathfrak{p}_k(a, b, c)^{(r)}]_n \leq r(r + 1)/2$ (the equality holds for $n >> 0$), we know

$$(C3) \Longrightarrow (C2) \Longrightarrow (C1).$$

One can show that, if (C1) holds with $r \leq 2$ over a field $k$ of characteristic 0, (C2) is satisfied. The examples due to Goto-Nishida-Watanabe have negative curves with $r = 1$. Therefore, they satisfy the condition (C2).

**Theorem 6.** *Let $a$, $b$ and $c$ be pairwise coprime integers such that $(a, b, c) \neq (1, 1, 1)$, and $k$ be a field. Then, we have the following:*

*(1) Assume that $\sqrt{abc} \notin \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > abc$.*

*(2) Assume that $\sqrt{abc} \in \mathbb{Z}$. Then, (C3) holds if and only if $(a + b + c)^2 > 9abc$.*

*(3) If $(a + b + c)^2 > abc$, then, (C2) holds. In particular, a negative curve exists in this case.*

If $(a+b+c)^2 > abc$, then $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian by a result of Cutkosky.

If $(a+b+c)^2 > abc$ and $\sqrt{abc} \notin \mathbb{Z}$, then the existence of negative curves follows from the cone theorem of Mori. (Our proof is simple and purely algebraic.)

If $(a,b,c) = (1,1,1)$, there is no negative curve in this case.

Using a computer, we know that (C2) is not satisfied in some cases, for example, $(a,b,c) = (9,10,13), (13,14,17)$.

Using a computer, we can prove the following:

**Theorem 7.** *Let $a$, $b$ and $c$ be pairwise coprime positive integers such that $(a,b,c) \neq (1,1,1)$. Assume that the characteristic of $k$ is zero.*

*If all of $a$, $b$ and $c$ are at most 300, then there exists a negative curve on $X_k(a,b,c)$.*

Then, we obtain the following corollary immediately.

**Corollary 8.** *Let $a$, $b$ and $c$ be pairwise coprime positive integers such that all of $a$, $b$ and $c$ are at most 300. Assume that the characteristic of $k$ is positive.*

*Then the symbolic Rees ring $R_s(\mathfrak{p}_k(a,b,c))$ is Noetherian.*

# Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism

Shigeru Kuroda

Let $k$ be a field, and $n$ a natural number. The polynomial ring $k[\mathbf{x}] = k[x_1, \dots, x_n]$ in $n$ variables over $k$ is one of the best-known commutative algebras, yet little is known about the structure of the group $\mathrm{Aut}_k\, k[\mathbf{x}]$ of automorphisms in general. Let $F : k[\mathbf{x}] \to k[\mathbf{x}]$ be an endomorphism over $k$. We identify $F$ with the $n$-tuple $(f_1, \dots, f_n)$ of elements of $k[\mathbf{x}]$, where $f_i = F(x_i)$ for each $i$. Then, $F$ is an automorphism if and only if the $k$-algebra $k[\mathbf{x}]$ is generated by $f_1, \dots, f_n$. Note that the sum $\deg F := \sum_{i=1}^n \deg f_i$ of the total degrees of $f_1, \dots, f_n$ is at least $n$ whenever $F$ is an automorphism. We say that $F$ is *affine* if $\deg F = n$, and *elementary* if there exist $l \in \{1, \dots, n\}$ and $\phi \in k[x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n]$ such that $f_l = x_l + \phi$ and $f_i = x_i$ for each $i \neq l$. The subgroup $\mathrm{T}_k\, k[\mathbf{x}]$ of $\mathrm{Aut}_k\, k[\mathbf{x}]$ generated by affine automorphisms and elementary automorphisms is called the *tame subgroup*. An automorphism is said to be *tame* if it belongs to $\mathrm{T}_k\, k[\mathbf{x}]$.

It is a fundamental question whether $\mathrm{T}_k\, k[\mathbf{x}] = \mathrm{Aut}_k\, k[\mathbf{x}]$ holds for each $n$. The equality is obvious if $n = 1$. This also holds true if $n = 2$. It was shown by Jung [1] in 1942 when $k$ is of characteristic zero, and by van der Kulk [2] in 1953 when $k$ is an arbitrary field. This follows from the fact that each automorphism of $k[x_1, x_2]$ but an affine automorphism admits an elementary reduction. Here, we say that $F$ admits an *elementary reduction* if $\deg(F \circ E) < \deg F$ for some elementary automorphism $E$.

When $n = 3$, the structure of $\mathrm{Aut}_k\, k[\mathbf{x}]$ becomes far more difficult. In 1972, Nagata [6] conjectured that the automorphism

$$F = (x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2 x_3,\ x_2 + (x_1x_3 + x_2^2)x_3,\ x_3) \tag{8.1}$$

is not tame. This famous conjecture was finally solved in the affirmative by Shestakov-Umirbaev [8] in 2003 for a field $k$ of characteristic zero. Therefore, $\mathrm{T}_k\, k[\mathbf{x}] \neq \mathrm{Aut}_k\, k[\mathbf{x}]$ if $n = 3$. However, the question remains open for $n \geq 4$.

Shestakov-Umirbaev used an inequality [7, Theorem 3] concerning the total degrees of polynomials as a crucial tool. In this talk, we generalize their inequality from the original point of view. Using this result, we reconstruct the Shestakov-Umirbaev theory on polynomial automorphisms, and give a more precise tameness criterion. We also generalize Jung's theorem by means of the generalized Shestakov-Umirbaev inequality.

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# Non-commutative desingularizations


Graham J. Leuschke


What should it mean for a commutative ring, or an algebraic variety, to have a "non-commutative" resolution of singularities? I will discuss a possible answer to this question, along with related ideas, with particular attention to work of O. Iyama, M. Van den Bergh, and joint work of my own with Van den Bergh and R.-O. Buchweitz.

# Doset Hibi rings and application

Mitsuhiro Miyazaki

Let $K$ be a field and $H$ a finite distributive lattice. Hibi [Hib] defined the ring $\mathcal{R}_K(H)$ which is now called the Hibi ring. It is an algebra with straightening law (ASL for short, ordinal Hodge algebra in the terminology of [DEP]) over $K$ generated by $H$. The straightening relation in $\mathcal{R}_K(H)$ for $\alpha$, $\beta \in \mathcal{R}_K(H)$ with $\alpha \not\sim \beta$ is

$$\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta)$$

and $\mathcal{R}_K(H)$ is a normal affine semigroup ring.

On the other hand, De Concini, Eisenbud and Procesi [DEP] defined the notion of a doset as follows. Let $P$ be a finite poset. A subset $D$ of $P \times P$ is called a doset of $P$ if

1. $\{(\alpha, \alpha) \mid \alpha \in P\} \subset D \subset \{(\alpha, \beta) \in P \times P \mid \alpha \le \beta\}$.
2. for $\alpha$, $\beta$, $\gamma \in P$ with $\alpha \le \beta \le \gamma$, $(\alpha, \gamma) \in D$ if and only if $(\alpha, \beta) \in D$ and $(\beta, \gamma) \in D$.

Now let $L$ be a lattice and $\varphi \colon H \to L$ is a lattice homomorphism, that is, a map such that $\varphi(\alpha \wedge \beta) = \varphi(\alpha) \wedge \varphi(\beta)$, and $\varphi(\alpha \vee \beta) = \varphi(\alpha) \vee \varphi(\beta)$. We set

$$D := \{(\alpha, \beta) \in H \times H \mid \alpha \le \beta, \varphi(\alpha) = \varphi(\beta)\}.$$

It is easy to verify that $D$ is a doset of $H$.

**Definition 1.** We call the subalgebra of $\mathcal{R}_K(H)$ generated by $\{\alpha\beta \mid (\alpha, \beta) \in D\}$ the doset Hibi ring with respect to $\varphi$.

We show that a doset Hibi ring is a normal affine semigroup ring.

As an application of the notion of doset Hibi rings, we generalize the result of De Concini and Procesi [DP] on formal $O(m)$ invariants to the matrices defining the Schubert subvarieties of Grassmannians. And show that the ring of formal $O(m)$ invariants is isomorphic to subalgebra of a polynomial ring whose initial algebra is a doset Hibi ring. It follows form [CHV] that the ring of formal $O(m)$ invariants is a normal domain and has rational singularity if $\mathrm{char} K = 0$ or F-rational if $\mathrm{char} K > 0$.

If time allows, we also talk about the ring of formal $SO(m)$ invariants.

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# An upper bound on the reduction number of an ideal

Koji Nishida

Let $Q$, $I$ and $J$ be ideals of a commutative Noetherian ring $A$ such that $Q \subseteq I \subseteq J$. As is noted in [GNO, Proposition 2.6], if $J/I$ is cyclic as an $A$-module and $J^2 = QJ$, then we have $I^3 = QI^2$. The purpose of this talk is to generalize this fact. We will show that if $J/I$ is generated by $v$ elements as an $A$-module and $J^2 = QJ$, then $I^{v+2} = QI^{v+1}$. In particular, we see that if $(A, \mathfrak{m})$ is a 2-dimensional regular local ring and $I$ is an $\mathfrak{m}$-primary ideal, then

$$\mathrm{r}_Q(I) \leq \mu_A(\overline{I}/I) + 1$$

for any minimal reduction $Q$ of $I$, where $\overline{I}$ denotes the integral closure of $I$. We get this result as a corollary of the following theorem which generalizes Rossi's assertion stated in the proof of [2, Theorem 1.3].

**Theorem 2.** *Let $A$ be a commutative ring and $\{F_n\}_{n\geq 0}$ a family of ideals in $A$ such that $F_0 = A$ and $IF_n \subseteq F_{n+1}$ for any $n \geq 0$. Let $k$ be a non negative integer and $\mathfrak{A}$ an ideal in $A$ such that $I^{k+1} \subseteq QF_k + \mathfrak{a}F_{k+1}$. Suppose that $F_n/(QF_{n-1} + I^n)$ is generated by $v_n$ elements for any $n \geq 0$ and $v_n = 0$ for $n \gg 0$. We put $v = \sum_{n\geq 0} v_n$. Then we have*

$$I^{v+k+1} = QI^{v+k} + \mathfrak{a}I^{v+k+1}\,.$$

We mainly take an interest in the case where $F_n = IF_{n-1}$ for $n \gg 0$ and $Q$ is a minimal reduction of $I$. As a typical example of such $\{F_n\}_{n\geq 0}$, we find $\{\widetilde{I^n}\}_{n\geq 0}$ when $I$ contains a non-zerodivisor, where $\widetilde{I^n}$ denotes the Ratliff-Rush closure of $I^n$ (cf. [3]). If $A$ is an analytically unramified local ring, then $\{\overline{I^n}\}_{n\geq 0}$ is also an important example. It is obvious that $\{J^n\}_{n\geq 0}$ always satisfies the required condition on $\{F_n\}_{n\geq 0}$ for any ideal $J$ with $I \subseteq J \subseteq \overline{I}$.

We prove Theorem 2 following Rossi's argument in the proof of [2, Theorem 1.3]. However we do not assume that $I$ is $\mathfrak{m}$-primary. And furthermore, by setting $\mathfrak{a} = \mathfrak{m}$ in the case where $(A, \mathfrak{m})$ is a local ring, we can deduce the following corollary from Theorem 2.

**Corollary 3.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $\{F_n\}_{n\geq 0}$ a family of ideals in $A$ such that $F_0 = A$, $IF_n \subseteq F_{n+1}$ for any $n \geq 0$, and $I^{k+1} \subseteq QF_k + \mathfrak{m}F_{k+1}$ for some $k \geq 0$. Then we have*

$$\mathrm{r}_Q(I) \leq 1 + \mu_A(F_1/I) + \sum_{n\geq 2} \mu_A(F_n/QF_{n-1})\,.$$

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# The structure of Sally module of rank one


Kazuho Ozeki


My talk is a joint work with S. Goto and K. Nishida.

Let $A$ be a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of $A$ is infinite. Let $I$ be an $\mathfrak{m}$-primary ideal in $A$ and choose a minimal reduction $Q = (a_1, a_2, \cdots, a_d)$ of $I$. Let

$$R = \mathrm{R}(I) := \bigoplus_{n \geq 0} I^n, \quad T = \mathrm{R}(Q) := \bigoplus_{n \geq 0} Q^n \quad \text{and} \quad G = \mathrm{G}(I) := \bigoplus_{n \geq 0} I^n/I^{n+1}$$

respectively denote the Rees algebras of $I$ and $Q$ and the associated graded ring of $I$. We then define $S = IR/IT$ and call it the Sally module of $I$ with respect to $Q$. Let $B = T/\mathfrak{m}T$, which is the polynomial ring with $d$ indeterminates over the field $k = A/\mathfrak{m}$.

The main result of my talk is the following, which is a complete structure theorem of the Sally modules of $\mathfrak{m}$-primary ideals $I$ satisfying the equality $e_1 = e_0 - \ell_A(A/I) + 1$, where $e_i = \mathrm{e}_I^i(A)$ denots the $i$-th Hilbert coefficient of $I$.

**Theorem 4** ([GNO]). *The following three conditions are equivalent to each other.*

(1) $e_1 = e_0 - \ell_A(A/I) + 1$.

(2) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.

(3) $S \cong (X_1, X_2, \cdots, X_c)B$ *as graded* $T$*-modules for some* $0 < c \leq d$*, where* $\{X_i\}_{1 \leq i \leq c}$ *are linearly independent linear forms of the polynomial ring* $B$.

*When this is the case,* $c = \ell_A(I^2/QI)$ *and* $I^3 = QI^2$*, and the following assertions hold.*

*(i)* $\mathrm{depth}\, G \geq d - c$ *and* $\mathrm{depth}_T\, S = d - c + 1$.

*(ii)* $\mathrm{depth}\, G = d - c$*, if* $c \geq 2$.

*(iii) Suppose* $c < d$*. Then* $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ *for all* $n \geq 0$*. Hence*

$$e_i = \begin{cases} 0 & \text{if } i \neq c+1, \\ (-1)^{c+1} & \text{if } i = c+1 \end{cases}$$

*for* $2 \leq i \leq d$.

*(iv) Suppose* $c = d$*. Then* $\ell_A(A/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1}$ *for all* $n \geq 1$*. Hence* $e_i = 0$ *for* $2 \leq i \leq d$.

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# Irreducible affine space curves and the uniform Artin-Rees property on the prime spectrum

Francesc Planas-Vilanova

We give a negative answer to the problem, open for twenty years, as to whether the full Uniform Artin-Rees Property holds on the prime spectrum of an excellent ring (it was known to hold locally on the prime spectrum of such a ring). This is a joint work with Liam O'Carroll.

# Blowups and fibers of morphisms

Claudia Polini

This is a report on ongoing work with Kustin and Ulrich. I will describe properties of the blowup algebras of any ideal generated by forms of the same degree in a polynomial ring in two variables over a field.

# Fontaine rings defined by noetherian rings

Paul C. Roberts

Recent work on the homological conjectures has concentrated on the study of the absolute integral closure of a ring, particularly for rings of mixed characteristic. A construction of Fontaine gives, for a ring $R$ of mixed characteristic, a ring $E(R)$ of positive characteristic from which, under certain assumptions, the original ring can be reconstructed up to $p$-adic completion. However, while this construction works well for the absolute integral closure of a Noetherian ring, the assumptions mentioned above are not satisfied for Noetherian rings themselves. In this talk we will describe Fontaine rings, discuss their relations with other problems, and discuss how to approach the problem of finding an appropriate construction for Noetherian rings.

# Extremal Betti numbers of ideals in a local regular ring

Maria Evelina Rossi

Let $I$ be an ideal of a local regular ring $(R, m, k)$. It is well known that if we start with a minimal free resolution of $\mathrm{gr}_{m/I}(R/I)$ as $\mathrm{gr}_m(R)$-module, then we can build up a free resolution of $A = R/I$ as $R$-module which is not necessarily minimal. Hence for the Betti numbers of $A$ and $\mathrm{gr}_{m/I}(A)$ one has $\beta_i(\mathrm{gr}_{m/I}(A) \geq \beta_i(A)$. The local ring $A$ is called of *homogeneous type* if $\beta_i(A) = \beta_i(\mathrm{gr}_{m/I}(A))$. In [J. Herzog, M. E. Rossi, G. Valla, On the depth of the Symmetric Algebra, Trans. A.M.S. 296, (1986)] the problem to find classes of rings of homogeneous type is related to get information on the depth of the Symmetric Algebra. Notice that $\beta_1(A) = \beta_1(\mathrm{gr}_{m/I}(A))$, i.e. the ideal $I$ is minimally generated by a $m$-standard base, does not imply that $A$ is of homogeneous type. In this talk we characterize large classes of local rings $A$ of homogeneous type, in particular we prove that if the ideal $I$ is minimally generated by a $m$-standard base and $I^*$ is componentwise linear, then $A$ is of homogeneous type.

Joint work with Leila Sharifan.

# Frobenius structures on injective modules and tight closure test ideals

Rodney Y. Sharp

Let $R$ be a commutative Noetherian local ring of prime characteristic $p$, with maximal ideal $\mathfrak{m}$; denote the Frobenius homomorphism of $R$ by $f$. In this talk, I shall discuss the situation in which the injective envelope $E$ of $R/\mathfrak{m}$ has a 'nilpotent-free Frobenius structure', that is (in alternative language), a structure as a $T$-torsion-free left module over the 'Frobenius' skew polynomial ring $R[T, f]$ associated to $R$ and $f$, in the indeterminate $T$. Earlier work on graded annihilators of left $R[T, f]$-modules will be used, and I shall relate the test ideal of $R$ to the smallest '$E$-special' ideal of $R$ of positive height.

A byproduct is an analogue of a result of Janet Cowden Vassilev: she showed, in the case where $R$ is an $F$-pure homomorphic image of an $F$-finite regular local ring, that there exists a strictly ascending chain $0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_t = R$ of radical ideals of $R$ such that, for each $i = 0, \ldots, t-1$, the reduced local ring $R/\tau_i$ is $F$-pure and its test ideal (has positive height and) is exactly $\tau_{i+1}/\tau_i$. I shall present an analogous result in the case where $R$ is complete (but not necessarily $F$-finite) and $E$ has a structure as a $T$-torsion-free left $R[T, f]$-module.

# Finite Schur filtration dimension for modules over an algebra with Schur filtration

Vasudevan Srinivas

This is a report on some joint work with W. van der Kallen.

Let $G = \mathrm{GL}_N$ or $\mathrm{SL}_N$ as a reductive linear algebraic group over a field $k$ of characteristic $p > 0$. We prove several results that were previously established (by van der Kallen) when $N \leq 5$ or $p > 2^N$.

Let $G$ act rationally on a finitely generated commutative $k$-algebra $A$ and let $\mathrm{gr}(A)$ be the Grosshans graded ring. We show that the cohomology algebra $H^*(G, \mathrm{gr}(A))$ is finitely generated over $k$. If moreover $A$ has a good filtration and $M$ is a noetherian $A$-module with compatible $G$ action, then $M$ has finite good filtration dimension and the $H^i(G, M)$ are noetherian $A^G$-modules. To obtain results in this generality, we employ a functorial resolution of the ideal of the diagonal in a product of Grassmannians.

# Finiteness properties of rings with finite F-representation type

Shunsuke Takagi

This is a joint work with Ryo Takahashi (Shinshu University).

Let $R$ be a Noetherian ring of prime characteristic $p$. For each integer $e \geq 1$, the $e$-times iterated Frobenius map $F^e : R \to R$ is defined by sending $r$ to $r^{p^e}$ for all $r \in R$. We denote by ${}^eR$ the module $R$ with its $R$-module structure determined by $r \cdot s := r^{p^e}s$ for all $r, s \in R$. If $R = \bigoplus_{n \geq 0} R_n$ is a graded ring, then ${}^eR$ carries a $\mathbb{Q}$-graded $R$-module structure: grade ${}^eR$ by putting $[{}^eR]_\alpha = [R]_{p^e\alpha}$ if $\alpha$ is a multiple of $1/p^e$, otherwise $[R]_\alpha = 0$. We denote by $I^{[p^e]}$ the ideal of $R$ generated by the $p^{e\text{th}}$ powers of elements of an ideal $I \subseteq R$ and by $R^\circ$ the set of elements of $R$ which are not in any minimal prime ideal.

Rings with finite F-representation type were first introduced by Smith and Van den Bergh [4].

**Definition 1** ([4, Definition 3.1.1]). (i) We say that $R$ has *finite F-representation type* by finitely generated $R$-modules $M_1, \ldots, M_s$ if for every $e \geq 0$, the $R$-module ${}^eR$ is isomorphic to a finite direct sum of the $R$-modules $M_1, \ldots, M_s$, that is, there exist non-negative integers $n_{e1}, \ldots, n_{es}$ such that

$$ {}^eR \cong \bigoplus_{i=1}^{s} M_i^{\oplus n_{ei}}. $$

We simply say that $R$ has finite F-representation type if there exist finitely generated $R$-modules $M_1, \ldots, M_s$ by which $R$ has finite F-representation type.

(ii) Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring of prime characteristic $p$. We say that $R$ has *finite graded F-representation type* by finitely generated $\mathbb{Q}$-graded $R$-modules $M_1, \ldots, M_s$ if for every $e \geq 0$, the $\mathbb{Q}$-graded $R$-module ${}^eR$ is isomorphic to a finite direct sum of the $\mathbb{Q}$-graded $R$-modules $M_1, \ldots, M_s$, that is, there exist non-negative integers $n_{ei}$ and rational numbers $\alpha_{ij}^{(e)}$ for all $1 \leq i \leq s$ and $1 \leq j \leq n_{e,i}$ such that

$$ {}^eR \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_{ei}} M_i(\alpha_{ij}^{(e)}) $$

as a $\mathbb{Q}$-graded R-module. We simply say that $R$ has finite graded F-representation type if there exist finitely generated $\mathbb{Q}$-graded $R$-modules $M_1, \ldots, M_s$ by which $R$ has finite graded F-representation type.

We talk about three finiteness properties of rings with finite F-representation type. The first one is a generalization of a result due to Huneke and Sharp [3].

**Theorem 2.** *Let $R$ be a Cohen-Macaulay ring with finite F-representation type by finitely generated $R$-modules $M_1, \ldots, M_s$ admitting a canonical module $\omega_R$. Let $I$ be an ideal of $R$ and $n$ an integer. Then one has*

$$ \operatorname{Ass}_R H_I^n(\omega_R) \subseteq \bigcup_{i=1}^{s} \operatorname{Ass}_R \operatorname{Ext}_R^n(M_i/IM_i, \omega_R). $$

*In particular, $H_I^n(\omega_R)$ has only a finite number of associated primes.*

The second property is a generalization of a result due to Alvarez-Montaner, Blickle and Lyubeznik [1].

**Theorem 3.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian graded ring with $k := R_0$ a field of characteristic $p > 0$ and $K$ be the perfect closure of $k$. If $R_K := R \otimes_k K$ has finite graded F-representation type, then for any non-zerodivisor $x \in R$, $R_x$ is generated by $1/x$ as a $D_{R/k}$-module.*

**Definition 4.** Let $\mathfrak{a}$ be a fixed ideal in an excellent reduced ring $R$ of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $I$ be an arbitrary ideal in $R$.

(i) Given a real number $t > 0$, the $\mathfrak{a}^t$*-tight closure* $I^{*\mathfrak{a}^t}$ of $I$ is defined to be the ideal of $R$ consisting of all elements $x \in R$ for which there exists $c \in R^\circ$ such that $cx^q\mathfrak{a}^{\lceil tq \rceil} \subseteq I^{[q]}$ for all large $q = p^e$.

(ii) Given a real number $t > 0$, the *generalized test ideal* $\tau(\mathfrak{a}^t)$ is defined to be $\tau(\mathfrak{a}^t) = \bigcap_{J\subseteq R} J : J^{*\mathfrak{a}^t}$, where $J$ runs through all ideals of $R$.

(iii) A positive real number $t$ is said to be an *F-jumping exponent* of $\mathfrak{a}$ if $\tau(\mathfrak{a}^t) \subsetneq \tau(\mathfrak{a}^{t-\epsilon})$ for every $\epsilon > 0$.

The last property is a generalization of a result due to Blickle, Mustaţă and Smith [2].

**Theorem 5.** *Let $R = \bigoplus_{n\in\mathbb{N}} R_n$ be a F-regular graded ring with $R_0$ a field of characteristic $p > 0$ and let $\mathfrak{a}$ be a homogeneous ideal of $R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Assume in addition that $R$ has finite graded F-representation type. Then the set of F-jumping exponents of $\mathfrak{a}$ have no accumulation points.*

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# Approximating modules by resolving subcategories


Ryo Takahashi


The notion of an approximation of a finitely generated module by a subcategory of the category of finitely generated modules was first introduced over artin algebras by Auslander and Smalø [4] in connection with studying the problem of which subcategories admit almost split sequences. A subcategory by which every finitely generated module is approximated is called a contravariantly finite subcategory. Auslander and Bridger [1] defined a resolving subcategory in the study of modules of G-dimension zero, which are now also called totally reflexive modules. Auslander and Reiten [3] proved that contravariantly finite resolving subcategories are closely related to tilting theory, which has been playing an central role in the representation theory of algebras. There is also an application of contravariantly finite resolving subcategories to the study of the finitistic dimension conjecture.

In this talk, we will study contravariantly finite resolving subcategories over commutative rings. Let $R$ be a commutative noetherian henselian local ring. We denote by $\operatorname{mod} R$ the category of finitely generated $R$-modules, by $\mathcal{F}(R)$ the full subcategory of free $R$-modules, by $\mathcal{C}(R)$ the full subcategory of maximal Cohen-Macaulay $R$-modules, and by $\mathcal{G}(R)$ the full subcategory of totally reflexive $R$-modules. The subcategory $\mathcal{F}(R)$ is always a contravariantly finite resolving subcategory, and so is $\mathcal{C}(R)$ if $R$ is Cohen-Macaulay. The latter fact follows from the Cohen-Macaulay approximation theorem, which was proved by Auslander and Buchweitz [2]. The subcategory $\mathcal{G}(R)$ is always resolving, and coincides with $\mathcal{C}(R)$ if $R$ is Gorenstein.

The main result of this talk is the following.

**Theorem 1.** *If $R$ is Gorenstein, then all the contravariantly finite resolving subcategories of $\operatorname{mod} R$ are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and $\operatorname{mod} R$.*

Actually we can prove several more general results than the above theorem. One of them yields another proof of the following theorem, which is a main result of [5].

**Theorem 2.** *Suppose that there is a nonfree $R$-module in $\mathcal{G}(R)$. If $\mathcal{G}(R)$ is contravariantly finite in* $\operatorname{mod} R$*, then $R$ is Gorenstein.*

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# Arithmetical rank of Stanley-Reisner ideals with 2-linear resolution


Naoki Terai


We consider the arithmetical rank of monomial ideals. Let $S = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field $k$. Let $I$ be an ideal of $S$.We define the arithmetical rank $\mathrm{ara} I$ of $I$ by

$$\mathrm{ara} I := \min\{r; \exists a_1, a_2, \ldots a_r \in I \text{ such that } \sqrt{(a_1, a_2, \ldots a_r)} = \sqrt{I}\ \}.$$

When $I$ is a squarefree monomial ideal, the following theorem is known:

**Theorem (Lyubeznik).** *Let $I$ be a squarefree monomial ideal. Then we have*

$$\mathrm{ara}\ I \geq \mathrm{projdim}(S/I).$$

*In particular, if $I$ is a set-theoretic complete intersection, then $S/I$ is Cohen-Macaulay.*

Then we consider the following problem:

**Problem.** Let $I$ be a squarefree monomial ideal. Under what conditions do we have $\operatorname{ara} I = \operatorname{projdim}(S/I)$?

We have the following result on the above problem:

**Theorem.** *Suppose the base field $k$ has a positive characteristic. Let $I$ be a squarefree monomial ideal with 2-linear resolution. Then we have*

$$\operatorname{ara} I = \operatorname{projdim}(S/I).$$

# Combinatorial characterizations of normal monomial ideals

Ngo Viet Trung

We will present two combinatorial characterizations of normal monomial ideals. The first one is by means of the integer rounding property of the exponent matrix of the vertices of the Newton polygon. The second one is by means of the normality of a convex polytope associated with the Newton polygon. These characterizations lead to several new classes of normal squarefree monomial ideals.

# Criteria for Integral Dependence

Bernd Ulrich

This is a report on joint work with Javid Validashti. We provide multiplicity based criteria for integral dependence of modules. This type of criteria goes back to Rees, who treated the case of zero-dimensional ideals in universally catenary equidimensional local rings. Much later, Flenner and Manaresi were able to extend Rees's work to arbitrary ideals, using the notion of j-multiplicity instead of the classical Hilbert-Samuel multiplicity. We consider the general case of finitely generated modules. Our proof is self-contained and relatively short, and implies the earlier algebraic results as special cases.

# The Chern Coefficients of Local Rings

Wolmer V. Vasconcelos

The Chern numbers of the title are the first coefficients (after the multiplicities) of the Hilbert functions of various filtrations of ideals of a local ring $(R, \mathfrak{m})$. For a Noetherian (good) filtration $\mathcal{A}$ of $\mathfrak{m}$-primary ideals, the positivity and bounds for $e_1(\mathcal{A})$ are well-studied if $R$ is Cohen-Macaulay, or more broadly, if $R$ is a Buchsbaum ring or mild generalizations thereof. For arbitrary geometric local domains, we introduce techniques based on the theory of maximal Cohen-Macaulay modules and of extended multiplicity functions to establish the meaning of the positivity of $e_1(\mathcal{A})$, and to derive lower and upper bounds for $e_1(\mathcal{A})$.

# Hilbert polynomials and powers of ideals

Jugal K. Verma

This is a joint work with Jürgen Herzog and Tony J. Puthenpurakal.

The growth of Hilbert coefficients for powers of ideals are studied. For a graded ideal $I$ in the polynomial ring $S = K[x_1, \ldots, x_n]$ and a finitely generated graded $S$-module $M$, the Hilbert coefficients $e_i(M/I^kM)$ are polynomial functions. Given two families of graded ideals $(I_k)_{k\geq 0}$ and $(J_k)_{k\geq 0}$ with $J_k \subset I_k$ for all $k$ with the property that $J_kJ_\ell \subset J_{k+\ell}$ and $I_kI_\ell \subset I_{k+\ell}$ for all $k$ and $\ell$, and such that the algebras $A = \bigoplus_{k\geq 0} J_k$ and $B = \bigoplus_{k\geq 0} I_k$ are finitely generated, we show the function $k \to e_0(I_k/J_k)$ is of quasi-polynomial type, say given by the polynomials $P_0, \ldots, P_{g-1}$. If $J_k = J^k$ for all $k$, for a graded ideal $J$, then we show that all the $P_i$ have the same degree and the same leading coefficient. As one of the applications it is shown that if $I$ is a monomial ideal then $\lim_{k\to\infty} \lambda(\Gamma_{\mathfrak{m}}(S/I^k))/k^n$ is a rational number. We also study analogous statements in the local case.

# Squarefree modules over a toric face ring

Kohji Yanagawa

This is a joint work with Ryota Okazaki (Osaka University).

Recently, Bruns, Römer and their coworkers are studying *toric face rings* (over a field $K$). This notion is a generalization of both Stanley-Reisner rings and affine semigroup rings. Roughly speaking, as a Stanley-Reisner ring is associated with a simplicial complex, a toric face ring is associated with a "complex" which locally comes from a rational polyhedral fan. So, for a toric face ring $R$, we have a regular cell complex $\Sigma$ which is an "underlying space" of $R$. There is a one-one correspondence between maximal cells of $\Sigma$ and minimal primes of $R$. More generally, a cell $\sigma \in \Sigma$ gives a prime ideal $P_\sigma$ of $R$ such that $K[\sigma] := R/P_\sigma$ is an affine semigroup ring with $\dim K[\sigma] = \dim \sigma + 1$.

Under the assumption that $K[\sigma]$ is normal for all $\sigma \in \Sigma$, we will show the following.

(1) A toric face ring $R$ admits "Ishida complex", that is, the dualizing complex of $R$ is quasi-isomorphic to a complex of the form

$$0 \to \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma=d}} K[\sigma] \to \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma=d-1}} K[\sigma] \to \cdots \to \bigoplus_{\substack{\sigma\in\Sigma \\ \dim\sigma=0}} K[\sigma] \to K \to 0.$$

Here $d = \dim \Sigma$.

(2) We can extend the theory of squarefree modules ([3]) to a toric face ring $R$. In particular, Cohen-Macaulay property, Gorenstein* property, and Buchsbaum property of $R$ is a topological property of (the underlying space) of the regular cell complex $\Sigma$.

The difficulty of the proofs comes from the fact that "while each $K[\sigma]$ is an affine semigroup ring (i.e., a multi-graded ring), there is no good multi-grading on $R$ itself in general".

If I will have enough time, I will introduce a few applications of the notion of toric face rings.

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# Some properties of generalized test ideals


Ken-ichi Yoshida


This is a joint work with Mircea Mustaţă (University of Michigan).

The *generalized test ideals* introduced by Hara and I in [*A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), 3143–3174] are related to multiplier ideals via reduction to characteristic $p$. In addition, they satisfy many of the subtle properties of the multiplier ideals, which in characteristic zero follow via vanishing theorems. In this talk we give several examples to emphasize the different behavior of test ideals and multiplier ideals. Our result is

**Theorem 1.** *Let $(R, \mathfrak{m})$ be an $F$-finite regular local ring of characteristic $p > 0$. Then for any nonzero ideal $I$ in $R$, there exists a rational number $c > 0$ and $f \in I$ such that $\tau(f^c) = I$.*

*Moreover, if $I$ is $\mathfrak{m}$-primary, then we show that we may write also $I = \tau(\mathfrak{a}^{c'})$ for some $\mathfrak{m}$-primary ideal $I$ and some $c' > 0$.*

Let $\mathrm{fpt}(I)$ denote the *$F$-pure threshold*, which is the smallest jumping number (coefficient), of an ideal $I$. Using the above theorem and an idea of its proof, we can illustrate concrete counter-examples (in dimension 2) to the following questions:

(1) Is the ideal $\tau(f^c)$ integrally closed ?
(2) Is the ideal $\tau(f^{\mathrm{fpt}(f)})$ radical ?
(3) Suppose that $\tau(f^c)$ is $\mathfrak{m}$-primary. Is there an $\mathfrak{m}$-primary ideal $\mathfrak{b}$ such that $f \in \mathfrak{b}$ and $\mathrm{fpt}(f) = \mathrm{fpt}(\mathfrak{b})$ ?

Note that the corresponding question for multiplier ideals is always true.

# Deforming a chain complex to non-commutative direction

Yuji Yoshino

Let $k$ be a field and let $R$ be an arbitrary associative $k$-algebra. When we say $\mathbb{F} = (F, d)$ is a chain complex (or simply a complex) of $R$-modules, we mean that $F = \oplus_{i \in \mathbb{Z}} F_i$ is a graded left $R$-module and $d : F \to F[-1]$ is a graded homomorphism satisfying $d^2 = 0$. A projective complex $\mathbb{F} = (F, d)$ is just a complex where the underlying graded module $F$ is a projective $R$-module.

We introduce the category $\mathcal{A}_k$ which will be a base of infinitesimal deformation. The objects of the category $\mathcal{A}_k$ are artinian local $k$-algebras with residue field $k$ and the morphisms are $k$-algebra homomorphisms. (Note that the objects are not necessarily commutative rings, but they are of finite dimension over $k$.)

For a projective complex $\mathbb{F} = (F, d)$ of left $R$-modules, a chain complex $(F \otimes_k A, \Delta)$ over $R \otimes_k A^{op}$ is said to be a lift of $\mathbb{F}$ to $A \in \mathcal{A}_k$ if $(F \otimes_k A, \Delta) \otimes_A k = (F, d)$.

The aim of this talk is to construct the universal lift of a given projective chain complex $\mathbb{F} = (F, d)$ which dominates any lifts to any noncommutative artinian $k$-algebras, and to investigate the properties of its parameter algebra.

We should note that such a universal lift is no longer defined on an artinian algebra, but defined on a 'pro-artinian' local $k$-algebra. We call such a pro-artinian algebra a complete local $k$-algebra by an abuse of terminology for commutative ring theory. The noncommutative formal power series ring $k\langle\langle t_1, \dots, t_r \rangle\rangle$ with noncommutative variables $t_1, \dots, t_r$ is an example of complete local $k$-algebra. This is actually complete and separated in the $(t_1, \dots, t_r)$-adic topology. And a complete local $k$-algebra is defined to be a residue ring of the noncommutative formal power series ring by a closed ideal. In particular any artinian algebras in $\mathcal{A}_k$ is a complete local $k$-algebra. But the difficulty here is that complete local $k$-algebras are not necessarily noetherian rings.

Let $\mathbb{F} = (F, d)$ be a projective complex of left $R$-modules and we fix it. Then we define a covariant functor

$$\mathbb{F} : \mathcal{A}_k \to (Sets)$$

by setting as $\mathbb{F}(A)$ the set of chain-isomorphism classes of lifts of $\mathbb{F}$ to $A$ for any $A \in \mathcal{A}_k$.

The first main result of my talk is about the existence and the uniqueness of universal lifts, which actually says that the functor $\mathbb{F}$ is pro-representable.

**Theorem 1.** *Let $\mathbb{F} = (F, d)$ be a projective complex of left $R$-modules. We assume that it satisfies $r = \dim_k \operatorname{Ext}^1_R(\mathbb{F}, \mathbb{F}) < \infty$. Then the following statements hold true.*

(1) *There exists a universal lift $\mathbb{L}_0 = (F\widehat{\otimes}_k P_0, \Delta_0)$ of $\mathbb{F}$ defined over a complete local $k$-algebra $P_0$.*

*That is, the natural transformation of functors*

$$\phi_{\mathbb{L}_0} : \operatorname{Hom}_{k-alg}(P_0, -) \to \mathbb{F}$$

*defined by*

$$\phi_{\mathbb{L}_0}(f) = (F \otimes_k A, \Delta_0 \otimes_{P_0} {}_f A),$$

*for any $A \in \mathcal{A}_k$ and any $f \in \operatorname{Hom}_{k-alg}(P_0, A)$, is an isomorphism of functors. In this case, we say that $P_0$ is a parameter algebra.*

(2) *A parameter algebra $P_0$ is unique up to $k$-algebra isomorphisms.*

(3) *Fixing a parameter algebra $P_0$, a universal lift $\mathbb{L}_0$ is unique up to chain isomorphisms of complexes of left $R\widehat{\otimes}_k P_0^{op}$-modules.*

(4) *The parameter algebra has a description $P_0 \cong T/I$, where $T = k\langle\langle t_1, \ldots, t_r\rangle\rangle$ is a noncommutative formal power series ring of $r$ variables and $I$ is a closed ideal which is contained in the square of the unique maximal ideal of $T$.*

For example, if $P$ is a complete local $k$-algebra with maximal ideal $\mathfrak{m}_P$, and if $\mathbb{F} = (F, d)$ is a free resolution of a left $P$-module $k = P/\mathfrak{m}_P$, then we can see that there exists a universal lift of $\mathbb{F}$ of the form $\mathbb{L} = (F\widehat{\otimes}_k P, \Delta)$. In particular, $P$ itself is a parameter algebra of $\mathbb{F}$.

In my talk, I plan to give several properties of parameter algebras in connection with obstruction theory.

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# Micro-invariants of a one dimensional Cohen-Macaulay ring and invariants of its tangent cone

Santiago Zarzuela

Given a one-dimensional equicharacteristic Cohen-Macaulay local ring $A$, J. Elias introduced in 2001 the set of micro-invariants of $A$ in terms of its first neighbourhood ring. We give a new formula for these micro-invariants that allows to extend them to any one-dimensional Cohen-Macaulay local ring. On the other hand, the authors have recently defined a new set of invariants for the tangent cone of one-dimensional Cohen-Macaulay local rings (or more in general, for the fiber cone of any regular ideal with analytic spread one). We compare both sets of invariants by giving an explicit formula relating them and prove that, in fact, they coincide if and only if the tangent cone is Cohen-Macaulay. Some explicit computations will also be given.

*This is based on a joint work in progress with Teresa Cortadellas.*

# List of Participants

**Yuho Abukawa**
Meiji University

**Sarfraz Ahmad**
Abdus Salam School of Mathematical Sciences

**Josep Alvarez Montaner**
Universitat Politècnica de Catalunya

**Mutsumi Amasaki**
Hiroshima University

**Mohammad Hassan Anjom Shoa**
University of Payme Noor

**Tokuji Araya**
Nara University of Education

**Holger Brenner**
Universität Osnabrück

**Winfried Bruns**
Universität Osnabrück

**Gemma Colomè-Nin**
Universitat de Barcelona

**Alberto Corso**
University of Kentucky

**Nguyen Tu Cuong**
Institute of Mathematics, Hanoi

**Emanuela De Negri**
University of Genova

**Juan Elias**
Universitat de Barcelona

**Kazufumi Eto**
Nippon Institute of Technology

**Mukasa James Fredrick**
Sempa Institute of Sciences-Wobulenzi

**Laura Ghezzi**
New York City College of Technology (CUNY)

**Shiro Goto**
Meiji University

**Tadahito Harima**
Hokkaido University of Education, Kushiro

**Mitsuyasu Hashimoto**
Nagoya University

**Futoshi Hayasaka**
Meiji University

**Le Tuan Hoa**
Institute of Mathematics, Hanoi

**Jun Horiuchi**
Meiji University

**Yosuke Horiuchi**
Meiji University

**Craig Huneke**
University of Kansas

**Eero Hyry**
University of Tampere

**Shin-ichiro Iai**
Hokkaido University of Education, Sapporo

**Kei-ichiro Iima**
Okayama University

**Cristodor Ionescu**
Institute of Mathematics of the Romanian Academy

**Osamu Iyama**
Nagoya University

**Yuji Kamoi**
Meiji University

**Yuta Kanazawa**
Meiji University

**Tadashi Kanzo**
Kanto Gakuin University

**Kiriko Kato**
Osaka Prefecture University

**Ken-ichiroh Kawasaki**
Nara University of Education

**Takeshi Kawasaki**
Tokyo Metropolitan University

**Kyouko Kimura**
Nagoya University

**Satoru Kimura**
Meiji University

**Asuki Koura**
Meiji University

**Kazuhiko Kurano**
Meiji University

**Shigeru Kuroda**
Tokyo Metropolitan University

**Sijong Kwak**
KAIST

**Graham J. Leuschke**
Syracuse University

**Kuei-Nuan Lin**
Purdue University

**Keivan Borna Lorestani**
University of Tehran

**Tatsuo Matsubara**
Meiji University

**Naoyuki Matsuoka**
Meiji University

**Chikashi Miyazaki**
Saga University

**Mitsuhiro Miyazaki**
Kyoto University of Education

**Reza Naghipour**
University of Tabriz

**Koji Nishida**
Chiba University

**Jun-ichi Nishimura**
Osaka Electro-Communication University

**Atsushi Noma**
Yokohama National University

**Satoshi Ohnishi**
Nihon University

**Hidefumi Ohsugi**
Rikkyo University

**Masahiro Ohtani**
Nagoya University

**Ryota Okazaki**
Osaka University

**Akira Ooishi**
Yokohama National University

**Khalid Ouarghi**

**Kazuho Ozeki**
Meiji University

**Euisung Park**
Korea University

**Francesc Planas-Vilanova**
Universitat Politècnica de Catalunya

**Claudia Polini**
University of Notre Dame

**Paul Calvin Roberts**
University of Utah

**Maria Evelina Rossi**
University of Genova

**Hideto Sakurai**
Meiji University

**Rodney Yorke Sharp**
University of Sheffield

**Takafumi Shibuta**
Kyushu University

**Yasuhiro Shimoda**
Kitasato University

**Vasudevan Srinivas**
Tata Institute

**Hidenori Suzuki**
Meiji University

**Shunsuke Takagi**
Kyushu University

**Ryo Takahashi**
Shinshu University

**Yukihide Takayama**
Ritsumeikan University

**Naoki Terai**
Saga University

**Massoud Tousi-Ardakani**
Shahid Beheshti University

**Ngo Viet Trung**
Institute of Mathematics, Hanoi

**Bernd Ulrich**
Purdue University

**Wolmer V. Vasconcelos**
Rutgers University

**Jugal Kishore Verma**
Indian Institute of Technology Bombay

**Kei-ichi Watanabe**
Nihon University

**Shuhei Yamada**
Meiji University

**Kikumichi Yamagishi**
Himeji Dokkyo University

**Kohji Yanagawa**
Kansai University

**Ken-ichi Yoshida**
Nagoya University

**Yuji Yoshino**
Okayama University

**Santiago Zarzuela**
Universitat de Barcelona